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Solutions of the Callan-Symanzik equation in a complex neighborhood of zero coupling*

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In this paper we investigate some of the consequences of having the vertex functions in a field theory or a model, that satisfy the Callan-Symanzik equation, also satisfy some analyticity and uniformity properties in the coupling constant g. The solutions of the Callan-Symanzik equation in a complex neighborhood of the origin are studied. The implications of analyticity in g to scaling, anomalous dimensions, and Borel summability for large Euclidean momenta are pointed out. The input we start with, though not yet established in four-dimensional field theories, has been proved for two-dimensional ϕ^4 theories. It also happens to be true in many "models" discussed in the literature in connection with Bjorken scaling.

I. INTRODUCTION

It is generally recognized that perturbation expansions in field theory are most probably divergent. For certain superrenormalizable, self-coupled, boson field theories this divergence has actually been proved.[1] At best, the hope is that in renormalizable field theories perturbation expansions are asymptotic expansions.

Asymptotic series are useful for calculating physical quantities for small values of the coupling constant. However, in many cases one wants to use perturbation theory in a deeper way as a guide to the full theory. Much hard work has been done on proving properties of the Feynman perturbation series order by order. One would like to have a mechanism that could make a perturbative equation exact in an actual theory. In general there is an infinite class of functions which lead to the same asymptotic series, and perturbation expansions may not define a unique function. Several years ago, Simon[2] showed that summability techniques could give us a mechanism which will not only lead us in a unique way from perturbation theory to the full answer, but will also allow us to show that properties that were proved order by order in perturbation theory are valid in the full theory. Using a classical theorem by Watson and Carleman,[3] he formulated what was called the *strong asymptotic condition,* which leads to a form of Borel summability that guarantees a unique way of recovering the function from the perturbation expansion. A basic ingredient of the strong asymptotic condition is that the functions one is dealing with have to be analytic in the coupling constant in a sectorial region around the origin of the form $\{g \mid 0<|g|<r,\ |\arg g|<\frac{1}{2}\pi+\delta,\ \delta>0\}$. Another ingredient is the uniformity of the estimate of the error one gets in taking the first N terms in the perturbation expansion to approximate the function in the sectorial region.

Recently, Glimm, Jaffe, and Spencer[4] proved that in a $g\phi^4$ field theory in two dimensions the Euclidean Green's functions are analytic in g in a sectorial domain defined as above but with $|\arg g|<\frac{1}{2}\pi$. For g in that sector they also showed that the Green's functions remain tempered in x space even for complex values of g. Eckmann, Magnen, and Seneor[5] extended this sectorial domain to $|\arg g|<\frac{1}{2}\pi+\delta,\ \delta>0$. They also were able to establish all the estimates needed for the strong asymptotic condition to hold and thus proved the unique Borel summability of the Euclidean Green's function in this case. This much desired relationship between perturbation theory and the full theory is then true at least in one example in two dimensions.

One important physical example of the use of properties known only order by order in perturbation theory as properties of the full theory is the Callan-Symanzik procedure for calculating the asymptotic behavior of one-particle-irreducible Green's functions in the deep Euclidean region.[6,7] In this procedure certain mass insertion terms are ignored and assumed small even though one has only been able to prove this fact order by order in perturbation theory. Without the assumption that the full mass insertion terms are also negligible, one cannot get any useful results from the Callan-Symanzik procedure. Although this is rarely explicitly stated, the hope is presumably that some kind of summability method will enable us to prove this crucial assumption. If, for example, the Borel summability established by Eckmann *et al.* in two dimensions can be also proved in four dimensions, then one can actually prove that ignoring the mass insertion terms in the Callan-Symanzik (CS) equation can indeed be justified for the purpose of calculating asymptotic behavior.

Very little is known yet about analyticity in the coupling constant in *four* dimensions. However, if the singularity structure of Euclidean Green's functions in the neighborhood of the origin is too complicated, that in itself will throw considerable doubt on the validity of dropping the mass insertion terms in the CS procedure. Guided by this fact, and by the analogy with the two-dimensional case, we shall make certain simple analyticity assumptions in the coupling constant for the truncated, one-particle-irreducible Green's functions at Euclidean, nonexceptional momenta. The assumptions we make allow for the divergence of perturbation expansions and they are only limited to a small, finite sectorial neighborhood of the origin whose size may depend on the momenta.

Given this type of analyticity in g, the purpose of this paper is then to study the solution of the Callan-Symanzik equation in a complex neighborhood of the origin, $g=0$. We do this mainly for $g\phi^4$ theory in four dimensions, but the generalization of our results to other cases is quite simple.

Our first result shows that even though $g\phi^4$ is not asymptotically free for real $g>0$, nevertheless there exists a domain of complex initial values of g inside the assumed analyticity domain for which the vertex functions, $\Gamma^{(n)}(\lambda p_1, \ldots, \lambda p_n; g, m)$, behave as $\lambda \to \infty$ as they do in an asymptotically free theory. This is perhaps not so surprising. But what is surprising is that this domain of asymptotic freedom turns out to intersect with the region $\mathrm{Re}g>0$, and to have a boundary curve in that region that starts from the origin with zero slope. Thus for small enough $|g|$, initial values with $|g_i/g_r|>C\sin^2\epsilon$, $\arg g \equiv \epsilon$, will be shown to lie in the asymptotically free domain for any small $\epsilon>0$, as long as $|g|/\sin\epsilon$ is small. In other words, points arbitrarily close to the positive real g axis have $\Gamma^{(n)}(\lambda p_i; g, m)$ with asymptotically free behavior as $\lambda \to \infty$.

When one couples this result with the temperedness of $\Gamma^{(n)}(\lambda p_1, \ldots, \lambda p_n; g, m)$ in momentum space for complex g inside the analyticity domain, a property that has been established in Ref. 4 in the two-dimensional case, then one is led to severe restrictions on ϕ^4 field theory in four dimensions. If the theory has a nontrivial ultraviolet-stable fixed point at $g=g_\infty$, with $\beta(g_\infty)=0$ and $\beta'(g_\infty)<0$, then one is forced under our assumptions to choose between one of the following two alternatives: (I) The anomalous dimension at the first nontrivial fixed point vanishes, i.e., $\gamma(g_\infty)\equiv 0$, and the leading power behavior of the $\Gamma^{(n)}(\lambda p_i; g, m)$ for large λ and small real $g>0$ is canonical. (II) As $\lambda\to\infty$, the functions $\Gamma^{(n)}(\lambda p_i; g, m)$ must have a cut or a line of dense singularities in the g plane that approach the origin as λ grows and get denser in spacing. This line of singularities will be arbitrarily close to the positive real axis and will start off with zero slope from the origin. Alternative I can only be valid if the fixed point is sufficiently singular so that one cannot write a CS equation at the point $g=g_\infty$, but has to solve the differential equation for $g<g_\infty$ and then take the limit of the solution as $g\to g_\infty$. If the CS equation holds at $g=g_\infty$, then as has been shown by Parisi,[8] and Callan and Gross,[9] $\gamma(g_\infty)=0$ implies $g_\infty\equiv 0$ or a free-field theory. Alternative II means that for large λ one has not only to give up Borel summability for $\Gamma^{(n)}(\lambda p_i; g, m)$ except for very small values of g that shrink as $\lambda\to\infty$, but more seriously the existence of a line of singularities close to the positive real axis puts the Callan-Symanzik assumption about the mass insertion terms itself in doubt.

One is left with the possibility that ϕ^4 field theory has no nontrivial fixed points. In this case the usual real g Callan-Symanzik approach does not give us any information on the asymptotic behavior of $\Gamma^{(n)}$. Our present approach will in that case lead to the result that for all g in the domain of analyticity, including real positive values of g, the asymptotic behavior of $\Gamma^{(n)}(\lambda p_i; g, m)$ has the canonical powers in λ as $\lambda\to\infty$, i.e., λ^{4-n} times terms which increase (or decrease) slower than any power of λ. This of course is true only if the dense singularities of alternative II are not present.

In Sec. II we review the results of two-dimensional field theories and state our assumptions precisely. The Callan-Symanzik equation with complex g is then solved in Sec. III for g in a neighborhood of the origin. A domain of asymptotic freedom for $g\phi^4$ theory in four dimensions is found and certain properties of the solutions for g in this domain are derived. In Sec. IV we show how the results of Sec. III lead to the above-mentioned restrictions on the theory. We then briefly discuss the generalization of our results to gauge type theories and to non-Abelian gauge theories in Sec. V. In these cases one has to use the variable $\alpha=g^2$, and for non-Abelian gauge theories the line of singularities of alternative II lies in the second quadrant of the α plane and close to the negative real axis, leading to no inconsistency with Borel summability for $\mathrm{Re}\alpha>0$.

In Appendix B we briefly comment on the implication of our results for Bjorken scaling.

II. ANALYTICITY IN COUPLING CONSTANT

The purpose of this paper is to investigate the consequences of having the vertex functions in a field theory or a model satisfy certain general analyticity and uniformity properties in the coupling constant g. In this section we shall briefly state our analyticity assumptions and discuss the motivation for making them.

