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Spontaneous symmetry breaking in O(N)-symmetric ϕ^6 theory in the $1/N$ expansion*

P. K. Townsend
Department of Physics, Brandeis University, Waltham, Massachusetts 02154
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An O(N)-symmetric scalar field theory with ϕ^4 and ϕ^6 coupling is considered to leading order in the $1/N$ expansion for one, two, and three space-time dimensions. The effective potential is derived and conditions for spontaneous symmetry breaking studied. In one and two dimensions symmetry breaking is impossible. In three dimensions symmetry breaking can occur, and the vacuum expectation values of the fields are given by the tree-approximation result for an appropriate choice of renormalized parameters. There are possible first- and second-order transitions to an O(N)-asymmetric vacuum and a tricritical point. Critical behavior is not discussed in detail in this paper, however. A technique is introduced which allows the derivation of all the Green's functions of the theory to leading order in $1/N$. The propagators in the asymmetric vacuum are derived and found to be well behaved and, in three dimensions, qualitatively similar to $\lambda\phi^4$ theory. These conclusions are checked by independent methods, verifying to leading order the consistency of the $1/N$ expansion for this model.

I. INTRODUCTION

The model we consider, ϕ^6 theory, is a field theory with N real scalar fields and an O(N)-symmetric Lagrangian density,

$$\mathcal{L}(\phi)=\tfrac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a)-\tfrac{1}{2}m_0^{\,2}(\phi_a\phi_a)-\frac{\lambda_0}{4!N}(\phi_a\phi_a)^2-\frac{\eta_0}{6!N^2}(\phi_a\phi_a)^3\,. \tag{1.1}$$

[The Minkowski metric has the signature $(+,-,-,-)$, and summation over repeated indices is assumed throughout.]

Many authors[1-6] have studied a similar model, $\lambda\phi^4$ theory, where the Lagrangian density is as above but without the ϕ^6 coupling. Interest has centered on the spontaneous breaking of the O(N) symmetry to a residual O(N − 1) symmetry. This happens when one of the fields develops a nonzero vacuum expectation value (VEV), and in practice we always take this to be the Nth field. There are then $N-1$ generators of the O(N) symmetry which do not annihilate the vacuum, so that the particle spectrum contains $N-1$ massless Goldstone bosons and one remaining massive boson, in a new vacuum found by shifting the fields such that all VEV's vanish. The Lagrangian in the shifted fields governs the O(N − 1)-symmetric theory. The same considerations apply to ϕ^6 theory.

Typically, for various ranges of the parameters either the O(N)- or O(N − 1)-symmetric theory will have a stable vacuum. Varying these parameters from one range to another will produce a transition from one vacuum to another. This transition may be first or second order (we do not encounter higher orders), depending on whether the VEV's undergo a continuous or discontinuous change at the critical value of the parameters for which the transition occurs.

Let us make some remarks relevant to the tree approximation before considering quantum effects. (In this approximation the bare parameters are finite and we may drop the subscripts on them.) Spontaneous symmetry breaking occurs when the classical potential, $U(\phi^2/N)$, develops an absolute minimum for $\phi^2\neq 0$,[7] where,

$$U\left(\frac{\phi^2}{N}\right)=\tfrac{1}{2}m^2\phi^2+\frac{\lambda\phi^4}{4!N}+\frac{\eta\phi^6}{6!N^2}\,. \tag{1.2}$$

(We have used the notation $\phi_a\phi_a=\phi^2$.) The values of ϕ_a for which this minimum occurs are the VEV's of the fields. These values are real, and so we must restrict our study of $U(\phi^2/N)$ to positive ϕ^2.

A necessary condition for an absolute minimum is

$$\partial U/\partial\phi_a=0 \tag{1.3}$$

or

$$\left[m^2+\frac{\lambda}{6}\left(\frac{\phi^2}{N}\right)+\frac{\eta}{5!}\left(\frac{\phi^2}{N}\right)^2\right]\phi_a=0\,. \tag{1.4}$$

The nonzero solution of this equation, which leads to an absolute minimum, is

$$\frac{\phi^2}{N}=\frac{-10\lambda}{\eta}+\left[\left(\frac{10\lambda}{\eta}\right)^2-\frac{5!m^2}{\eta}\right]^{1/2}, \tag{1.5}$$

where we argue for the plus sign below.

Note that ϕ^2 is proportional to N. This motivates the choice of the factors of N in the Lagrangian, since when the O(N) symmetry is broken each term is proportional to N. We will say that the Lagrangian is of order N.

In Fig. 1 we plot the function, $U(x)$, for various ranges of parameters. A perusal of this figure shows that we must choose the positive square root in Eq. (1.5) for a *minimum* of $U(\phi^2/N)$. We also require that ϕ^2 be positive and real, or the O(N) symmetry will be unbroken, and that $\eta > 0$, otherwise there will be no absolute minimum to $U(\phi^2/N)$ even in the O(N)-symmetric theory. Thus, Eq. (1.5) provides three possibilities for symmetry breaking. These are

$$\begin{aligned} &1.\ \lambda \geq 0,\quad m^2 < 0 \\ &2.\ \lambda < 0,\quad m^2 < 0 \\ &3.\ \lambda < 0,\quad m^2 \geq 0\,. \end{aligned} \tag{1.6}$$

However, Fig. 1 shows that in region 3, $|\lambda|$ must be sufficiently large. Precisely, $U(x)$ must have two real positive zeros. We deduce from Eq. (1.2) that this restricts the parameters in region 3 to

$$\frac{5\lambda^2}{8\eta} > m^2\,. \tag{1.7}$$

(The reality of ϕ^2 imposes the condition, $5\lambda^2/6\eta > m^2$, which is less restrictive.)

In $\lambda\phi^4$ theory where $\eta = 0$, a necessary condition for an absolute minimum of the potential is $\lambda > 0$, so that only region 1 is accessible. The transition from the O(N)-symmetric to the O(N)-*asymmetric* [O($N-1$)-symmetric] theory at $m^2 = 0$ is first order.

