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Some formal aspects of the Melosh and generalized unitary transformations of a class of massive-particle wave equations*

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The Melosh, Foldy-Wouthuysen, Cini-Touschek, and Majorana transformations of the Dirac Hamiltonian are discussed in a parallel way as special cases of two general unitary transformations for spin $\frac{1}{2}$, which generalize to include all integer and half-integer spins for the Weaver-Hammer-Good class of massive-particle wave equations. It is further shown how to construct a matrix $\tilde{\alpha}_3$ for all spins, the generalization of α_3 for spin $\frac{1}{2}$; and a discussion of "good" and "bad" components of operators and wave functions is given.

SPIN $\frac{1}{2}$

It is well known that one may transform the Dirac Hamiltonian for a free particle and antiparticle,

$$H_{1/2} = \vec{\alpha}\cdot\vec{p} + m\beta \tag{1}$$

[where m and $\vec{p}$ are the mass and momentum, related to the energy $E=(m^2+\vec{p}\cdot\vec{p})^{1/2}$, and $\vec{\alpha}$ and β are a set of 4×4 matrices], to the form $E\beta$ with a unitary transformation first studied by Pryce[1] and later by Tani[2] and Foldy and Wouthuysen[3] and which is commonly called the Foldy-Wouthuysen (FW) transformation,

$$U_{FW}^\dagger H_{1/2} U_{FW} = E\beta \ . \tag{2a}$$

The motivation for the transformation is dependent on the representation of Dirac matrices appropriate for considering the nonrelativistic limit, the Dirac-Pauli[4] representation, i.e.,

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3a}$$

with $\vec{\sigma}$ the usual Pauli matrices. One may imagine diagonalizing $H_{1/2}$ or making a unitary transformation to the rest-frame representation of matrix operators, or any other motivation to remove $\vec{\alpha}$ from the Hamiltonian so that the upper two components of wave functions are decoupled from the lower two components. The result for the unitary operator is

$$U_{FW}^\dagger = \exp[\tfrac{1}{2}\beta\vec{\alpha}\cdot\hat{p}\tan^{-1}(p/m)] . \tag{4a}$$

Another standard choice of a representation of Dirac matrices is the extreme relativistic or spinor representation[5] given by

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3b}$$

This representation is appropriate for discussing high-energy or massless Dirac particles because $H_{1/2}$ becomes $\vec{\alpha}\cdot\vec{p}$ when m/p goes to zero. Then the two kinds of two-component spinors in the Dirac wave function are not mixed. For this case one has the Cini-Touschek (CT) transformation[6] given by

$$U_{CT} H_{1/2} U_{CT}^\dagger = E\vec{\alpha}\cdot\hat{p} \ , \tag{2b}$$

with

$$U_{CT} = \exp[\tfrac{1}{2}\vec{\alpha}\cdot\hat{p}\beta\tan^{-1}(m/p)] , \tag{4b}$$

where $\hat{p}$ is a unit vector. In this representation, the matrix that has this form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is $-\gamma_5 = -i\alpha_1\alpha_2\alpha_3$ so that $\frac{1}{2}(1\pm\gamma_5)$ project out the upper/lower two-components of wavefunctions just as $\frac{1}{2}(1\pm\beta)$ does in the Dirac-Pauli representation.

Another aspect of the CT transformation is its connection with the infinite-momentum frame limit for evaluating certain sum rules, suggested by Fubini and Furlan,[7] and the related ideas of lightlike charges.[8] In connection with these ideas one is interested in operators $A(O)$ having the form

$$A(O) = \int d^3x\,\psi^\dagger(x)O\psi(x) = \int d^4x\,\delta(t)\psi^\dagger(x)O\psi(x) \ , \tag{5}$$

but Lorentz-transformed to a frame in which the speed v approaches the speed of light. It turns out that although the Lorentz transformation matrix that appears in

$$\mathfrak{U}_L\psi(x)\mathfrak{U}_L^{-1} = \exp[\tfrac{1}{2}\vec{\alpha}\cdot\hat{v}\tanh^{-1}(v)]\psi(x)$$

becomes infinite as $v\to1$, the bilinear combination appearing in Eq. (5) remains finite, becoming ($\vec{v}=v\hat{z}$)

$$\lim_{v\to 1}\mathfrak{U}_L A(O)\mathfrak{U}_L^{-1} = \int d^4x\, \delta(t+z) \times \psi^\dagger(x)\frac{1+\alpha_3}{\sqrt{2}}\, O\, \frac{1+\alpha_3}{\sqrt{2}}\,\psi(x). \tag{6}$$

It is clear that by restricting $\psi(x)$ to contain only the positive eigenvalue solution of Eq. (1), the operator form $(1+\alpha_3)/\sqrt{2}$ can be obtained from Eq. (2c). The utility of this operator form will be apparent in the representation of Dirac matrices given below in Eq. (3d).