For simplicity, and for having a concrete example, we shall consider a massive $g\phi^4$ theory in four dimensions, or a model based on summing a certain subclass of graphs in ϕ^4 theory. Following the standard notation, we let $\Gamma^{(n)}(p_1,\ldots,p_n;g,m)$ denote the one-particle-irreducible, amputated part, of the corresponding connected Green's function, with $\sum_{i=1}^n p_i=0$. We are only interested in Euclidean, nonexceptional momenta, p_i. Our main assumptions are the following:

(A) For Euclidean, nonexceptional p_i, $\Gamma^{(n)}(p_1,\ldots,p_n;g,m)$ are analytic functions of g in a small cut circle domain D:

$$D=\{g\mid 0<|g|<r_0(p_i),\ |\arg g|<\pi\}. \tag{2.1}$$

A similar domain is assumed for the Callan-Symanzik functions $\beta(g)$ and $\gamma(g)$.

(B) The asymptotic properties, for small $|g|$,

$$\begin{aligned}\beta(g)&=cg^2+O(g^3),\quad c>0\\ \gamma(g)&=bg^2+O(g^3),\end{aligned} \tag{2.2}$$

hold *uniformly* for all $g\in D$.

There are several reasons for starting with such simple but perhaps strong assumptions and investigating their consequences. We list some of them here.

(1) Properties similar to (A) and (B) have been proved for two-dimensional ϕ^4 field theories.[4,5] Although everyone recognizes that the transition from two dimensions to four is not easy, still (A) and (B) have not been ruled out and give us simple but admittedly strong starting points. In fact one of the lessons of this paper, as we shall see below, is that assuming too much analyticity will lead to almost trivial theories, and even given (A) and (B) one possibility we show is that singularities start approaching the origin as $p_i\to\infty$ in the Euclidean region.

(2) Analyticity and uniformity in a sector with opening angle larger than π are necessary for Borel summability. Borel summability is one way one can prove the Callan-Symanzik assumption about dropping the mass insertion term in the Callan-Symanzik equation.

(3) Assumption (A) is the simplest one can make even if it is strong. It allows for a singularity at $g=0$ which forces perturbation expansions to diverge. We choose the cut related to this singularity along the negative real axis. What we are assuming is that the next singularity nearest the origin is at least of modulus $r_0(p_i)$. We have *not* excluded the possibility that $r_0(p_i)$ might collapse to zero as $p_i\to\infty$ in the Euclidean region. No uniformity in p_i is assumed.

(4) Finally, even if (A) and (B) are not true in a field theory, they are true in many models that have been used extensively in the literature to study the problem of Bjorken scaling. There are even models in which the $\Gamma^{(n)}$'s have a finite radius of convergence in g, and which also satisfy a Callan-Symanzik equation with a nontrivial $\beta(g)$.[10] For such models our results point out that it is the analyticity in g which is sufficient to explain the absense of power deviations from scaling, and not the specific dynamical details of the models. The fact that renormalization-group type equations are also being used in other areas such as the Reggeon calculus, where one again has sometimes relatively simple models, provides an additional incentive for this investigation.

Following the standard procedure we define the asymptotic forms at nonexceptional momenta which are denoted by $\Gamma_{\text{as}}^{(n)}(p_1,\ldots,p_n,g,m)$. Formally, these are obtained from $\Gamma^{(n)}(\lambda p_i;g,m)$ by first using perturbation theory in g and then expanding each order in a double power series in λ^{-1} and $\ln\lambda$. The formal sum obtained by discarding all but the leading terms in powers of λ^{-1} is $\Gamma_{\text{as}}^{(n)}$. It was shown by Symanzik[6] that the $\Gamma_{\text{as}}^{(n)}$ are the vertex functions of a theory with massless particles.

Even for nonasymptotic values of λ, the functions $\Gamma_{\text{as}}^{(n)}(\lambda p_i;g,m)$ satisfy the homogeneous Callan-Symanzik equation[6,7]

$$\left[-\lambda\frac{\partial}{\partial\lambda}+\beta(g)\frac{\partial}{\partial g}+4-n(1+\gamma(g))\right]\times\Gamma_{\text{as}}^{(n)}(\lambda p_1,\ldots,\lambda p_n;g,m)=0. \tag{2.3}$$

The functions $\Gamma^{(n)}(\lambda p_i;g,m)$ satisfy an equation of the form (2.3) with an additional term on the right. For large λ that term has been shown to be negligible only order by order in perturbation theory, and it is an open question whether one can prove that the sum is also negligible. If that is not so then $\Gamma_{\text{as}}^{(n)}$ will not give the asymptotic form of $\Gamma^{(n)}$ for large λ even at nonexceptional Euclidean momenta. One way to guarantee that the Callan-Symanzik procedure is valid would be to prove Borel summability in the sense of Ref. 2 for $\Gamma^{(n)}$, $\Gamma_{\text{as}}^{(n)}$, and the mass insertion terms.

We shall now further assume that $\Gamma_{\text{as}}^{(n)}(p_1,\ldots,p_n;g,m)$ have similar analyticity properties to those stated under assumption (A). We shall denote the cut circle domain for $\Gamma_{\text{as}}^{(n)}$ also by D even though in principle it might be different from the domain for $\Gamma^{(n)}$. Since one by dealing with $\Gamma_{\text{as}}^{(n)}$ is effectively dealing with a zero mass theory, this is an independent assumption which is excluding the possibility that a dense set of singularities collapse toward the origin as one goes to the zero-mass limit. In two-dimensional $g\phi^4$ theory, g has dimension of $(\text{mass})^2$ and only the dimensionless coupling constant (g,μ^{-2}) has an analyticity radius independent of some mass. However, in four dimensions g is dimensionless.

Thus we can consider Eq. (2.3) as holding for all $g \in D$, assuming that differentiation with respect to λ does not destroy the analyticity in g. In the next section we study the solutions of Eq. (2.3) for complex $g \in D$.

III. SOLUTIONS OF THE CALLAN-SYMANZIK EQUATION FOR $g \in D$

Our task in this section is to study the solution of Eq. (2.3) for complex $g \in D$. Before we proceed we briefly review the method of solving Eq. (2.3) for real g.

For real g, $g > 0$, the solution of the Callan-Symanzik equation is well known,

$$\Gamma^{(n)}_{\rm as}(\lambda p_i; g, m) = \lambda^{4-n}\Gamma^{(n)}_{\rm as}(p_i; \bar{g}(t,g), m)\exp\left[-n\int_g^{\bar{g}(t,g)}\frac{\gamma(x)}{\beta(x)}dx\right], \tag{3.1}$$

where $t \equiv \ln\lambda$, and the effective coupling constant $\bar{g}(t,g)$ is defined as the solution of the differential equation,

$$\frac{d\bar{g}}{dt}(t,g) = \beta(\bar{g}(t,g)), \tag{3.2}$$

with initial value

$$\bar{g}(0,g) = g.$$

In Eq. (2.2), $c > 0$, and starting with a small $g > 0$, $\bar{g}(t,g)$ will grow as $t \to \infty$. If we assume the existence of a fixed point, g_∞, with $\beta(g_\infty) = 0$, and $\beta'(g_\infty) < 0$, such that $\bar{g}(t,g) \to g_\infty$ for $t \to \infty$, then if $\Gamma^{(n)}_{\rm as}(p_i; g, m)$ and $\gamma(g)$ are left continuous at g_∞, Eq. (3.1) gives us the asymptotic behavior for large λ, and nonexceptional momenta:

$$\Gamma^{(n)}_{\rm as}(\lambda p_i; g, m) \cong \lambda^{4-n[1+\gamma(g_\infty)]}\Gamma^{(n)}_{\rm as}(p_i; g_\infty, m) \times \exp[-n r(t,g)], \tag{3.3}$$

where

$$r(t,g) = \int_0^t dt'[\gamma(\bar{g}(t',g)) - \gamma(g_\infty)] = o(\ln\lambda). \tag{3.4}$$

Thus in this case the field ϕ has the anomalous dimension $d = 1 + \gamma(g_\infty)$. If $\beta(g)$ on the other hand has no zero in the interval $0 < g < \infty$, then the CS approach tells us nothing about the behavior of $\Gamma^{(n)}(\lambda p_i; g, m)$ for large λ.

To extend the solution (3.1) to complex initial values $g \in D$, one has at first to find solutions of the differential equation (3.2) with real t and complex initial values $g \in D$ such that

(a) $\bar{g}(t,g)$ are analytic in g for $g \in D$ for all t, $0 < t < \infty$;

(b) for all t, $0 \leq t < \infty$, $\bar{g}(t,g) \in D$, i.e.,

$$0 < |\bar{g}(t,g)| < r_0(p_i), \quad |\arg\bar{g}(t,g)| < \pi. \tag{3.5}$$

Otherwise, if (b) is not satisfied, a $\bar{g}(t,g)$ which for some value of $t = t_0$ has a value outside D will be useless since then the term on the right-hand side of (3.1), $\exp\{-n\int_g^{\bar{g}}[\gamma(x)/\beta(x)]dx\}$, will not in general be defined if $\bar{g}$ is outside the domain of analyticity of $\gamma(g)$. [We are assuming that $\beta(z)$ has no zeros for $z \in D$ other than at the origin; g_∞ is taken to be outside D.]