In ϕ^6 theory we also have a second-order transition for $m^2 > 0$ at $5\lambda^2/8\eta = m^2$. We sketch these regions and the transitions in Fig. 2 where we see that the point $m^2 = 0$, $\lambda = 0$ is a tricritical point[8] in that the loci of critical parameters for both the first- and second-order transitions meet here. We do not discuss, in this paper, how these parameters may be related to physical variables such as temperature.[9]

Synopsis of remaining sections

In going beyond the tree approximation we make use of the $1/N$ expansion, which has received much attention recently.[1-6,10,11] This expansion has the nice property that the leading-order quantum corrections are of the same order as the classical quantities. Consequently, the leading order, which adequately characterizes the theory in the large-N limit, preserves much of the nonlinear structure of the full theory.

In Sec. II we derive the effective action to leading order in $1/N$ for three dimensions, and construct the effective potential, which is the quantum generalization of the classical potential in that its absolute minimum determines the vacuum, as has been extensively discussed in the literature.[1,4,6,7,12]

In Sec. III we introduce a modification of the composite field method of Refs. 2 and 5 to obtain the particle propagators in the asymmetric vacuum to leading order in $1/N$ for three dimensions.

In Sec. IV we discuss the massive boson propagator and demonstrate the stability of the vacuum to leading order in $1/N$. This is unlike $\lambda\phi^4$ theory in four dimensions, which cannot be consistently studied in the $1/N$ expansion because the theory

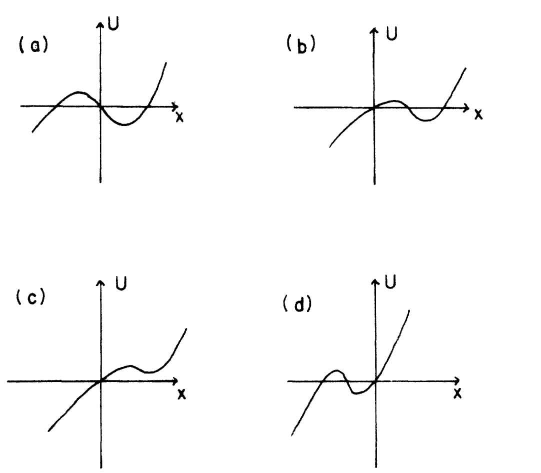


FIG. 1. The classical potential as a function of $\phi^2/N = x$. Only $x > 0$ should be considered. The O(N) symmetry is broken only in (a) and (b). (a) $m^2 < 0$, $\lambda \gtrless 0$. This corresponds to regions 1 and 2. (b), (c) $m^2 > 0$, $\lambda < 0$. This corresponds to region 3 for $|\lambda|$ sufficiently large. (d) $m^2 > 0$, $\lambda > 0$.

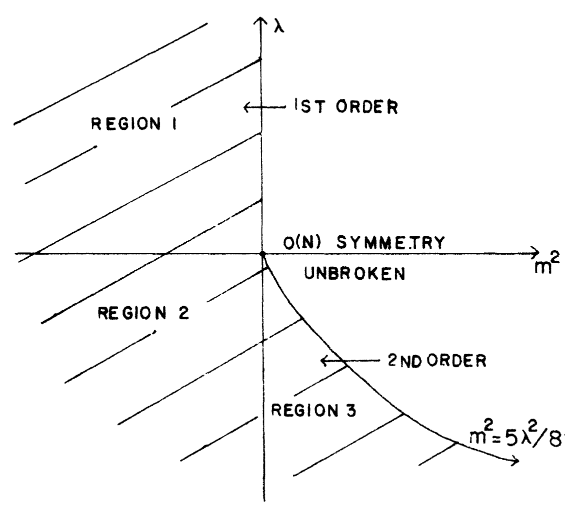


FIG. 2. The regions in λ, m^2 of symmetry breaking for fixed η, showing the first- and second-order transitions.

develops a tachyon and the vacuum becomes unstable,[2,3,5] features which persist in higher order.[3] In three dimensions the $1/N$ expansion is consistent for $\lambda\phi^4$ theory,[2] but this is not the most general renormalizable theory of its type. This is the motivation for adding a ϕ^6 coupling. The resulting ϕ^6 theory is the most general renormalizable scalar field theory in three dimensions and possesses a much richer critical behavior.[14] (This aspect has been studied by other authors.[8,13])

We study one and two dimensions in Sec. V and draw conclusions in Sec. VI. We leave to appendices independent derivations of the stability of the vacuum and the particle propagators.

II. EFFECTIVE ACTION TO LEADING ORDER IN $1/N$

A. Formalism

Our model has the quantum action

$$I(\Phi)=\int d^nx\left[\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}m_0{}^2\Phi^2-\frac{\lambda_0\Phi^4}{4!N}-\frac{\eta_0\Phi^6}{6!N^2}\right]. \tag{2.1}$$

This governs the behavior of the quantum field $\Phi(x)$ via the Euler-Lagrange equations,

$$\frac{\delta I(\Phi)}{\delta\Phi_a(x)}=0,\quad a=1,2,\ldots,N. \tag{2.2}$$

We are interested in the effective action $\Gamma(\phi)$ which governs the behavior of the expectation values $\phi_a(x)$ of the quantum field via the Euler-Lagrange equations,

$$\frac{\delta\Gamma(\phi)}{\delta\phi_a(x)}=0,\quad a=1,2,\ldots,N. \tag{2.3}$$

$\Gamma(\phi)$ can be shown to be the sum of one-particle irreducible (1PI) Feynman graphs with a factor $\phi_a(x)$ on external lines. It is easily seen that the tree approximation to $\Gamma(\phi)$ is the classical action. All quantum effects are contained in graphs with closed loops (each loop carries a factor of $\hbar$).

The number of 1PI graphs is large and unmanageable in the present case. Consequently, we make use of the formalism studied by Cornwall, Jackiw, and Tomboulis,[15] who reduce the problem to summing two-particle irreducible (2PI) Feynman graphs by defining a generalized effective action, $\Gamma(\phi,G)$, which is a functional not only of $\phi_a(x)$, but also of the expectation values $G_{ab}(x,y)$ of the time-ordered product of quantum fields, $T\{\Phi_a(x)\Phi_b(y)\}$.[16] We quote their result,

$$\Gamma(\phi,G)=I(\phi)+\tfrac{1}{2}i\hbar\,\mathrm{tr}\ln G^{-1}+\tfrac{1}{2}i\hbar\,\mathrm{Tr}\mathfrak{D}^{-1}(\phi)G+\Gamma_2(\phi,G)+\text{constant}. \tag{2.4}$$

G is shorthand for $G_{ab}(x,y)$. $\mathfrak{D}$ is defined by

$$i\mathfrak{D}^{-1}=\frac{\delta^2I(\phi)}{\delta\phi_a(x)\delta\phi_b(y)} \tag{2.5}$$

and is shorthand for $\mathfrak{D}_{ab}(\phi;x,y)$. $\Gamma_2(\phi,G)$ is the sum of all 2PI vacuum graphs with propagator G, and ϕ-dependent vertices given by the interaction part of $\mathfrak{L}(\Phi+\phi)$ [which we call $\mathfrak{L}_{\rm int}(\phi;\Phi)$, where $\mathfrak{L}$ was given in Eq. (1.1)]. The trace and logarithm are functional.