In general, there are two other distinct representations of Dirac matrices in which the matrix form of β in Eqs. (3a) and (3b) is interchanged with one of the α's. The first is a variant of the Majorana representation[9] in which the role of β and one of the α's in Eq. (3b) is interchanged. The original purpose of this representation was to simplify discussion of the charge conjugation properties of the Dirac equation and so α_2 was chosen to interchange with β. In the present paper, however, it is more useful to single out α_3, so the representation is

$$\vec{\alpha}_T = \begin{pmatrix} \vec{\sigma}_T & 0 \\ 0 & -\vec{\sigma}_T \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \tag{3c}$$

where $\vec{\sigma}_T = (\sigma_1, \sigma_2, 0)$. Because of its analogy with the extreme relativistic representation, the roles of m and p_3 should be interchanged, and one expects this representation to be appropriate for discussing $p_3/(m^2+p_\perp{}^2)^{1/2} \ll 1$ particles, i.e., those particles with large transverse momenta, although it has not been used so far for such a discussion. In this representation the matrix $-i\alpha_1\alpha_2\beta$ has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and one expects to find a unitary operator that eliminates α_3 from Eq. (1) so that the transformed $H_{1/2}$ commutes with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

just as in the previous two representations. In detail, one has

$$U_{\text{Maj}} H_{1/2} U_{\text{Maj}}^\dagger = \alpha_1 p_1 + \alpha_2 p_2 + (m^2 + p_3{}^2)^{1/2}\beta, \tag{2c}$$

where the unitary operator has the matrix form

$$U_{\text{Maj}} = \exp\left[\tfrac{1}{2}\beta\alpha_3 \tan^{-1}\frac{p_3}{m}\right]. \tag{4c}$$

The fourth and final independent kind of representation of Dirac matrices is the one in which the role of β in Eq. (3a) is interchanged with one of the α's, taken here to be α_3. One has

$$\vec{\alpha}_T = \begin{pmatrix} 0 & -\vec{\sigma}_T \\ -\vec{\sigma}_T & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{3d}$$

Because of the analogy with Eq. (3a), one expects this representation to be important in describing physics with $(m^2+p_\perp{}^2)^{1/2}/p_3 \ll 1$, i.e., when infinite-momentum frame considerations are important and useful. Following the previous discussion, one would expect to be able to transform $H_{1/2}$ to a form that commutes with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(in this case α_3), i.e., to a form proportional to α_3. In fact, it is not possible to formulate such a unitary transformation in the simple manner of the previous three transformations. It takes a more complicated operator to perform the transformation, one example of which is

$$U' \boldsymbol{H}_{1/2} U'^\dagger = E\alpha_3, \tag{2d}$$

where

$$U' = \exp[\tfrac{1}{2}\alpha_3\beta(\pi/2)]\, U_{\text{FW}}^\dagger. \tag{4d}$$

In fact, the representation given in Eq. (3d) has seen much activity in the past two years in the context of the Melosh transformation.[10] This is the unitary transformation that is proposed to relate the constituent-quark and current-quark pictures of elementary-particle symmetries and interactions,[11] at least in the context of free quarks. In detail, the Melosh transformation is

$$U_{\text{M}} H_{1/2} U_{\text{M}}^\dagger = \alpha_3 p_3 + (m^2 + p_\perp{}^2)^{1/2}\beta, \tag{7}$$

with

$$U_{\text{M}} = \exp[\tfrac{1}{2}\beta\vec{\alpha}\cdot\hat{p}_\perp \tan^{-1}(p_\perp/m)]. \tag{8}$$