We shall concentrate our attention on a small neighborhood of the origin in which the first term of Eq. (2.2) gives a good approximation to $\beta(g)$. Namely, we choose a value κ_0,

$$\kappa_0 < r_0(p_i), \tag{3.6}$$

such that for all $g = \kappa e^{i\phi}$ and $\kappa < \kappa_0$,

$$\beta(\kappa e^{i\phi}) \approx c\kappa^2 e^{2i\phi}, \quad |\phi| < \pi. \tag{3.7}$$

Let us define the domain $D_{\rm II}$ in the second quadrant of the g plane as

$$D_{\rm II} = \{g \mid 0 < |g| < \kappa_0, \ \tfrac{1}{2}\pi < \arg g < \pi\}. \tag{3.8}$$

The following theorem can be proved if the properties (A) and (B) are valid:

Theorem 1. For any $g \in D_{\rm II}$ there exists a unique solution $\bar{g}(t,g)$ of Eq. (3.2) which for any $t > 0$ is (a) analytic in g, (b) satisfies $|\bar{g}(t,g)| < \kappa_0$, $\frac{1}{2}\pi < \arg\bar{g}(t,g) < \pi$, and (c)

$$\lim_{t\to\infty}|\bar{g}(t,g)| = 0, \quad \lim_{t\to\infty}(\arg\bar{g}(t,g))\pi, \quad g \in D_{\rm II} \tag{3.9}$$

where the last limit is approached from above the negative real axis, i.e., for any finite $t > 0$, $\arg\bar{g}(t,g) < \pi$.

Hence, if we choose our initial g to be complex and to lie in $D_{\rm II}$, then $\bar{g}(t,g)$, the effective coupling constant, shrinks in modulus as $t \to \infty$ and $\Gamma^{(n)}_{\rm as}(\lambda p_i; g, m)$ behave as in an asymptotically free field theory for all $g \in D_{\rm II}$.

Proof of theorem 1. We shall leave the proof of the analyticity of $\bar{g}(t,g)$ in g for Appendix A. It consists of a simple extension of Picard's classical theorem for the existence of solutions to differential equations to the case of complex initial conditions.

The proof of assertions (b) and (c) is quite simple. Equation (3.2) is equivalent to the integral equation

$$\bar{g}(t,g) = g + \int_0^t dt'\,\beta(\bar{g}(t',g)). \tag{3.10}$$

We take $t=\Delta t$, then for $g=\kappa e^{i\phi}$, $\kappa<\kappa_0$,

$$\bar{g}(\Delta t, g)=\kappa e^{i\phi}+c\kappa^2 e^{2i\phi}\Delta t+O(\kappa^3). \tag{3.11}$$

But since $g \in D_{II}$, then $\pi/2<\phi<\pi$, and hence

$$|\bar{g}(\Delta t, g)| \cong \kappa\,|1-c\kappa e^{i\phi}\Delta t| < \kappa,$$

and we see that the modulus shrinks,

$$|\bar{g}(\Delta t, g)| < |\bar{g}(0,g)|. \tag{3.12}$$

Similarly, one can show that the phase increases,

$$[\arg\bar{g}(\Delta t, g)-\arg g]=c\kappa\Delta t\,\sin\phi+O(\kappa^2),\quad \tfrac{1}{2}\pi<\phi<\pi. \tag{3.13}$$

In general if $\bar{g}(t,g)\in D_{II}$, then

$$|\bar{g}(t+\Delta t, g)| < |\bar{g}(t,g)|, \tag{3.14}$$

and

$$\arg\bar{g}(t+\Delta t, g)-\arg\bar{g}(t,g)\cong c\,|\bar{g}(t,g)|\,\Delta t\,\sin\phi(t,g), \tag{3.15}$$

where $\phi(t,g)\equiv\arg\bar{g}(t,g)$. By induction one sees that as $t\to\infty$, $|\bar{g}(t,g)|\to 0$. Furthermore, the argument of $\bar{g}(t,g)$ for $g\in D_{II}$ increases with t and approaches π. It is clear from Eq. (3.15) that the increase in the phase of $\bar{g}$ for each interval $t\to t+\Delta t$ gets smaller and smaller as $\phi(t,g)$ gets closer to π since the right-hand side in (3.15) is proportional to $\sin\phi(t,g)$. Thus the flow $\bar{g}(t,g)$ approaches zero as $t\to\infty$ along a curve above the negative real axis whose tangent at the origin is zero.

For large t, one can easily show that $\bar{g}(t,g)$ approaches the solution of the differential equation $d\bar{g}_0/dt=c\bar{g}_0^2(t,g)$, which is given by

$$\bar{g}_0(t,g)\equiv\frac{-g}{cgt-1}. \tag{3.16}$$

At the end of Appendix A we show that for large t

$$\bar{g}(t,g)-\bar{g}_0(t,g)=O\left(\frac{1}{t^2}\right),\quad g\in D_{II} \tag{3.17}$$

where the error on the right is real and the imaginary part of the error is $O(t^{-3})$. This gives

$$\mathrm{Re}\,\bar{g}(t,g)\underset{t\to\infty}{\simeq}-\frac{1}{ct}+O\left(\frac{1}{t^2}\right), \tag{3.18}$$

$$\mathrm{Im}\,\bar{g}(t,g)\underset{t\to\infty}{\simeq}\frac{\mathrm{Im}\,g}{c^2|g|^2t^2}+O\left(\frac{1}{t^3}\right),\quad g\in D_{II}. \tag{3.19}$$

So far our results are not surprising since had we had analyticity in a full circle around the origin, then it is obvious that we would get $\bar{g}(t,g)\to 0$ as $t\to\infty$ if g is real and $g<0$. The domain we have considered up till now has $\mathrm{Re}\,g<0$.

For $g\in D_{II}$, we have $\bar{g}(t,g)\to 0$ as $t\to\infty$. On the other hand, for real $g, g>0$, the flow is such that $\bar{g}(t,g)$ increases with g and if a fixed point exists, $\bar{g}(t,g)\to g_\infty$ as $t\to\infty$. The question is: What happens to $\bar{g}(t,g)$ for initial values g that lie in the part of D in the first quadrant? In other words, can the domain of asymptotic freedom, D_{II}, be extended into the first quadrant? The surprising answer is that not only can D_{II} be extended, but the extension includes points which for small enough $|g|$ have a phase that can come as close as we want to zero. The domain we get is given by, for $g=|g|\exp(i\phi)$,

$$D_F=\left\{g\;\middle|\;0<|g|<\frac{\kappa_0}{a},\ \frac{\pi}{2}<\phi<\pi\right\}\cup\left\{g\;\middle|\;0<|g|<\frac{\kappa_0}{a}\sin\phi,\ 0<\phi\leq\frac{\pi}{2}\right\}, \tag{3.20}$$

with

$$a=1+\pi/2.$$

Thus in the second quadrant the domain D_F is bounded by the quarter circle with radius $|g|=\kappa_0/a$; and in the first quadrant the boundary is given by $|g|=(\kappa_0/a)\sin\phi$ where $\phi=\arg g$. The domain D_F is shown in Fig. 1.

We now prove the following theorem again accepting the assumptions (A) and (B) for $\beta(g)$:

Theorem 2. For any $g\in D_F$, there exists a unique solution of Eq. (3.2), $\bar{g}(t,g)$, which for any $t>0$ is (a) analytic in g, (b) has the property $\bar{g}\in D_F$, and (c) for large t, $\lim_{t\to\infty}\bar{g}(t,g)=0$ and $\lim_{t\to\infty}\arg\bar{g}(t,g)=\pi$ as in theorem 1.

Proof of theorem 2. Obviously, we do not have to prove the theorem if $\pi>\arg g>\pi/2$. This was already done in theorem 1. We consider an initial

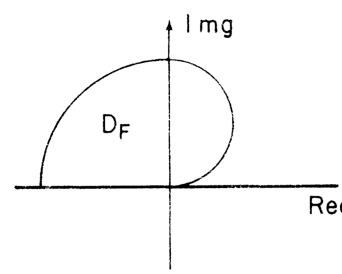

FIG. 1. The domain of asymptotic freedom, D_F.

g in the first quadrant with $g=|g|e^{i\phi}$ such that

$$|g|<\frac{\kappa_0}{a}\sin\phi, \quad a=1+\frac{\pi}{2}, \quad 0<\phi\leq\frac{\pi}{2}, \tag{3.21}$$

where ϕ can be as small as we please as long as the inequality (3.21) is satisfied. We define the sequence $\{g_n\}$ as

$$\begin{aligned} g_1 &= g+c|g|^2e^{2i\phi}\Delta t, \\ g_n &= g_{n-1}+c|g_{n-1}|^2e^{2i\phi_{n-1}}\Delta t, \end{aligned} \tag{3.22}$$

where Δt is small and $\arg g_n\equiv\phi_n$. As long as $|g_n|<\kappa_0$ the above sequence gives a good approximation to $\bar{g}(t,g)$, namely:

$$\bar{g}(n\Delta t,g)=g_n+O(n|g|^3). \tag{3.23}$$

This can be checked by approximately solving the integral equation (3.10) in discrete steps $t_n=n\Delta t$.