The effective action as usually defined is found by solving for G in the equation

$$\frac{\delta\Gamma(\phi,G)}{\delta G_{ab}(x,y)}=0 \tag{2.6}$$

and substituting in $\Gamma(\phi,G)$.

Rather than repeat previous work we refer the reader to the above-mentioned paper[15] for the proofs and details of these statements.

B. $1/N$ expansion

A little algebra shows that for our model

$$\begin{aligned}\mathfrak{L}_{\rm int}(\phi;\Phi)=&-\frac{1}{2}\left(\frac{\lambda_0\phi_a}{3N}+\frac{\eta_0\phi^2\phi_a}{30N^2}\right)\Phi_a\Phi^2-\left(\frac{8\eta_0\phi_a\phi_b\phi_c}{6!N^2}\right)\Phi_a\Phi_b\Phi_c-\frac{1}{4!N}\left(\lambda_0+\frac{\eta_0\phi^2}{10N}\right)\Phi^4\\&-\left(\frac{12\eta_0\phi_a\phi_b}{6!N^2}\right)\Phi_a\Phi_b\Phi^2-\frac{1}{5!}\left(\frac{\eta_0\phi_a}{N^2}\right)\Phi_a\Phi^4-\frac{\eta_0\Phi^6}{6!N^2},\end{aligned} \tag{2.7}$$

$$i\mathfrak{D}^{-1}(\phi)=-\left(\Box+m_0{}^2+\frac{\lambda_0\phi^2}{6N}+\frac{\eta_0\phi^4}{5!N^2}\right)\delta_{ab}\delta^n(x-y)-\left(\frac{\lambda_0}{3N}+\frac{\eta_0\phi^2}{30N^2}\right)\phi_a\phi_b\delta^n(x-y). \tag{2.8}$$

The vertices in Eq. (2.7) contain factors of $1/N$ or $1/N^2$, but a Φ loop gives a factor of N provided the O(N) isospin flows around it alone and not into another part of the graph. We will call such loops bubbles.[1] Then to leading order in $1/N$ the vacuum graphs are bubble trees[1] with two or three bubbles at each vertex. Closed rings of bubbles do not contribute, since simply opening the ring will produce an extra factor of N at least. Figure 3 shows some typical graphs. Of these, the only ones that are 2PI are shown in Fig. 4, and the Feynman rules for the relevant vertices are shown in Fig. 5. Using these rules (recall that re-

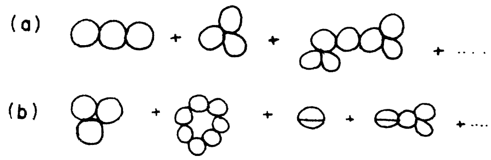


FIG. 3. Some typical vacuum graphs for $\mathcal{L}_{\text{int}}(\phi\,;\Phi)$. (a) These are order N. (b) These are order 1.

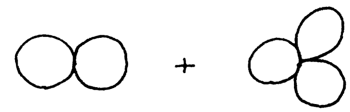


FIG. 4. The 2PI vacuum graphs.

peated indices are summed, and that there is a factor of $\hbar$ for each loop) we obtain

$$\Gamma_2(\phi, G) = \frac{-\hbar^2}{4!N}\int\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)[G_{aa}(x,x)]^2 d^n x - \frac{\eta_0\hbar^3}{6!N^2}\int[G_{aa}(x,x)]^3 d^n x\,. \tag{2.9}$$

Therefore, Eq. (2.6) becomes

$$\frac{\delta\Gamma(\phi,G)}{\delta G_{ab}(x,y)} = 0 = -\tfrac{1}{2}(G^{-1})_{ab}(x,y) + \tfrac{1}{2}i\hbar\mathfrak{D}^{-1}(\phi) - \frac{\hbar^2}{12N}\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)[\delta_{ab}G_{cc}(x,x)]\delta^n(x-y)$$
$$- \frac{3\eta_0}{6!N}\hbar^3\delta_{ab}[G_{cc}(x,x)]^2\delta^n(x-y)\,. \tag{2.10}$$

Rewriting this equation,

$$(G^{-1})_{ab}(x,y) = \mathfrak{D}^{-1}(\phi) + \frac{i\hbar}{6N}\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)\delta_{ab}G_{cc}(x,x)\delta^n(x-y) + \frac{i\eta_0\hbar^2}{5!N^2}\delta_{ab}[G_{cc}(x,x)]^2\delta^n(x-y)\,. \tag{2.11}$$

Hence,

$$\tfrac{1}{2}i\hbar\,\mathrm{Tr}\,\mathfrak{D}^{-1}G = \text{constant} + \frac{\hbar^2}{12N}\int\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)[G_{cc}(x,x)]^2 d^n x + \frac{3\eta_0\hbar^3}{6!N^2}\int[G_{cc}(x,x)]^3 d^n x\,. \tag{2.12}$$

Using Eqs. (2.11) and (2.12) in (2.9) we find the effective action,

$$\Gamma(\phi) = I(\phi) + \tfrac{1}{2}i\hbar\,\mathrm{Tr}\ln G^{-1} + \frac{\hbar^2}{4!N}\int\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)[G_{aa}(x,x)]^2 d^n x + \frac{2\eta_0\hbar^3}{6!N^2}\int[G_{aa}(x,x)]^3 d^n x\,, \tag{2.13}$$

where G_{ab} is given implicitly by Eq. (2.11). We may simplify by separating G_{ab} into transverse and longitudinal components with respect to an arbitrary unit vector, $\hat\phi$, in the N-dimensional isospin space,

$$G_{ab} = (\delta_{ab} - \hat\phi_a\hat\phi_b)g + \hat\phi_a\hat\phi_b\tilde g\,. \tag{2.14}$$