The right-hand sides of Eq. (7) and Eq. (2d) are identical in the limit of large $p_3/(m^2+p_\perp{}^2)^{1/2}$, but Eq. (8) has the advantage over Eq. (4d) of containing only matrices that commute with α_3. This means that U_{M} transforms "good" operators into "good" operators.[12] The unitary operator U_{M} also has the interesting property that when the Dirac Hamiltonian is generalized to include the effect of a constant, external magnetic field in the z direction and a Pauli-type anomalous-magnetic-moment interaction, the unitary transformation equivalent to Eq. (8) brings $H_{1/2}$ into a form whose square is diagonal.[13]

Some aspects of the constituent-quark and current-quark transforms have been studied by Palmer and Rabl,[14] in particular their Fock-space realizations. They found no overlap between current- and constituent-quark states in the free quark in the absence of a momentum cutoff. Bell and Ruegg[15] have investigated the null-plane dynamics of hydrogenlike atoms in a nonrelativistic approximation. Through second order in v/c they find an $SU(2)_W$ symmetry which is broken in third order. The symmetry-breaking term can, however, be transformed away by the Melosh transformation. Their investigation is important because it involves transforming between representations in the presence of an interaction. Of course, as mentioned above, an exact transformation can be carried out with a constant field and anomalous moment.

It should be pointed out that the unitary transformations of $H_{1/2}$ may be defined and discussed in a general way[16] by defining the unitary operator $U_{1/2}(\hat{e}, \theta)$ according to

$$U_{1/2}(\hat{e}, \theta) = \exp(\tfrac{1}{2}\vec{\alpha}\cdot\hat{e}\beta\theta) = \cos(\tfrac{1}{2}\theta) + \vec{\alpha}\cdot\hat{e}\beta\sin(\tfrac{1}{2}\theta) , \quad (9)$$

where $\hat{e}$ is a unit vector and θ is a number, both to be specified by the particular transformation. In general, one has

$$\exp(\tfrac{1}{2}\vec{\alpha}\cdot\hat{e}\beta\theta)(\vec{\alpha}\cdot\vec{p}+m\beta)\exp(-\tfrac{1}{2}\vec{\alpha}\cdot\hat{p}\beta\theta) = \vec{\alpha}\cdot\vec{p} + (m\sin\theta + \hat{e}\cdot\vec{p}\cos\theta - \hat{e}\cdot\vec{p})\vec{\alpha}\cdot\hat{e} + (m\cos\theta - \hat{e}\cdot\vec{p}\sin\theta)\beta . \quad (10)$$

Not all the wanted unitary transformations of $H_{1/2}$ can be generated by Eq. (11) [see, for example, Eqs. (2d) and (4d)]. One, in fact, needs two forms of $H_{1/2}$ to reach all the unitary transformations, i.e., one has also

$$\exp(\tfrac{1}{2}\vec{\alpha}\cdot\hat{e}\beta\theta)E\beta\exp(-\tfrac{1}{2}\vec{\alpha}\cdot\hat{e}\beta\theta) = E\sin\theta\,\vec{\alpha}\cdot\hat{e} + E\cos\theta\,\beta. \quad (11)$$

Many of the same transformations of $H_{1/2}$ can be obtained, both by the form (10) and the form (11), but, for example, the form

$$H'_{1/2} = \vec{\alpha}\cdot\hat{p}_\perp(m^2+p_\perp{}^2)^{1/2} + \alpha_3 p_3 \quad (12)$$

obtained from Eq. (10) with $\hat{e}=\hat{p}_\perp$ and $\theta = \tan^{-1}(m/p_\perp)$ cannot be reached by Eq. (11). The transformed Hamiltonian in Eq. (12) is a kind of Melosh-transformed $H_{1/2}$ in that the operator $U(\hat{p}_\perp, \tan^{-1}(m/p_\perp))$ commutes with α_3, although the operations under which $H'_{1/2}$ are invariant require commutation with α_3 and $\vec{\alpha}\cdot\hat{p}_\perp$, rather than α_3 and β.

As a final point on the spin-$\frac{1}{2}$ transformations, it is worth noting that, in general, when one transforms $H_{1/2}$ according to Eq. (10) to another form, the simplification or modification that one gains in the Hamiltonian is paid for by the more complicated Lorentz-transformation properties of the modified wave functions.