Starting with any ϕ such that $0<\phi<\pi/2$, we note that as long as $\phi_{n-1}<\pi/2$,

$$|g|<|g_1|<\cdots<|g_n|, \tag{3.24}$$

$$\phi<\phi_1<\cdots<\phi_n. \tag{3.25}$$

One can also show by repeatedly using the triangle inequality that

$$|g_n|\leq|g|\,(1+nc|g|\Delta t\,\cos\phi)+O(n|g|^3). \tag{3.26}$$

Hence if n is not too large, $|g_n|$ remains small. In fact if we define an $N_{\max}$ as

$$N_{\max}\equiv\frac{\pi}{2c|g|\Delta t\,\sin\phi}, \tag{3.27}$$

then for all $k\leq N_{\max}$,

$$|g_k|\leq|g|+\frac{\pi}{2}\frac{|g|}{\sin\phi}+O(N_{\max}|g|^3)$$

and using Eq. (3.21) we get

$$|g_k|\leq\kappa_0+O(\kappa_0{}^2), \quad k\leq N_{\max}. \tag{3.28}$$

The increase in phase at each step is given by

$$\delta_k=\phi_k-\phi_{k-1}\cong c|g_{k-1}|\Delta t\sin\left(\phi+\sum_{j=1}^{k-1}\delta_j\right)+O(|g|^2). \tag{3.29}$$

We claim that there must exist an integer $N_0\leq N_{\max}$ such that

$$\begin{aligned} \phi_{N_0} &= \frac{\pi}{2}+\epsilon, \quad \epsilon>0 \\ \phi_{N_0-1} &< \frac{\pi}{2}, \end{aligned} \tag{3.30}$$

for if no such N_0 exist then $g_{N_{\max}}$ will have a phase less than or equal to $\pi/2$, as will all the g_k's for $k\leq N_{\max}$. But this gives a contradiction since as long as all the g_k's lie in the first quadrant, δ_k will have a minimum which follows from Eq. (3.29):

$$\delta_k\geq c|g|\Delta t\,\sin\phi+O(|g|^2), \tag{3.31}$$

where we have used the inequalities $|g|<|g_k|$ and $\sin\phi<\sin(\phi+\sum_{j=1}^{k-1}\delta_j)$, both of which are true as long as $\phi_k<\pi/2$ for all $k\leq N_{\max}$. The lower bound (3.31) leads to a lower bound on $\phi_{N_{\max}}$,

$$\begin{aligned} \phi_{N_{\max}} &= \phi+\sum_{k=1}^{N_{\max}}\delta_k\geq\phi+N_{\max}\delta_{\min} \\ &\geq\phi+\frac{\pi}{2}. \end{aligned} \tag{3.32}$$

This contradicts the statement that $\arg g_{N_{\max}}<\pi/2$, and proves our assertion about the existence of N_0.

Thus starting with g in the first quadrant with any positive phase ϕ but satisfying Eq. (3.21), the sequence $g_1, g_2, \ldots, g_k, \ldots, g_{N_0}$ will increase in phase and modulus until finally the phase of g_{N_0} is bigger than $\pi/2$. However, even though the modulus increases it remains bounded for all $k\leq N_0\leq N_{\max}$ such that $|g_k|<\kappa_0$, and we never get to a value of $|g_k|$, where our approximation for $\beta(g_k)\approx cg_k{}^2$ is not good. Once the phase passes $\pi/2$, i.e., for g_{N_0+1}, g_{N_0+2}, etc. then we are in the domain of theorem 1, and from then on the modulus starts shrinking and we approach the origin in the same way as in theorem 1. Namely, we have for $k>N_0$ $\lim_{k\to\infty}|g_k|\to 0$ and $\lim_{k\to\infty}\arg g_k=\pi$. It is obvious that the asymptotic properties for large t of $\bar{g}(t,g)$ given by Eqs. (3.17), (3.18), and (3.19) also hold in this case. This completes the proof of assertion (c) of theorem 2. Assertion (a) on the analyticity of $\bar{g}(t,g)$ in g will also be proved in Appendix A.

We are left with the task of proving that if $g\in D_F$ then $\bar{g}(t,g)\in D_F$, i.e., assertion (b). To do this we first show that if $g\in D_F$ then $\bar{g}(\Delta t,g)\in D_F$. We are only interested in the case $0<\arg g<\pi/2$, since those values of g in the second quadrant part of D_F have already been dealt with in theorem 1. We have then for $g\in D_F$ and $\arg g<\pi/2$

$$|g|<\frac{\kappa_0}{a}\sin\phi, \tag{3.33}$$

and from the integral equation (3.10),

$$\bar{g}(\Delta t,g)=|g|\,e^{i\phi}(1+c|g|\Delta t\,e^{i\phi})+O(|g|^3), \tag{3.34}$$

which gives

$$|\bar{g}(\Delta t,g)|=|g|\,(1+c|g|\Delta t\,\cos\phi)+O(|g|^3), \tag{3.35}$$

$$\arg\bar{g}(\Delta t,g)\equiv\phi(\Delta t,g)=\phi+c|g|\Delta t\,\sin\phi+O(|g|^2). \tag{3.36}$$

Using the above four equations, we get

$$\frac{\kappa_0}{a}\sin\phi(\Delta t,g)\cong\frac{\kappa_0}{a}\sin(\phi+c|g|\Delta t\cos\phi)$$
$$\cong\frac{\kappa_0}{a}\sin\phi+\frac{\kappa_0}{a}\sin\phi c|g|\Delta t\cos\phi$$
$$\geqslant|g|+c|g|^2\Delta t\cos\phi\geqslant|\bar{g}(\Delta t,g)|, \tag{3.37}$$

where in the above inequalities we have ignored terms of higher order in $|g|$. It is clear from (3.37) that $\bar{g}(\Delta t,g)\in D_F$. In a similar way one can show that if for any t

$$\bar{g}(t,g)\in D_F,$$

then

$$\bar{g}(t+\Delta t,g)\in D_F.$$

This completes the proof of our theorem.

Perhaps we should here stress the fact that the D_F we have found in theorem 2 is not necessarily the maximal domain for asymptotic freedom. The boundary in the first quadrant, $|g|=(\kappa_0/a)\sin\phi$, is not, in the neighborhood of the origin, the "separatrix" or "critical line"[11] of the differential equation (3.2) but gives just an upper bound to it. However, for the purposes of this paper the domain D_F is sufficient.

For any complex $g\in D_F$ we can at this stage write the solution of the Callan-Symanzik equation as in Eq. (3.1), namely

$$\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)=\lambda^{4-n}\Gamma_{\rm as}^{(n)}(p_i;\bar{g}(t,g),m)\exp\left[-n\int_g^{\bar{g}(t,g)}\frac{\gamma(x)}{\beta(x)}dx\right]. \tag{3.1$'$}$$

The integral on the right-hand side in (3.1′) is a contour integral along a curve lying fully in D_F. We assume $\beta(z)$ has no zeros for $z\in D_F$, and it is then clear that the exponential term in Eq. (3.1′) is analytic in g for all $g\in D_F$ since $\bar{g}(t,g)$ is analytic. The size of the domain D_F for this term depends only on the size of the domain for $\beta(g)$ and is independent of $t=\ln\lambda$ and of the p_i's. We recall that $|\bar{g}(t,g)|<\kappa_0/a$ for all t and shrinks to zero for large t. The term $\Gamma_{\rm as}^{(n)}(p_i;\bar{g}(t,g),m)$, being an analytic function of an analytic function, is analytic in g for all $g\in D_F$. Again in getting the radial size of D_F here we only have to satisfy the inequality $|g|<(\kappa_0/a)\sin\phi$, with $\kappa_0<r_0(p_i)$, and κ_0 does not depend on λ for fixed p_i, i.e., we do *not* need an inequality of the form $\kappa_0<r_0(\lambda p_i)$. For fixed Euclidean p_i, the whole right-hand side of Eq. (3.1′) is analytic in a domain D_F whose size is determined by κ_0 and does not shrink as $\lambda\to\infty$.