In this form we can invert G_{ab} directly to obtain

$$(G^{-1})_{ab} = (\delta_{ab} - \tilde\phi_a\tilde\phi_b)g^{-1} + \hat\phi_a\hat\phi_b\tilde g^{-1}\,. \tag{2.15}$$

Take the trace on a, b,

$$G_{aa} = Ng + O(1), \quad (G^{-1})_{aa} = Ng^{-1} + O(1)\,. \tag{2.16}$$

We take G_{ab} and g to be of order 1. Equation (2.11) shows that

$$G^{-1} = \mathfrak{D}^{-1}(\phi) + \text{quantum corrections}\,. \tag{2.17}$$

But $\mathfrak{D}^{-1}$ is of order 1 so this assumption will be consistent with our later results. Now we substitute Eq. (2.16) into (2.13) and (2.11) and keep only the leading order. G_{ab} is diagonal in a, b, to leading order in $1/N$, so the trace and logarithm on these indices simply give a factor of N:

$$\Gamma(\phi) = I(\phi) + \frac{N}{2}i\hbar\,\mathrm{Tr}\ln g^{-1} + \frac{N\hbar^2}{4!}\int\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)[g(x,x)]^2 d^n x + \frac{2N\eta_0\hbar^3}{6!}\int[g(x,x)]^3 d^n x + O(1)\,, \tag{2.18}$$

$$g^{-1}(x,y) = i\left[\Box + m_0{}^2 + \frac{\lambda_0\phi^2}{6N} + \frac{\eta_0\phi^4}{5!N^2} + \frac{\hbar}{6}\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)g(x,x) + \frac{\hbar^2}{5!}\eta_0[g(x,x)]^2\right]\delta^n(x-y) + O\left(\frac{1}{N}\right)\,. \tag{2.19}$$

These two equations determine the effective action to leading order in the $1/N$ expansion.

C. Effective potential

If we restrict ϕ to be constant, the effective action takes the form

$$\Gamma(\phi) = -V(\phi)\int d^n x\,. \tag{2.20}$$

This defines the effective potential, $V(\phi)$. ϕ is now a *vacuum* expectation value of $\phi(x)$ and $G_{ab}(x,y)$ a *vacuum* expectation value of $T\{\Phi_a(x)\Phi_b(y)\}$. We now have translational invariance of $\mathcal{L}(\Phi+\phi)$ so $g(x,y)$ is a function only of $(x-y)$. We may write

$$g(x,y) = g(x-y) = \int \hat{g}(p)e^{ip(x-y)}d^n p/(2\pi)^n\,. \tag{2.21}$$

And we define a function of ϕ, $B(\phi)$, by

$$B(\phi) = g(0) = \int \hat{g}(p)d^n p/(2\pi)^n\,. \tag{2.22}$$

Then from Eq. (2.19) and rearranging terms,

$$\frac{1}{\hat{g}(p)} = i\left[-p^2 + m_0{}^2 + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N}+\hbar B\right) + \frac{\eta_0}{5!}\left(\frac{\phi^2}{N}+\hbar B\right)^2\right], \tag{2.23}$$

$$B(\phi) = \int \frac{d^n p}{(2\pi)^n}\,\frac{-i}{-p^2 + m_0{}^2 + (\lambda_0/6)(\phi^2/N+\hbar B) + (\eta_0/5!)(\phi^2/N+\hbar B)^2}\,. \tag{2.24}$$

And from Eq. (2.18)

$$V(\phi) = V_0(\phi) - \frac{iN\hbar}{2}\int \frac{d^n p}{(2\pi)^n}\left\{\ln\left[-p^2 + m_0{}^2 + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N}+\hbar B\right) + \frac{\eta_0}{5!}\left(\frac{\phi^2}{N}+\hbar B\right)^2\right]\right\} - \frac{N\hbar^2}{4!}\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)B^2 \frac{-2N\hbar^3\eta_0 B^3}{6!}\,. \tag{2.25}$$

We use the notation V_0 in place of U to indicate that we are using bare parameters. In all other respects $V_0(\phi)$ is the classical potential.

We now specialize to three space-time dimensions, leaving one and two dimensions for Sec. V. We insert a momentum cutoff, Λ, into the divergent integrals to make them finite. We continue all momenta to Euclidean values by taking $p_0 = ip_4$. Thus, $p_0{}^2 - \vec{p}^2 = -(\vec{p}^2 + p_4{}^2)$ so with our metric we must make the replacement $p^2 \to -p^2$. We then rotate the integrals back to the real p_4 axis. This Wick rotation serves to define the boundary conditions on the integrals, which are otherwise undefined.[7] The net result of these operations on Eq. (2.24), for example, is

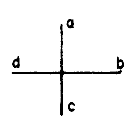
$$B(\phi) = \int_E^\Lambda \frac{d^3 p}{(2\pi)^3} \times \frac{1}{p^2 + m_0{}^2 + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N}+\hbar B\right) + \frac{\eta_0}{5!}\left(\frac{\phi^2}{N}+\hbar B\right)^2}\,. \tag{2.26}$$

We have introduced the subscript, E, to indicate that the integral and momenta have been continued to the Euclidean region. We will use this notation throughout.

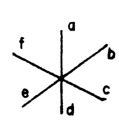
Performing the integration in Eq. (2.26),

$$B(\phi) = \frac{\Lambda}{2\pi^2} - \frac{1}{4\pi}\left[m_0{}^2 + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N}+\hbar B\right) + \frac{\eta_0}{5!}\left(\frac{\phi^2}{N}+\hbar B\right)^2\right]^{1/2}. \tag{2.27}$$

We could apply the same operations to the effective potential in Eq. (2.25), but we prefer to work with its derivative. Calculating $\partial V/\partial\phi_a$ using Eq. (2.24) we find that the terms with a factor of $\partial B/\partial\phi_a$ cancel. This was arranged when we set $\delta\Gamma(\phi,G)/\delta G = 0$ and so we will omit the explicit



$$\frac{-i}{3N}\left(\lambda_0 + \frac{\eta_0\phi^2}{10N}\right)\left(\delta_{ab}\delta_{cd} + \text{permutations}\right)$$



$$\frac{-i}{15N^2}\eta_0\left(\delta_{ab}\delta_{cd}\delta_{ef} + \text{permutations}\right)$$