SPIN 1

To generalize these results to higher spin, one requires a massive-particle description that has a well-defined Hamiltonian operator and a wave function with simple Lorentz-transformation properties that is not subject to auxiliary conditions. Such a description has been investigated in detail by Weaver, Hammer, and Good[17] and will be used here. Following Ref. 17, the wave function ψ, representing a particle and antiparticle of mass m and spin s, is $2(2s+1)$-dimensional and satisfies the wave equation

$$H_s\psi(\vec{x}, t) = i(\partial/\partial t)\psi(\vec{x}, t) , \quad (13)$$

where $H_{1/2}$ depends on E, m, $\vec{p}$, and the $2(2s+1)$-dimensional matrices $\vec{\alpha}$ and β which have the form, in the spinor representation,

$$\vec{\alpha} = \frac{1}{s}\begin{pmatrix}\vec{s} & 0\\ 0 & -\vec{s}\end{pmatrix}, \quad \beta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad (14)$$

with $\vec{s}$ being the Hermitian spin-s spin matrices. There is a well-defined prescription for finding H_s, and the spin-1 specialization is

$$H_1/E = \beta + \frac{2E\vec{\alpha}\cdot\vec{p}}{E^2+p^2} - \frac{2(\vec{\alpha}\cdot\vec{p})^2\beta}{E^2+p^2} . \quad (15)$$

In analogy with Eq. (9), define the unitary operator $U_1(\hat{e}, \phi)$ by

$$U_1(\hat{e}, \phi) = \exp(\vec{\alpha}\cdot\hat{e}\beta\phi) = 1 - (\vec{\alpha}\cdot\hat{e})^2 + \cos\phi\,\vec{\alpha}\cdot\hat{e} + \sin\phi\,\vec{\alpha}\cdot\hat{e}\beta . \quad (16)$$

Then

$$U_1(\hat{e}, \phi)\beta U_1^\dagger = \beta + \sin(2\phi)\vec{\alpha}\cdot\hat{e} + [\cos(2\phi) - 1](\vec{\alpha}\cdot\hat{e})^2\beta . \quad (17)$$

Choosing $\hat{e}=\hat{p}$ and $\tan\phi = p/E$ defines the unitary spin-1 FW transformation[18] relating β and H_1/E. Choosing $\hat{e}=\hat{z}$ and $\tan 2\phi = p_3/(m^2+p_\perp{}^2)^{1/2}$ defines the operator that transforms β to

$$U_1\left(\hat{z}, \tfrac{1}{2}\tan^{-1}\frac{p_3}{(m^2+p_\perp{}^2)^{1/2}}\right)\beta U^\dagger = \beta(1-\alpha_3{}^2) + \frac{(m^2+p_\perp{}^2)^{1/2}}{E}\beta\alpha_3{}^2 + \frac{p_3}{E}\alpha_3. \quad (18)$$

If the restriction to eigenstates of α_3 with eigenvalues ±1 is made, Eq. (18) becomes identical in form to Eq. (7), the Melosh Hamiltonian. Finally, the parameters $\hat{e}=\hat{p}$, $\phi=\frac{1}{4}\pi$ give the high-energy $(p/E\to 1)$ limit of H_1/E. In detail,

$$U_1(\hat{p},\tfrac{1}{4}\pi)=1-(\vec{\alpha}\cdot\hat{p})^2+\frac{1}{\sqrt{2}}(\vec{\alpha}\cdot\hat{p})^2+\frac{1}{\sqrt{2}}\vec{\alpha}\cdot\hat{p}\beta\,, \tag{19}$$

$$U_1(\hat{p},\tfrac{1}{4}\pi)\beta U_1^\dagger=\vec{\alpha}\cdot\hat{p}+\beta[1-(\vec{\alpha}\cdot\hat{p})^2]\,. \tag{20}$$

Equation (20) is identical to Eq. (2b) if the restriction to helicity ±1 states is made.