The following properties of the solution $\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)$ can be stated:

(i) $\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)$ is analytic in g for all $g\in D_F$. The size of this domain, for fixed p_i, is independent of λ and does not shrink as $\lambda\to\infty$. The parameter κ_0 which determines the size of D_F in (3.20) is determined by the behavior of $\beta(g)$ near the origin and by the condition $\kappa_0<r_0(p_i)$. Thus if $\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)$ has any singularities in the upper half g plane that approach the origin as $\lambda\to\infty$, these singularities must approach the origin along a path in the first quadrant that lies between the positive real axis and the curve $|g|=(\kappa_0/a)\sin\phi$. This last curve starts out at the origin with a zero slope.

(ii) From Eq. (3.1′) one can read off the asymptotic behavior of $\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)$ as $\lambda\to\infty$, $g\in D_F$:

$$\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)\to\lambda^{4-n}\Gamma_{\rm as}^{(n)}\left(p_i;\frac{-1}{ct}+i\epsilon(t),m\right)\times\exp\left[-n\int_0^t\gamma(\bar{g}(t',g))\,dt'\right]. \tag{3.38}$$

The integral in the exponential is convergent for large t. The imaginary part of $\bar{g}(t,g)$ for large t, $\epsilon(t)$, is positive and vanishes like $O(t^{-3})$. For Eq. (3.38) to be useful the limit as $g\to0$ of $\Gamma_{\rm as}^{(n)}(p_i;g,m)$ must exist along any curve that lies in the domain D. Furthermore, if we want to use perturbation theory to calculate $\Gamma_{\rm as}^{(n)}$ for $g\approx-1/ct+i\epsilon(t)$, then we have to extend assumption (B) on the uniformity of the perturbation expansion for any $g\in D$ to include $\Gamma_{\rm as}^{(n)}(p_i;g,m)$ in addition to $\beta(g)$ and $\gamma(g)$. For any $g\in D_F$, $\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)$ behave for large λ as if the theory was asymptotically free.

(iii) For large enough λ the vertex functions $\Gamma_{\rm as}^{(n)}(p_i;g,m)$ have no zeros for $g\in D_F$. Suppose, for fixed p_i, we have a zero at $g=z(t)$, $z(t)\in D_F$, i.e.,

$$\Gamma_{\rm as}^{(n)}(e^tp_i;z(t),m)\equiv0. \tag{3.39}$$

Since the p_i's are fixed and arbitrary the position of the zero will in general depend on $t=\ln\lambda$. But using Eq. (3.1′) we get then

$$\Gamma_{\rm as}^{(n)}(p_i;\bar{g}(t,z(t)),m)=0. \tag{3.40}$$

But $\bar{g}(t,z(t))$ either defines a curve in D_F as t varies, or $g(t,z(t))=\text{const}\equiv b$. The first possibility is clearly ruled out since it would imply that $\Gamma_{\rm as}^{(n)}(p_i;g,m)\equiv0$ for all $g\in D_F$. The second possibility can also be ruled out. We write

$$\bar{g}(t,z(t))=b. \tag{3.41}$$

As noted earlier for all $g\in D_F$ and large t, $\bar{g}(t,g)$ is well approximated by the solution $\bar{g}_0$ as in Eqs.

(3.16) and (3.17). We then have, ignoring terms of order $1/t^2$,

$$\frac{-z(t)}{cz(t)t-1} \cong b. \tag{3.42}$$

Solving this equation for $z(t)$ we get

$$z(t)=\frac{b}{cbt-1}. \tag{3.43}$$

But for large enough t, this gives $z(t)\approx 1/ct$ which lies definitely outside the domain D_F. Thus for Eq. (3.41) to hold, $z(t)\notin D_F$ for large values of t.

This last argument is somewhat heuristic. It does not treat the situation where $z(t)t$ does not grow as $t\to\infty$. But for such zeros $z(t)$ will approach the origin as $t\to\infty$ and will present no problem to our later argument. Again a sequence of zeros, $z_i(t)$, all of which satisfy $z_i(t)\to 0$ as $t\to\infty$, is not ruled out by the above argument. However, all we need is to have a part of D_F in the first quadrant free of zeros. Thus in the next section we shall assume that $\Gamma^{(2)}$ has no zeros in D_F. Our arguments can be easily modified to take care of the above exceptions leaving the main conclusions unchanged.

IV. RESTRICTIONS ON ϕ^4 FIELD THEORY AND MODELS

So far the results we have established, though potentially useful, do not seem to be directly relevant to physical field theories or models with real positive g. In order to show how these results restrict physical ϕ^4 field theory, we need to use a third assumption again taken from the analogy with the two-dimensional case. This has to do with the fact that in Ref. 4, it was not only shown that the Euclidean Green's functions are analytic in a small half circle with Re$g>0$, but it was also established that for g in this domain the Green's functions are tempered in x space even for complex g.

For the purposes of this section it suffices to consider the inverse propagator, $\Gamma^{(2)}(\lambda p;g,m)$. For real $g>0$, the propagator and the inverse propagator are tempered for large p^2. Our third assumption states that temperedness is not lost for complex g in a domain in the right half plane. We shall assume, in addition to (A) and (B), the following:

(C) For any g in the domain $\{g|\ |g|<\kappa_0,\ |\arg g|<\pi/2\}$, and fixed p, the following upper and lower bounds hold for large λ:

$$C_1\lambda^{-M'}\leqslant|\Gamma^{(2)}_{\rm as}(\lambda p;g,m)|\leqslant C_2\lambda^{M''}, \tag{4.1}$$

where M' and M'' might depend on g but are finite and positive. For real $g>0$ this is not an assumption since it follows from positivity and the Källén-Lehmann representation for the propagator. Furthermore, for $g\in D_F$, the results of the last section tell us that the power behavior of $\Gamma^{(2)}_{\rm as}(\lambda p;g,m)$ is canonical and given by λ^2. Thus we only need assumption (C) in the wedge between the positive real g axis and the curve $|g|=(\kappa_0/a)\sin\phi$.

We define two domains D_a and D_w as follows:

$$D_a=\{g|\ 0<|g|<\kappa_0/a,\ 0<\arg g<\pi\}, \tag{4.2}$$

and

$$D_a\equiv D_w\cup D_F, \tag{4.3}$$

where D_F is the domain defined in Eq. (3.20) of the preceding section. D_w is then the wedge bounded by the curve $|g|=(\kappa_0/a)\sin\phi$ in the first quadrant, the real interval $0<g<\kappa_0/a$, and part of the circle $|g|=\kappa_0/a$, as shown in Fig. 2.

From the results of Sec. III, we know that $\Gamma^{(2)}_{\rm as}(\lambda p;g,m)$ for large λ has no zeros or singularities for any $g\in D_F$. However, for $g\in D_W$ singularities in g for $\Gamma^{(2)}_{\rm as}(\lambda p;g,m)$ might begin to appear in D_W as $\lambda\to\infty$ for fixed p. There are two cases to consider separately:

Case 1. $\Gamma^{(2)}_{\rm as}(\lambda p;g,m)$ has no zeros or singularities for $g\in D_w$. We ignore the trivial zero at $g=0$. We consider a crescent-shaped finite, closed domain, Σ, as shown in Fig. 3, such that $D_w\cap\Sigma$ is not empty and $D_F\cap\Sigma$ is not empty and such that Σ includes a piece of the positive real axis in the interval $0<g<\kappa_0/a$.[13] The origin is taken to be outside Σ.

We take a sequence $\{\lambda_j\}$, $\lambda_j\to\infty$ as $j\to\infty$, and with λ_1 large enough. A sequence of functions, $\alpha_j(g)$, is defined as follows:

$$\alpha_j(g)\equiv-\frac{[\ln(\Gamma^{(2)}_{\rm as}(\lambda_j p;g,m)\lambda_j^{-2})]}{2\ln\lambda_j},\quad g\in\Sigma. \tag{4.4}$$

By hypothesis this sequence is analytic in g for all $g\in\Sigma$. Furthermore, for all $g\in\Sigma$ the bounds (4.1) gives us a uniform bound

$$|\alpha_j(g)|\leqslant M,\quad g\in\Sigma \tag{4.5}$$

where $M=\text{Max}_{g\in\Sigma}(M',M'')$. The limit as $j\to\infty$

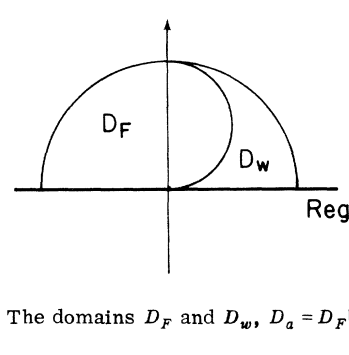

FIG. 2. The domains D_F and D_w, $D_a=D_F\cup D_w$.