FIG. 5. The relevant Feynman rules for $\mathcal{L}_{\rm int}(\phi;\Phi)$ (Minkowski momentum).

verification. The remaining terms give simply

$$\frac{\partial V}{\partial \phi_a} = \left[m_0^{\ 2} + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N} + \hbar B\right) + \frac{\eta_0}{5!}\left(\frac{\phi^2}{N} + \hbar B\right)^2 \right]\phi_a . \tag{2.28}$$

D. Renormalization

We can absorb all cutoff-dependent terms into the definition of finite renormalized quantities by the prescription

$$\begin{aligned} \overline{B} &= B - \frac{\Lambda}{2\pi^2}, \\ m^2 &= m_0^{\ 2} + \frac{\lambda_0}{6}\left(\frac{\hbar\Lambda}{2\pi^2}\right) + \frac{\eta_0}{5!}\left(\frac{\hbar\Lambda}{2\pi^2}\right)^2, \\ \lambda &= \lambda_0 + \frac{\eta_0}{10}\left(\frac{\hbar\Lambda}{2\pi^2}\right), \\ \eta &= \eta_0 . \end{aligned} \tag{2.29}$$

Note that there is no renormalization of the ϕ^6 coupling because there are no logarithmically divergent graphs to leading order in $1/N$. This fact also ensures that there is no field renormalization.

With these substitutions Eq. (2.27) becomes

$$\overline{B}(\phi) = -\frac{1}{4\pi}\left[m^2 + \frac{\lambda}{6}\left(\frac{\phi^6}{N} + \hbar\overline{B}\right) + \frac{\eta}{5!}\left(\frac{\phi^2}{N} + \hbar\overline{B}\right)^2 \right]^{1/2} \tag{2.30}$$

which demonstrates that $\overline{B}(\phi)$ is finite.

Equation (2.28) becomes

$$\frac{\partial V}{\partial \phi_a} = \left[m^2 + \frac{\lambda}{6}\left(\frac{\phi^2}{N} + \hbar\overline{B}\right) + \frac{\eta}{5!}\left(\frac{\phi^2}{N} + \hbar B\right)^2 \right]\phi_a . \tag{2.31}$$

So that $V(\phi)$ is finite, up to an unimportant constant.

E. Spontaneous symmetry breaking

From Eq. (2.31) we see that if there is a nonzero minimum of $V(\phi)$, then

$$m^2 + \frac{\lambda}{6}\left(\frac{\phi^2}{N} + \hbar\overline{B}\right) + \frac{\eta}{5!}\left(\frac{\phi^2}{N} + \hbar\overline{B}\right)^2 = 0 \tag{2.32}$$

must be satisfied. But from Eq. (2.30) this implies that at this value of ϕ^2 we have $\overline{B}(\phi) = 0$, so Eq. (2.32), which characterizes the extrema, becomes

$$m^2 + \frac{\lambda}{6}\left(\frac{\phi^2}{N}\right) + \frac{\eta}{5!}\left(\frac{\phi^2}{N}\right)^2 = 0 . \tag{2.33}$$

This is the same equation as for the tree approximation, but in terms of renormalized parameters.

Of course, this is really the tree-approximation result only if the finite parameters m^2, λ, η are the coupling constants in the absence of symmetry breaking defined by

$$\begin{aligned} \left.\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}\right|_{\phi=0} &= m_{\text{tree}}^{\ 2}\delta_{ab} , \\ \left.\frac{\partial^4 V}{\partial\phi_a\partial\phi_b\partial\phi_c\partial\phi_d}\right|_{\phi=0} &= \lambda_{\text{tree}}(\delta_{ab}\delta_{cd} + \text{permutations}) , \\ \left.\frac{\partial^6 V}{\partial\phi_a\partial\phi_b\partial\phi_c\partial\phi_d\partial\phi_e\partial\phi_f}\right|_{\phi=0} &= \eta_{\text{tree}}(\delta_{ab}\delta_{cd}\delta_{ef} + \text{permutations}) . \end{aligned} \tag{2.34}$$

In fact they are not the same. For example, at $\phi = 0$ we have

$$\begin{aligned} \left.\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}\right|_{\phi=0} &= \left(m^2 + \frac{\lambda}{6}\hbar B(0) + \frac{\eta}{5!}[\hbar\overline{B}(0)]^2 \right)\delta_{ab} , \\ \overline{B}(0) &= -\frac{1}{4\pi}\left(m^2 + \frac{\lambda}{6}\hbar\overline{B}(0) + \frac{\eta}{5!}[\hbar\overline{B}(0)]^2 \right)^{1/2} . \end{aligned} \tag{2.35}$$

These equations show that $\overline{B}(0) \neq 0$ and so $m_{\text{tree}}^{\ 2} \neq m^2$. This means that there is a finite renormalization connecting the coupling constants defined by Eqs. (2.34) and the finite parameters defined by Eqs. (2.29). But we choose to use our parameters m^2, λ, η to discuss symmetry breaking conveniently.

Because Eq. (2.33) is the same as the tree approximation, our conclusions of Sec. I remain valid to leading order in $1/N$, and the three regions of symmetry breaking of (1.6) are unchanged, except for the restriction of the parameters in region 3 given in Eq. (1.7). While we have shown that the extrema of the classical potential coincide with those of the effective potential, it is the *effective* potential which must be minimized. But Eq. (1.7) was found by requiring that the minimum of the classical potential in region 3 be absolute. Consequently, this condition will be modified by quantum corrections and the values of the parameters at the critical transition will be altered. We have not calculated this correction since the qualitative features remain, in particular, the existence of region 3 as shown in Fig. 2.

In Sec. III we verify these results by an independent, but related, method and calculate the particle propagators in the presence of symmetry breaking.