The form of Eq. (20) suggests a generalization of the matrix α_3 to $\tilde{\alpha}_3$ given by

$$\tilde{\alpha}_3\equiv\alpha_3+\beta(1-\alpha_3^2)\,. \tag{21}$$

The matrix $\tilde{\alpha}_3$ reduces to α_3 when $\alpha_3=\pm1$ eigenstates are considered, and it is nonsingular with eigenvalue ±1, so that it may be put in the diagonal form

$$\tilde{\alpha}_3^{\text{diag}}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,. \tag{22}$$

In detail, if s_3 is diagonal, then in the spinor representation

$$\tilde{\alpha}_3^{\text{diag}}=V\tilde{\alpha}_3V^\dagger\,, \tag{23}$$

where V is a unitary matrix having nonzero elements $V_{11}=V_{44}=V_{36}=V_{63}=1$, $V_{22}=V_{25}=V_{52}=-V_{55}=1/\sqrt{2}$. One may speak of "good" and "bad" components with respect to $\tilde{\alpha}_3$, just as one does for α_3 in spin-$\frac{1}{2}$ problems.

SPIN $\frac{3}{2}$

The complications of higher spin become clear when one considers $s=\frac{3}{2}$, as shown below. One has the Hamiltonian

$$H_{3/2}=[(2E^2+7p^2)m\beta+(6E^2+20p^2)\vec{\alpha}\cdot\vec{p}-9m(\vec{\alpha}\cdot\vec{p})^2\beta-18(\vec{\alpha}\cdot\vec{p})^3]/2(E^2+3p^2)\,, \tag{24}$$

but there is no unitary matrix with the form $\exp(\frac{3}{2}\vec{\alpha}\cdot\hat{p}\beta\phi)$ that transforms between $E\beta$ and $H_{3/2}$. Instead, one requires the more general form $U_s(\hat{e},\phi_0,\phi_1,\ldots)$ given by

$$U_s(\hat{e},\phi_0,\phi_1,\ldots)=\exp\left[\sum_{k=0}^{N}(s\vec{\alpha}\cdot\hat{e})^{2k+1}\beta\phi_k\right]\,, \tag{25}$$

with $N=s-1$ for integer spin and $s-\frac{1}{2}$ for half-integer spin. This form is the most general unitary matrix constructed from the matrices $\vec{\alpha}\cdot\hat{e}$, β, and multiples, since the spin matrices satisfy their characteristic equation and so limit the maximum power of $\vec{\alpha}\cdot\hat{e}$ for a given spin. For spin $\frac{1}{2}$ $(\vec{\alpha}\cdot\hat{e})^2=1$ and for spin 1 $(\vec{\alpha}\cdot\hat{e})^3=\vec{\alpha}\cdot\hat{e}$, so Eq. (25) contains $U_{1/2}$ and U_1 as the simplest cases.

For spin $\frac{3}{2}$ the specialization of Eq. (25) is

$$U_{3/2}(\hat{e},\phi_0,\phi_1)=\exp[\tfrac{3}{2}\vec{\alpha}\cdot\hat{e}\beta\phi_0+(\tfrac{3}{2}\vec{\alpha}\cdot\hat{e})^3\beta\phi_1] \tag{26}$$

and the transformation between $E\beta$ and $H_{3/2}$ is effected with the parameters

$$\hat{e}=\hat{p}\,,$$

$$\phi_0=\tfrac{9}{8}\tan^{-1}\frac{p}{m}-\tfrac{1}{24}\tan^{-1}\frac{p(3E^2+p^2)}{m^3}\,,$$

and

$$\phi_1=-\tfrac{1}{2}\tan^{-1}\frac{p}{m}+\tfrac{1}{6}\tan^{-1}\frac{p}{m}\,\frac{(3E^2+p^2)}{m^2}\,.$$

Letting $p\to\infty$ gives one the high-energy limit, $\hat{e}=\hat{p}$, $\phi_0=\frac{13}{24}\pi$, and $\phi_1=-\frac{1}{6}\pi$, resulting in

$$U_{3/2}(\hat{p},\tfrac{13}{24}\pi,-\tfrac{1}{6}\pi)\beta U_{3/2}^\dagger=\vec{\alpha}\cdot\hat{p}[\tfrac{13}{4}-\tfrac{9}{4}(\vec{\alpha}\cdot\hat{p})^2]\,, \tag{27}$$

the massless limit of $H_{3/2}/E$, as expected, and suggesting the form for $\tilde{\alpha}_3$ to be for spin $\frac{3}{2}$

$$\tilde{\alpha}_3\equiv\alpha_3(\tfrac{13}{4}-\tfrac{9}{4}\alpha_3^2)\,, \tag{28}$$