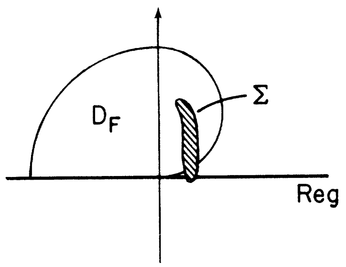

FIG. 3. A sketch of the domain Σ.

exists for all $g \in (\Sigma \cap D_F)$. Hence using Vitali's theorem,[12]

$$\lim_{j\to\infty} \alpha_j(g) \equiv \phi(g), \quad g \in \Sigma \tag{4.6}$$

where $\phi(g)$ is analytic for $g \in \Sigma$. But from Eq. (3.38) we already know that

$$\phi(g) = 0, \quad g \in (D_F \cap \Sigma). \tag{4.7}$$

Hence we must have

$$\phi(g) \equiv 0, \quad g \in \Sigma. \tag{4.8}$$

But Σ includes a segment of the positive real axis, $(\Sigma \cap R_+)$, and this result must then hold for real g, $g \in (\Sigma \cap R_+)$. If $\beta(g)$ has no nontrivial fixed points, then Eq. (4.8) tells us that the power behavior of $\Gamma^{(2)}(\lambda p; g, m)$ for g in some interval in $0 < g < \kappa_0/a$ must be canonical. Namely as $\lambda \to \infty$, $\ln(\Gamma^{(2)}(\lambda p; g, m)\lambda^{-2})/\ln\lambda$ must tend to zero.

However, if $\beta(g)$ has an ultraviolet-stable fixed point at $g = g_\infty$, then Eq. (4.8) cannot be consistent with Eq. (3.3) unless

$$\gamma(g_\infty) = 0. \tag{4.9}$$

This conclusion is true even if g_∞ is outside the domain D, as it most probably is.

There are well-known arguments by Parisi[8] and Callan and Gross[9] which show that if the anomalous dimension, $\gamma(g_\infty)$, is zero then this can only happen if $g_\infty \equiv 0$. The argument assumes that one can write the CS equation at the point $g = g_\infty$. With $\beta(g_\infty) = \gamma(g_\infty) = 0$, this equation for $\Gamma^{(2)}$ becomes trivial and will give a solution $\Gamma^{(2)}_{\rm as}(\lambda p; g_\infty, m) = \mathrm{const}(\lambda p)^2$. Thus the zero-mass theory, by the Federbush-Johnson theorem,[14] is free and $g_\infty = 0$.

This argument is not a proof that excludes Eq. (4.9) for $g_\infty \neq 0$. For example, all one needs to get around it is to have $\beta(g)$ or its derivatives singular at $g = g_\infty$. In such a case one solves the CS equation for $g < g_\infty$ and obtains an answer for $\Gamma^{(2)}_{\rm as}(\lambda p; g, m)$ which in the limit $g \to g_\infty$ will not necessarily lead to the free-field result.

The only way to get around our conclusion that the asymptotic power behavior in momentum space for $\Gamma^{(2)}$ is canonical is thus to allow zeros or singularities in the g plane for $g \in D_w$. This leads us to Case 2.

Case 2. As $\lambda \to \infty$, $\Gamma^{(2)}_{\rm as}(\lambda p; g, m)$ will begin to have zeros or singularities in the g plane in the domain $g \in D_w$. One zero or a finite number of zeros clearly will not be enough to avoid the conclusion of Case 1. In that case we can always find a sequence $\{\lambda_j\}$ and a domain Σ such that $\Gamma^{(2)}_{\rm as}(\lambda_j p; g, m)$ has no zeros for $g \in \Sigma$. One way to destroy the argument of Case 1 is to have a line of zeros (or singularities) in D_w that get more densely spaced as $\lambda \to \infty$ and approach the origin as λ grows. This will cut the domain D_w (or Σ) in two parts as $\lambda_j \to \infty$, and the functions $\alpha_j(g)$ as $j \to \infty$, will behave as the pressure does in the thermodynamic limit when we have a phase transition. The limit as $j \to \infty$ of $\alpha_j(g)$ will not then be an analytic function for all $g \in \Sigma$.

Another way to avoid the results of Case 1 would be to have a branch cut in D_w whose end point tends to the origin as $\lambda \to \infty$. This could be caused by even a finite number of branch points moving into D_w as λ grows in such a way that one cannot choose the cut structure so as to allow for a regular Σ that connects D_F with a segment of the positive real g axis which lies on the physical sheet of the g plane. The effect of such a cut on Borel summability will be the same as that of a line of zeros or singularities.

This cut or curve of zeros (or singularities) will have to lie in D_w and be between the real axis and the line $|g| = (\kappa_0/a)\sin\phi$. It will thus start with a zero slope, and for small $|g|$ and small ϕ will be at a distance less than $0\,(|g|\sin^2\phi)$ from the real axis.

Under Case 2 then, $\Gamma^{(2)}_{\rm as}(\lambda p; g, m)$ cannot be Borel-summable for large λ except for very small values of g that tend to zero as $\lambda \to \infty$.

For large λ, these singularities of $\Gamma^{(2)}_{\rm as}(\lambda p; g, m)$ or the propagator $\Delta((\lambda p)^2, g)$ are so close to the positive real g axis as to cast doubt on the validity of the Callan-Symanzik assumption about dropping the mass insertion terms. It is possible that this line of dense singularities for $\Gamma^{(2)}_{\rm as}$ is present for large λ and is exactly canceled by singularities in $\Gamma^{(2)}(\lambda p; g, m)$ so that the difference $[\Gamma^{(2)}(\lambda p, g, m) - \Gamma^{(2)}_{\rm as}(\lambda p, g, m)]$ has no singularities near the positive real axis and no inconsistency with Borel summability exists for it. However, we know of no argument for such a cancellation to occur. Why should the vertex functions $\Gamma^{(n)}$ have singularities in g for deep Euclidean momenta and then have these singularities suddenly disappear in the mass insertion terms $\Delta\Gamma^{(n)}$? It is hard to understand why adding the mass insertions will lead after resumming perturbation theory to a function in which the above-mentioned singularities disappear.

To summarize we are faced with three alternatives, none of which are without problems. Perhaps the simplest possibility consistent with (A), (B), and (C) is to let $\beta(g)$ have no zero except at $g=0$. In this case, for $0<g<\kappa_0$ our results give a $\Gamma^{(2)}(\lambda p;g,m)$ which has canonical power behavior in λ, but allows for additional powers of $\ln\lambda$ or even terms such as $\exp(\ln\lambda)^{1-\delta}$, etc. The second possibility is to assume an ultraviolet (UV) fixed point at $g=g_\infty$ with $\gamma(g_\infty)\neq 0$, and accept the line of dense singularities (zeros) in the g plane for $\Gamma_{\rm as}^{(2)}(\lambda p;g,m)$ with large λ. We then have to give up Borel summability except for very small values of g that approach zero as $\lambda\to\infty$. The relation between $\Gamma^{(2)}$ and $\Gamma_{\rm as}^{(2)}$ will in this case be speculative. The third possibility is to take the option $\gamma(g_\infty)=0$. This will give us scaling. But the arguments of Refs. 8 and 9 show that this cannot hold with $g_\infty\neq 0$, unless $\beta(g)$ is singular at $g=g_\infty$.

On the other hand, one of our assumptions, (A), (B), or (C), might be wrong. The analogy with two dimensions might fail. However, for simple "models" that satisfy (A), (B), (C), and the CS equation, the above results show that scaling cannot be broken by a power if the model has a finite radius of convergence in g whose size is independent of p, with $\beta(g)$ nontrivial.

Finally, it is important to stress the point that the arguments of this section could have been applied to any $\Gamma_{\rm as}^{(n)}$ and not just $\Gamma_{\rm as}^{(2)}$. In that case one needs again both sides of the inequality (4.1). Namely, we have to assume not only that $\Gamma_{\rm as}^{(n)}$ increases slower than some power of λ, but also that it does not decrease faster than some power of λ. For $\Gamma_{\rm as}^{(2)}$ and for real g this is obvious since $\Gamma_{\rm as}^{(2)}$ is an inverse propagator. For general $\Gamma_{\rm as}^{(n)}$ the latter part of this assumption, though reasonable, does not follow immediately from temperedness even for real g.

Before ending this section we would like to point to a possible mechanism that could be the cause of the line of singularities we are talking about above. Let us assume that there is a "critical curve" in D_w starting at the origin, and separating those values of $g\in D_a$ for which $\bar{g}(t,g)\to 0$ as $t\to\infty$, from the values of g for which $\bar{g}(t,g)\to g_\infty$ as $t\to\infty$. This curve will by definition lie below the spiral $|g|=(\kappa_0/a)\sin\phi$, which forms the boundary of D_F, and it will also have to start at $g=0$ with zero slope. A "critical curve" or "separatrix" will for large enough t have closely spaced singularities. To prove this let us take two points g_+ and g_-, $g_+\in D_a$, $g_-\in D_a$ one above the separatrix and one below it such that $|g_+-g_-|<\epsilon$, with ϵ arbitrarily small. We can always find a value of $t=T$ large enough such that $g(T,g_+)\approx 0$ and $g(T,g_-)\approx g_\infty$. For this value of $T=\ln\lambda$, we then have

$$\frac{|\bar{g}(T,g_+)-\bar{g}(T,g_-)|}{|g_+-g_-|}\geqslant\frac{g_\infty}{\epsilon}. \tag{4.10}$$

Thus derivatives at the separatrix for large enough λ become singular, and if we take any point near the separatrix then for large enough λ it must be near a singularity or a cut.