III. PARTICLE PROPAGATORS IN THE PRESENCE OF SYMMETRY BREAKING

A. Composite field

Equation (2.18) for the effective action may be written

$$\Gamma(\phi)=I(\phi)+\frac{N}{4!}\int d^nx\left(\lambda_0+\frac{\eta_0\phi^2}{10N}+\frac{\eta_0\hbar g(x,x)}{15}\right)[\hbar g(x,x)]^2+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln g^{-1}+O(1)\,. \tag{3.1}$$

With some algebra,

$$\Gamma(\phi)=I(\phi)+\frac{N}{4!}\int d^nx\left[\frac{\eta_0}{15}\left(\hbar g(x,x)+\frac{\phi^2}{N}\right)^2+\left(\lambda_0-\frac{\eta_0\phi^2}{10N}\right)\left(\hbar g(x,x)+\frac{\phi^2}{N}\right)^2-2\frac{\lambda_0\phi^2}{N}\left(\hbar g(x,x)+\frac{\phi^2}{N}\right)\right.$$
$$\left.+\lambda_0\left(\frac{\phi^2}{N}\right)^2+\frac{\eta_0}{30}\left(\frac{\phi^2}{N}\right)^3\right]+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln g^{-1}+O(1)\,. \tag{3.2}$$

We introduce the function $f(x)$, which depends on $\phi(x)$, by

$$f(x)=\hbar g(x,x)+\frac{\phi^2(x)}{N}\,. \tag{3.3}$$

In this notation the effective action is

$$\Gamma(\phi)=\int d^nx\left[\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0^2\phi^2+\frac{N\eta_0}{3\times 5!}f^3+\frac{N}{4!}\left(\lambda_0-\frac{\eta_0\phi^2}{10N}\right)f^2-\frac{\lambda_0\phi^2}{12}f\right]+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln g^{-1}+O(1)\,. \tag{3.4}$$

And g^{-1} is given by a rearrangement of Eq. (2.19) as

$$g^{-1}(x,y)=i\left(\Box+m_0^2+\frac{\lambda_0}{6}f(x)+\frac{\eta_0 f^2(x)}{5!}\right)\delta^n(x-y)\,. \tag{3.5}$$

It is tempting to think of $f(x)$ as a composite field with its own Feynman rules defined by the Lagrangian of Eq. (3.4). *But this is incorrect* since the presence of the nonquadratic f^3 term means that the theory defined by the Lagrangian,

$$\mathcal{L}(\phi,\chi)=\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0^2\phi^2+\frac{N\eta_0}{3\times 5!}\chi^3+\frac{N}{4!}\lambda_0\chi^2-\frac{\eta_0}{2\times 5!}\phi^2\chi^2-\frac{\lambda_0}{12}\phi^2\chi \tag{3.6}$$

is not equivalent to the original. We have used the *field* $\chi(x)$ to indicate that this is a different theory, and that $f(x)$ is not a field. This difference means that we cannot directly use the composite-field method of Refs. 2 and 5, which was found to be so convenient for $\lambda\phi^4$ theory.[2,3]

However, let us look at the theory defined by the Lagrangian of Eq. (3.6) is more detail. The Green's functions for this theory involving external ϕ particles can be derived from the functional integral,

$$\exp\left(\frac{1}{\hbar}W(j)\right)=\int(d\phi)(d\chi)\exp\left\{\frac{i}{\hbar}\int d^nx\left[\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0^2\phi^2+j\phi+\left(\frac{N}{4!}\lambda_0-\frac{\eta_0}{2\times 5!}\phi^2\right)\chi^2+\frac{N\eta_0\chi^3}{3\times 5!}-\frac{\lambda_0\chi\phi^2}{12}\right]\right\} \tag{3.7}$$

or

$$\exp\left(\frac{i}{\hbar}W(j)\right)=\int(d\phi)\exp\left(\frac{i}{\hbar}\int d^nx[\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0^2\phi^2+j\phi]\right)\int(d\chi)\exp\left(\frac{i}{\hbar}\int d^nx L(\phi,\chi)\right)\,, \tag{3.8}$$

where we have introduced $L(\phi,\chi)$,

$$L(\phi,\chi)=-\frac{\lambda_0\phi^2\chi}{12}+\frac{N}{4!}\left(\lambda_0-\frac{\eta_0\phi^2}{10N}\right)\chi^2+\frac{N\eta_0\chi^3}{3\times 5!}\,. \tag{3.9}$$

Expand this about a particular value, $\bar{\chi}$ of χ,

$$L(\phi,\chi)=L(\phi,\bar{\chi})+(\chi-\bar{\chi})\frac{\partial L}{\delta\chi}\bigg|_{\chi=\bar{\chi}}+\tfrac{1}{2}(\chi-\bar{\chi})^2\frac{\partial^2 L}{\delta\chi^2}\bigg|_{\chi=\bar{\chi}}+\tfrac{1}{6}(\chi-\bar{\chi})^3\frac{\partial^3 L}{\delta\chi^3}\bigg|_{\chi=\bar{\chi}}\ . \tag{3.10}$$

Choose $\bar{\chi}$ such that

$$\partial L/\delta\chi|_{\chi=\bar{\chi}}=0\ . \tag{3.11}$$

From Eq. (3.9) this condition is

$$(\lambda_0+\eta_0\bar{\chi}/10)(\bar{\chi}-\phi^2/N)=0. \tag{3.12}$$

We may choose

$$\bar{\chi}=\phi^2/N\ , \tag{3.13}$$

so with a little algebra and Eqs. (3.10) and (3.13),

$$L(\phi,\chi)=-\frac{\lambda_0\phi^4}{4!N}-\frac{\eta_0\phi^6}{6!N^2}+\frac{N}{4!}\left(\lambda_0+\frac{\eta_0\phi^2}{10N}\right)\left(\chi-\frac{\phi^2}{N}\right)^2+\frac{2N\eta_0}{6!}\left(\chi-\frac{\phi^2}{N}\right)^3. \tag{3.14}$$

Therefore, the functional integral of Eq. (3.8) becomes

$$\exp\left(\frac{i}{\hbar}W(j)\right)=\int(d\phi)\exp\left[\frac{i}{\hbar}\int d^nx\left(\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0{}^2\phi^2-\frac{\lambda_0\phi^4}{4!N}-\frac{\eta_0\phi^6}{6!N^2}+j\phi\right)\right]$$
$$\times\int(d\chi)\exp\left\{\frac{i}{\hbar/N}\int d^nx\left[\frac{1}{4!}\left(\lambda_0+\frac{\eta_0\phi^2}{10N}\right)\left(\chi-\frac{\phi^2}{N}\right)^2+\frac{2\eta_0}{6!}\left(\chi-\frac{\phi^2}{N}\right)^3\right]\right\}\ . \tag{3.15}$$