a nonsingular matrix with eigenvalues ±1, which may be put in the diagonal form of Eqs. (3d) and (22) using the 8×8 unitary matrix V, which for s_3 diagonal has the nonzero elements

$$V_{11}=V_{22}=V_{37}=V_{48}=V_{55}=V_{66}=V_{73}=V_{84}=1\,.$$

Finally, with the choice of parameters

$$\hat{e}=\hat{z}\,;$$

$$\phi_0=\tfrac{13}{12}\tan^{-1}\frac{p_3}{(m^2+p_\perp{}^2)^{1/2}}\,,$$

and

$$\phi_1=-\tfrac{1}{3}\tan^{-1}\frac{p_3}{(m^2+p_\perp{}^2)^{1/2}}$$

one transforms $E\beta$ for spin $\frac{3}{2}$ to the form

$$EU_{3/2}(\hat{z},\phi_0,\phi_1)\beta U_{3/2}^\dagger=(m^2+p_\perp{}^2)^{1/2}\beta+p_3\tilde{\alpha}_3 \tag{29}$$

in complete parallel to Eq. (7). A further parallel for the half-integer spins is the form of the high-energy limit of $U_{3/2}$ which may be written (for $\hat{e}=\hat{z}$)

$$U_{3/2}(\hat{z},\tfrac{13}{24}\pi,-\tfrac{1}{6}\pi)=\frac{1}{\sqrt{2}}(1+\tilde{\alpha}_3\beta)\,, \tag{30}$$

again in complete parallel with Eq. (4b) specialized to $\hat{p}=\hat{z}$, $\theta=\frac{1}{2}\pi$. One expects this kind of parallel construction to hold in general for half-integer spin, whereas for integer spin the parallel with Dirac theory will be less complete, as shown for spin 1.

For general spin, one solves the equation

$$U_s(\hat{p}, \phi_0, \phi_1, \ldots)\beta U_s^\dagger = H_s/E \tag{31}$$

for the parameters $\phi_0, \phi_1, \ldots,$ using the prescription of Ref. 17 to find H_s. This gives the unitary FW transformation for spin s. Letting $p \to \infty$ in $U_s(\hat{p}, \phi_0, \phi_1, \ldots)$ gives the unitary CT transformation for spin s, and the massless limit of H_s/E defines the structure of $\tilde{\alpha}_3$ appropriate for spin s. One then constructs the parallel of Eqs. (7) and (29) for half-integer spin and the more limited parallel of Eq. (18) for the integer spins which are limited because of the zero eigenvalue in the spin matrices.

DISCUSSION

This paper has shown how the various kinds of unitary transformations of the Dirac equation are related to the ways of representing Dirac matrices. In the particular case of the "good" representation, it is seen that the conventional kind of transformation to $E\alpha_3$ is not possible using only the simplest form of the unitary operator. In fact, this representation has been extensively used in quark systematics in the context of the Melosh transformation,[10] where one transforms with a simple unitary operator to the form $\beta(m^2 + p_\perp{}^2)^{1/2} + \alpha_3 p_3$, rather than $E\alpha_3$. The formal considerations leading to the Melosh transformation have been extended to higher spin for a particular class of massive-particle wave equations,[18] and the result is to identify matrices that could play the role of α_3 in the "good" representation, and Hamiltonians that have formal similarity to the Melosh Hamiltonian. Of course, one important question to be eventually answered is: "What happens to the unitary transformations and any associated physical content in the presence of interactions?" Very preliminary answers have been made in this paper for spin $\frac{1}{2}$ when the interaction takes the form of a constant external magnetic field, including an anomalous Pauli moment term, and by Bell and Ruegg.[15] Perhaps the next step to consider is a quark–vector-gluon model in which the interactions cannot be described simply as external fields. For example, one might have an interaction term of the form $\bar{\psi}\sigma_{\mu\nu}\psi(aF_{\mu\nu} + bG_{\mu\nu})$ or $\bar{\psi}\gamma_\mu\psi A_\mu$, where $F_{\mu\nu}$, $G_{\mu\nu}$, and A_μ are Proca-type descriptions of massive spin-1 fields. The present paper sets the framework and gives the free-particle discussion and leaves the interaction considerations to a future publication.

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