This gives us singularities in $\bar{g}(T,g)$ for g on the separatrix. However, the solution (3.1′) only holds for $g\in D_F$ and for real g. We do not know if we can write such a solution for $g\in D_w$, nontheless for g on the separatrix. But even if we were able to do that due to larger analyticity domains for $\beta(g)$, $\gamma(g)$, and $\Gamma_{\rm as}^{(n)}(\lambda p_i;g,m)$ than previously assumed, it still does not follow that a singularity in $\bar{g}(T,g)$ will lead to a singularity in $\Gamma_{\rm as}^{(n)}(e^T p_i;g,m)$. It is quite possible, for example, for $\bar{g}(T,g)$ to be singular at $g=g_0\in D_w$, but $\gamma(x)$ will be such that $\gamma(\bar{g}(T,g_0))$ is not singular at $g=g_0$. But if γ has no such singularity-eliminating mechanism then the singularities of $\bar{g}(T,g)$ could just be the source of our line of singularities.

In closing this section we should clarify what we mean by assumption (C) when singularities (or zeros) are allowed in D_w as in Case 2. Obviously, if (C) holds everywhere for $g\in D_w$ then there could be no zeros or poles in D_w. Under Case 2, we take the temperedness bounds (4.1) to hold for all $g\in D_w$ except for small neighborhoods of the singularities or zeros. As mentioned above the arguments of this section could fail because (C) is not true in four dimensions though it is true in two. This failure will have to occur at least along a line (or strip) in D_w along which $\Gamma_{\rm as}^{(2)}(\lambda p;g,m)$ will either grow faster than any power of λ or decrease faster than any power of λ.

V. REMARKS ABOUT GAUGE THEORIES AND TWO-DIMENSIONAL THEORIES

We shall make a few remarks about theories other than ϕ^4. Our analysis becomes more complicated for Yukawa type theories since there we will have more than one coupling constant and have to deal with coupled differential equations. We limit our remarks here to theories with one coupling constant.

a. Abelian gauge theories (quantum electrodynamics). Under similar assumptions the same results hold except one has to use the variable $\alpha=e^2$, and take analyticity in a cut circle in the α plane. Here again $\tilde{\beta}(\alpha)=c\alpha^2+O(\alpha^3)$ with $c>0$.

Theorems such as 1 and 2 hold for $\bar{\alpha}(t,\alpha)$ given the same analyticity assumptions.

b. Non-Abelian gauge theories.[15,16] For example, in the Yang-Mills case we again use the variable $\alpha = g^2$. Here, $\beta(g) = -cg^3 + O(g^5)$, with $c > 0$. Using $\tilde{\beta} \equiv 2g\beta$, we study the differential equation

$$\frac{d\bar{\alpha}}{dt} = \tilde{\beta}(\bar{\alpha}(t,\alpha)). \tag{5.1}$$

This theory is of course asymptotically free for $\alpha > 0$. Making the same assumptions as before gives an enlarged domain of asymptotic freedom, $\bar{D}_F$, which is a reflection about the imaginary axis of the domain D_F. Namely, the spiral part is now in the second quadrant. The line of singularities discussed in the preceding section will be in the proximity of the negative real α axis. This leads to no inconsistency with Borel summability for $\Gamma^{(n)}_{\text{as}}(\lambda p_i; \alpha, m)$ even for large λ. The analyticity one needs for Borel summability is just a sectorial domain with $|\arg\alpha| < \pi/2 + \delta$. This is yet another attractive feature of non-Abelian gauge theories.

c. Two dimensional theories (or $n < 4$). In these theories as in theories with dimension less than 4, the expansion of $\beta(g)$ has the form

$$\beta(g) = -cg + O(g^2), \tag{5.2}$$

where $c > 0$, and c is proportional to $(4-n)$. These theories are not only asymptotically free but also because of the term of first order in g, the equation

$$\frac{d\bar{g}}{dt} = \beta(\bar{g}(t,g))$$

will in the neighborhood of $g=0$ always give a flow that is directed radially toward the origin: $d\bar{g}/dt \approx -c\bar{g}$. Thus there could be *no* separatrix in this case in the neighborhood of $g=0$, nor will the line of singularities discussed earlier appear. The whole domain D will be a domain of asymptotic freedom.

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APPENDIX A

In this appendix we give the proof of the analyticity of $\bar{g}(t,g)$ in g for all $g \in D_F$ and any t in the interval, $0 \le t < \infty$. We also prove the asymptotic estimates for large t given in Eqs. (3.17), (3.18), and (3.19).

The first proof is a simple generalization to complex initial values of the proof of the Picard-Lindelöf theorem[17] on the existence of solutions to differential equations. Given any $\delta_0 > 0$ we consider a domain $D_F(\delta_0) \subset D_F$ which consists of all those points whose minimum distance to the boundary of D_F is greater than δ_0. Let us take any point $g_0 \in D_F(\delta_0)$. Then we define the domain $U_0 \subset D_F$,

$$U_0 = \{g \mid |g - g_0| < \delta_0\}. \tag{A1}$$

We chose δ_0 such that U_0 does not reach the boundary of D_F. For all $g \in D_F$, $\beta(g)$ is approximately given by $\beta(g) = cg^2 + O(g^3)$. Therefore, we can easily find a constant C, $C > c$, such that

$$\max_{g \in U_0} |\beta(g)| \le C(|g_0| + \delta)^2. \tag{A2}$$

But for all $g_0 \in D_F$ we have $|g_0| < \kappa_0/a$, hence

$$\max_{g \in U_0} |\beta(g)| \le C\left(\frac{\kappa_0}{a} + \delta\right)^2. \tag{A3}$$

At this point we consider the interval $0 \le t \le \tau_0$, with

$$\tau_0 = \frac{\delta_0}{C(\kappa_0/a + \delta)^2}. \tag{A4}$$

For t in this interval we define the sequence $\bar{g}_k(t,g_0)$, $k = 0, 1, 2, \ldots$,

$$\bar{g}_{k+1}(t,g_0) = g_0 + \int_0^t \beta(\bar{g}_k(t',g_0))\,dt', \quad 0 \le t \le \tau_0 \tag{A5}$$

where

$$\bar{g}_0(t,g_0) \equiv g_0. \tag{A6}$$

Following Ref. 17 one shows by induction that for all k

$$|\bar{g}_k(t,g_0) - g_0| \le \delta_0, \quad 0 \le t \le \tau_0. \tag{A7}$$

The functions $\bar{g}_k(t,g_0)$ are analytic functions of analytic functions in g_0 for any $g_0 \in D_F(\delta_0)$. For t such that $0 \le t \le \tau_0, \bar{g}_k \in U_0$ for all k. Using identical methods as those in Ref. 17 we can show that the sequence $\bar{g}_k(t,g_0)$ converges *uniformly* as $k \to \infty$ to a function $\bar{g}(t,g_0)$ for any $g_0 \in D_F(\delta_0)$ and $0 \le t \le \tau_0$. Thus $\bar{g}(t,g)$ for $0 \le t \le \tau_0$ is analytic for all $g \in D_F(\delta_0)$. We recall that the choice of δ_0 is arbitrary; any $\delta_0 > 0$ will do. Thus we have analyticity in a domain $D_F(\delta_0)$ with $D_F(\delta_0) \to D_F$ as $\delta_0 \to 0$.

We need of course to get to values of $t > \tau_0$. To do this we repeat the process again with an initial point taken at $\bar{g}(\tau_0, g_0)$. The arguments is essentially the same and we get analyticity for t in an interval $\tau_0 < t < \tau_1$. The process can be repeated for larger and larger values of t. The main ingredient that allows us to do this is that $|\bar{g}(t,g)|$ remains bounded as $t \to \infty$ and $\bar{g}$ flows away from the boundary of D_F.