Shift the integration variable in the χ integral by ϕ^2/N,

$$\exp\left(\frac{i}{\hbar}W(j)\right)=\int(d\phi)\exp\left[\frac{i}{\hbar}\int d^nx\left(\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0{}^2\phi^2-\frac{\lambda_0\phi^4}{4!N}-\frac{\eta_0\phi^6}{6!N^2}+j\phi\right)\right]$$
$$\times\int(d\chi)\exp\left\{\frac{i}{\hbar/N}\int d^nx\left[\frac{1}{4!}\left(\lambda_0+\frac{\eta_0\phi^2}{10N}\right)\chi^2+\frac{2\eta_0\chi^3}{6!}\right]\right\}\ . \tag{3.16}$$

The ϕ functional integral alone (without the integrand of the χ functional integral) generates the Green's functions of ϕ^6 theory, so this equation nicely demonstrates that the two theories are distinct.

It is crucial to note that the χ functional integral generates a loop expansion with expansion parameter $\hbar/N$, and with a ϕ-dependent propagator. There are no tree graphs generated since the translation of the χ integral by ϕ^2/N extracted these from $L(\phi,\chi)$. Schematically,

$$\int(d\chi)\exp\left\{\frac{i}{\hbar/N}\int d^nx\left[\left(\lambda_0+\frac{\eta_0\phi^2}{10N}\right)\chi^2+\frac{\eta_0\chi^3}{3\times5!}\right]\right\}=\exp\left\{\frac{i}{\hbar}\left[\frac{\hbar}{N}(1\ \chi\text{ loop})+\left(\frac{\hbar}{N}\right)^2(2\ \chi\text{ loops})+\cdots\right]\right\}\ , \tag{3.17}$$

so that Eq. (3.16) becomes

$$\exp\left(\frac{iW(j)}{\hbar}\right)=\int(d\phi)\exp\left[\frac{i}{\hbar}\int d^nx\left(\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0{}^2\phi^2-\frac{\lambda_0\phi^4}{4!N}-\frac{\eta_0\phi^6}{6!N^2}+j\phi+\frac{\hbar}{N}(1\ \chi\text{ loop})+\cdots\right)\right]\ . \tag{3.18}$$

The extra terms generated by χ loops are ϕ dependent, which is why this theory is not equivalent to ϕ^6 theory. But for calculations to leading order in $1/N$, the Green's functions for this theory are the same as those of the original theory. Obviously the same consideration applies to the effective potential. Thus, for our purposes we may replace the original ϕ^6 theory by the theory defined by the Lagrangian of Eq. (3.6), from which we obtain the Feynman rules shown in Fig. 6.

The χ propagator carries a factor of $1/N$, and ϕ bubbles contribute a factor of N. Therefore, the only graphs which contribute to leading order are those with one ϕ loop (or none). It is important to realize that we cannot go beyond the leading order for ϕ^6 theory with this substitute Lagrangian. Thus, while we have reduced the problem to summing a set of one-loop graphs, we are not subsequently allowed to look at higher-loop graphs.

As a check of this reasoning the reader may verify, using the Feynman rules of Fig. 6, that the one-ϕ-

loop graphs do indeed sum to the term $\frac{1}{2}Ni\hbar\,\mathrm{Tr}\ln g^{-1}$ of Eq. (3.4) and that $g^{-1}(x,y)$ is given by Eq. (3.5), but but with $f(x)$ replaced by $\chi(x)$.

The net result is that the effective action for the new theory, correct to leading order in $1/N$ is

$$\Gamma(\phi,\chi)=\int d^nx\left(\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m_0^2\phi^2+\frac{N\lambda_0}{4!}\chi_0^2+\frac{N\eta_0}{3\times5!}\chi_0^3-\frac{\eta_0\chi_0^2\phi^2}{2\times5!}-\frac{\lambda_0\chi_0\phi^2}{12}\right)+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln\left(\Box+m_0^2+\frac{\lambda_0\chi_0}{6}+\frac{\eta_0\chi_0^2}{5!}\right),\tag{3.19}$$

where we have written χ_0 in place of χ, anticipating an infinite shift in the χ field, necessary for renormalization.

It should be realized that although we relied on results in Sec. II to arrive at this result, once obtained, the new theory can be used to study ϕ^6 theory independently of any result in Sec. II. Thus this method constitutes an independent check of previous results as well as a means of obtaining the Green's functions.

B. Effective potential and renormalization

Restricting Eq. (3.19) to constant ϕ we obtain the effective potential,

$$V(\phi,\chi)=\tfrac{1}{2}m_0^2\phi^2-\frac{N\lambda_0\chi_0^2}{4!}-\frac{N\eta_0\chi_0^3}{3\times5!}+\frac{\eta_0\chi_0^2}{2\times5!}\phi^2+\frac{\lambda_0\chi_0\phi^2}{12}+\frac{N\hbar}{2}\int_E^\Lambda\frac{d^nk}{(2\pi)^n}\ln\left(k^2+m_0^2+\frac{\lambda_0}{6}\chi_0+\frac{\eta_0\chi_0^2}{5!}\right).\tag{3.20}$$

We remind the reader that the superscript Λ on the integral indicates the momentum cutoff, and the subscript E indicates that the integral and momentum k are Euclidean. We will look at the derivatives of this potential,

$$\frac{\partial V}{\partial\phi^2}=\frac{1}{2}\left(m_0^2+\frac{\lambda_0\chi_0}{6}+\frac{\eta_0\chi_0^2}{5!}\right),\tag{3.21}$$

$$\frac{\partial V}{\partial\chi_0}=\frac{-N}{12}\left(\lambda_0+\frac{\eta_0\chi_0}{10}\right)\chi_0+\frac{1}{12}\left(\lambda_0+\frac{\eta_0\chi_0}{10}\right)\phi^2+\frac{N\hbar}{2}\int_E^\Lambda\frac{d^nk}{(2\pi)^n}\,\frac{\frac{1}{6}(\lambda_0+\frac{1}{10}\eta_0\chi_0)}{k^2+m_0^2+(\lambda_0/6)\chi_0+(\eta_0/5!)\chi_0^2}.\tag{3.22}$$