Finally, we would like to prove the estimates (3.17), (3.18), and (3.19). To do this we consider the integral equation (3.10) for $t = T$ and $t = \alpha T$, where T is large and $\alpha \gg 1$. By subtracting we get

$$\bar{g}(T,g) - \bar{g}(\alpha T, g) = -\int_T^{\alpha T} dt' \, \beta(\bar{g}(t',g)), \quad g \in D_F. \tag{A8}$$

For large T we know that $|g(\alpha T, g)| < |g(T,g)|$, $\alpha > 1$. Hence for any $\alpha \gg 1$,

$$\left| \int_T^{\alpha T} dt' \, \beta(\bar{g}(t',g)) \right| \leq 2 |\bar{g}(T,g)|, \quad g \in D_F. \tag{A9}$$

Since α is arbitrary this gives us convergence of the integral $\int_T^\infty dt' \, \beta(\bar{g}(t',g))$ for any T. We can then take $\alpha \to \infty$ in (A8), and using the fact that $\bar{g}(\alpha T, g)$ vanishes in that limit we get

$$\bar{g}(T,g) = -\int_T^\infty dt' \beta(\bar{g}(t',g)), \quad g \in D_F. \tag{A10}$$

To get our estimates, we write an iteration scheme for (A10) which starts with $\bar{g}_0(t,g)$ the solution of the equation $d\bar{g}_0/dt = c\bar{g}_0^{\,2}$. We write

$$\bar{g}_{k+1}(T,g) = -\int_T^\infty dt' \, \beta(\bar{g}_k(t',g)), \quad g \in D_F \tag{A11}$$

with

$$\bar{g}_0(t,g) = \frac{-g}{cgt - 1}. \tag{A12}$$

It is a trivial matter to check now that

$$\bar{g}_1(T,g) - \bar{g}_0(T,g) \cong -b \int_T^\infty dt' \frac{g^3}{(cgt'-1)^3} \cong \frac{-b}{2c^2T^2}\left[1 + O\left(\frac{1}{cgT}\right)\right], \quad g \in D_F. \tag{A13}$$

Here b is real and is given by $\beta(g) = cg^2 + bg^3 + O(g^4)$. Thus the error between $\bar{g}_1$ and $\bar{g}_0$ is $O(1/T^2)$ and is real in that order. Complex contributions start at order $1/T^3$. Repeating the iteration we get

$$\bar{g}_n(T,g) - \bar{g}_0(T,g) = O\left(\frac{1}{T^2}\right) + i\,O\left(\frac{1}{T^3}\right). \tag{A14}$$

Since we have convergence in n this proves our assertion.

APPENDIX B

We briefly discuss the consequences of analyticity in g to the problem of Bjorken scaling. We limit ourselves to $g\phi^4$ theory, and use the notation of Ref. 15. As is well known Bjorken scaling is related to the asymptotic behavior of the c-number functions in the Wilson operator-product expansion, $C_i^{(n)}(q^2, g)$. The scaling structure functions are related to $C_i^{(n)}$ through

$$\int_0^1 dx\, x^{n-2} F_i(x, q^2) \underset{-q^2\to\infty}{\sim} C_i^{(n)}(q^2,g)\langle n | O^{(n)} | n\rangle, \tag{B1}$$

with $x = -q^2/2\nu$. The Wilson functions satisfy a CS type equation,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_n(g)\right] C_i^{(n)}\left(\frac{q^2}{\mu^2}, g\right) = 0. \tag{B2}$$

Even in two dimensions analyticity of $C_i^{(n)}(q^2/\mu^2, g)$ in g has not yet been established. But if we assume (A) and (B) for the $C_i^{(n)}$'s in four dimensions, then certainly the results of Sec. III will follow. However, to proceed further we need upper and lower bounds for large $(-q^2)$ and complex g similar to those of assumption (C), i.e.,

$$(-q^2)^{M'} \leq |C_i^{(n)}(q^2,g)| \leq (-q^2)^{M''}, \quad g \in D_a, \quad \mathrm{Re}\, g > 0. \tag{B3}$$

The right-hand side of this inequality is easy to understand in terms of temperedness of $C_i^{(n)}$ in q^2. But we also need the left-hand side, the lower bound, to carry through the arguments of Sec. IV. Unfortunately, here we do not have the Källén-Lehmann representation to guide us as in the case of $\Gamma^{(2)}$ in Sec. IV.

Just for the sake of discussion let us assume the validity of (B3) anyway. The same results as those in Sec. IV will follow. Namely, either

$$\lim_{-q^2\to\infty} \left[\frac{\ln C_i^{(n)}(q^2/\mu^2, g)}{\ln q^2}\right] = 0, \quad 0 < g < \frac{\kappa_0}{a}, \tag{B4}$$

or we have a dense line of singularities as before for $g \in D_w$ and large $(-q^2)$.

The alternative (B4) has some interesting consequences if the theory has a UV-stable fixed point at $g = g_\infty$. As we mentioned earlier, $\gamma_n(g_\infty) = 0$ in this case can only hold if the point $g = g_\infty$ is singular and thus $g_\infty \notin D$. For $g \in D_F$ one gets

$$C_i^{(n)}\left(\frac{q^2}{\mu^2}, g\right) \underset{-q^2\to\infty}{\sim} C_i^{(n)}(1,0)\, e^{-f^{(n)}(g)}, \tag{B5}$$

where

$$f^{(n)}(g)=\int_0^\infty \gamma_n(\bar{g}(t',g))\,dt',\quad g\in D_F. \tag{B6}$$

The function $f^{(n)}(g)$ is analytic for $g\in D_F$. But if $|C_i^{(n)}(q^2/\mu^2,g)|$ is uniformly bounded for all $g\in D_w$ then the limit as $(-q^2)\to\infty$ must give us an analytic function for all $g\in D_a$.[12] Hence, the functions $f^{(n)}(g)$ are analytic for all $g\in D_a$.

On the other hand, we can directly calculate the limit $(-q^2)\to\infty$ for real g, $g>0$,

$$C_i^{(n)}\left(\frac{q^2}{\mu^2},g\right)\underset{-q^2\to\infty}{\sim}C_i^{(n)}(1,g_\infty)e^{-f_r^{(n)}(g)},\quad g>0 \tag{B7}$$

with

$$f_r^{(n)}(g)=\int_0^\infty \gamma_n(\bar{g}_r(t',g))\,dt',\quad g>0. \tag{B8}$$

This last integral converges since we are assuming $\gamma_n(g_\infty)=0$. However, $f_r^{(n)}(g)$ is *not* an analytic function, since in the above integral $\bar{g}(t,g)$ for large t gets out of the domain of analyticity of $\gamma_n(z)$.

Comparing the result (B5) continued for real g with (B7) we get for real g, $g>0$

$$f^{(n)}(g)-f_r^{(n)}(g)=\ln\left[\frac{C_i^{(n)}(1,0)}{C_i^{(n)}(1,g_\infty)}\right]. \tag{B9}$$

One can see that if we had analyticity of $g=g_\infty$, and if $f_r^{(n)}(g)$ was the analytic continuation of $f^{(n)}(g)$, then the left-hand side of (B9) would vanish and we would get $C_i^{(n)}(1,0)=C_i^{(n)}(1,g_\infty)$. This is another indication of the results of Refs. 8 and 9. Namely, $\gamma_n(g_\infty)=0$ cannot hold with $g_\infty\neq 0$ unless the point $g=g_\infty$ is a singular point.

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[1]A. Jaffe, Commun. Math. Phys. 1, 127 (1965).

[2]B. Simon, in *Fundamental Interactions in Physics and Astrophysics*, edited by G. Iverson *et al*. (Plenum, New York, 1973), pp. 120–133.

[3]See for example, G. H. Hardy, *Divergent Series* (Oxford, London, England, 1956), p. 192, theorem 136.

[4]J. Glimm, A. Jaffe, and T. Spencer, in *Constructive Quantum Field Theory*, proceedings of the International School of Mathematical Physics "Ettore Majorana," Erice, Italy, 1973, edited by G. Velo and A. Wightman (Springer, New York, 1973), Vol. 25, pp. 199–242.

[5]J.-P. Eckmann, J. Magnen, and R. Seneor, Commun. Math. Phys. 39, 251 (1975).

[6]K. Symanzik, Commun. Math. Phys. 23, 49 (1971).

[7]C. G. Callan, Phys. Rev. D 5, 3202 (1972). The homogenous Callan-Symanzik equations are equivalent to the renormalization-group equations [see E. C. G. Stueckelberg and A. Petermann, Helv. Phys. Acta 26, 499 (1953); M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954)].

[8]G. Parisi, Nuovo Cimento Lett. 7, 84 (1973).

[9]C. G. Callan and D. J. Gross, Phys. Rev. D 8, 4383 (1973).

[10]For a list of some of these models see N. Christ, B. Hasslacher, and A. H. Mueller, Phys. Rev. D 6, 3543 (1972).

[11]A better calculation of the position of the "separatrix" would require some information on the sign of the g^3 term in β. The separatrix for (3.16) is just the line interval $0<g<1/ct$ on the real axis, but this is more of a degenerate case since for all complex g, $\bar{g}_0(t,g)\to 0$ as $t\to\infty$.

[12]See for example E. C. Titchmarsh, *The Theory of Functions*, second edition (Oxford, London, England, 1939), p. 168.

[13]The domain Σ as shown in Fig. 3 extends into the lower half g plane. However we can choose the intersection with the lower half plane as small as we please. In fact, with minor modifications of our argument, we can work with a Σ which does not dip into the lower half plane and has a piece of the real axis as a boundary line.

[14]P. Federbush and K. Johnson, Phys. Rev. 120, 1926 (1960). The generalization of this theorem for zero mass was carried through by K. Pohlmeyer, Commun. Math. Phys. 12, 204 (1969).

[15]D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973). See also Phys. Rev. D 8, 3633 (1973).

[16]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[17]E. A. Coddington and N. Levinson, *Theory of Differential Equations* (McGraw-Hill, New York, 1955), p. 12.