In three dimensions the integral is familiar from Sec. II,

$$\frac{\partial V}{\partial\chi}=-\frac{N}{12}\left(\lambda_0+\frac{\eta_0\chi_0}{10}\right)\left[\chi_0-\frac{\phi^2}{N}-\frac{\hbar\Lambda}{2\pi^2}+\frac{\hbar}{4\pi}\left(m_0^2+\frac{\lambda_0\chi_0}{6}+\frac{\eta_0\chi_0^2}{5!}\right)^{1/2}\right].\tag{3.23}$$

We can make both Eqs. (3.21) and (3.23) finite as $\Lambda\to\infty$ by the prescription

$$\begin{aligned}\chi_0&=\chi+\frac{\hbar\Lambda}{2\pi^2},\\ m^2&=m_0^2+\frac{\lambda_0}{6}\left(\frac{\hbar\Lambda}{2\pi^2}\right)+\frac{\eta_0}{5!}\left(\frac{\hbar\Lambda}{2\pi^2}\right)^2,\\ \lambda&=\lambda_0+\frac{\eta_0}{10}\left(\frac{\hbar\Lambda}{2\pi^2}\right),\\ \eta&=\eta_0.\end{aligned}\tag{3.24}$$

Equations (3.21) and (3.23) become

$$\frac{\partial V}{\partial\phi^2}=\frac{1}{2}\left(m^2+\frac{\lambda\chi}{6}+\frac{\eta\chi^2}{5!}\right),\tag{3.25}$$

$$\frac{\partial V}{\partial\chi}=-\frac{N}{12}\left(\lambda+\frac{\eta\chi}{10}\right)\left[\chi-\frac{\phi^2}{N}+\frac{\hbar}{4\pi}\left(m^2+\frac{\lambda\chi}{6}+\frac{\eta\chi^2}{5!}\right)^{1/2}\right]\tag{3.26}$$

Note that the renormalization prescription for the coupling constants is the same as that of Eq.

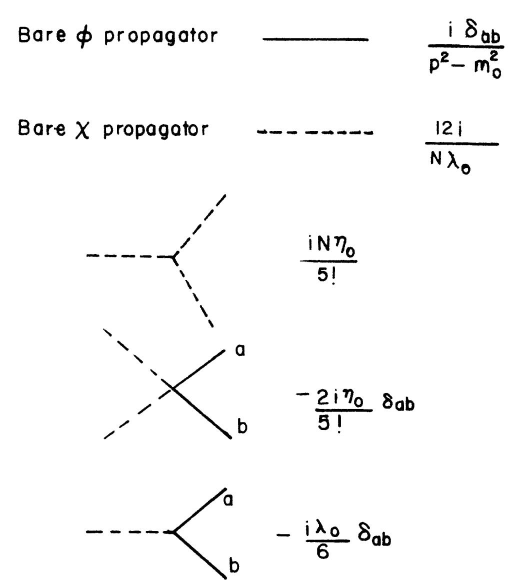


FIG. 6. The Feynman rules for $\mathcal{L}(\phi,\chi)$ (Minkowski momentum).

(2.29) so the finite parameters m^2, λ, η are the same as those of Sec. II.

C. Symmetry breaking

For a broken symmetry we require

$$\frac{\partial V}{\partial \phi^2}=0\,, \quad \frac{\partial V}{\partial \chi}=0\,. \tag{3.27}$$

From Eqs. (3.25) and (3.26) we see that this implies

$$m^2+\frac{\lambda\chi}{6}+\frac{\eta\chi^2}{5!}=0 \tag{3.28}$$

and

$$\left(\lambda+\frac{\eta\chi}{10}\right)\left(\chi-\frac{\phi^2}{N}\right)=0\,. \tag{3.29}$$

In Eq. (3.29) we choose the solution $\chi=\phi^2/N$. [Recall Eq. (3.12) where we also chose this solution for the stationary point of $L(\phi,\chi)$. The other choice would imply $\partial V/\partial\phi^2=m^2+10\lambda^2/\eta$ for all ϕ^2, which is unacceptable.] With this choice Eq. (3.28) implies

$$m^2+\frac{\lambda}{6}\left(\frac{\phi^2}{N}\right)+\frac{\eta}{5!}\left(\frac{\phi^2}{N}\right)^2=0\,. \tag{3.30}$$

This is the tree-approximation result again, verifying the conclusion of Sec. II.

D. Propagators

We shift the fields in the effective action of Eq. (3.19), defining new fields, π,σ,θ, by,

$$\chi=\theta+\langle\chi\rangle \tag{3.31}$$

and

$$\begin{aligned}&\phi_a=\pi_a\,,\quad a=1,2,\ldots,N-1\\ &\phi_N=\sigma+\langle\phi\rangle\,.\end{aligned} \tag{3.32}$$

This defines the effective action as a functional of the new fields, and as a function of the constants $\langle\chi\rangle,\langle\phi\rangle$. These constants are chosen such that

$$\frac{\delta\Gamma}{\delta\theta}=\frac{\delta\Gamma}{\delta\pi}=\frac{\delta\Gamma}{\delta\sigma}=0 \quad \text{for } \theta=\pi=\sigma=0\,. \tag{3.33}$$

This requirement means that $\langle\chi\rangle$ and $\langle\phi\rangle$ are the VEV's of χ and ϕ, and thus were derived previously. That is,

$$\langle\chi\rangle=\frac{\langle\phi\rangle^2}{N}, \tag{3.34}$$

$$m^2+\frac{\lambda}{6}\frac{\langle\phi\rangle^2}{N}+\frac{\eta}{5!}\frac{\langle\phi\rangle^4}{N^2}=0\,. \tag{3.35}$$

Recalling the relation between χ and χ_0 and using Eqs. (3.32) and (3.31), we also have

$$\chi_0=\theta+\frac{\hbar\Lambda}{2\pi^2}+\frac{\langle\phi\rangle^2}{N}\,, \tag{3.36}$$

$$\phi^2=\pi^2+(\sigma+\langle\phi\rangle)^2\,. \tag{3.37}$$

With these substitutions the effective action becomes

$$\begin{aligned}\Gamma(\pi,\sigma,\theta)=\int d^3x\,\Big[&\tfrac{1}{2}(\partial_\mu\pi)^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{2}m_0{}^2\pi^2-\tfrac{1}{2}m_0{}^2(\sigma+\langle\phi\rangle)^2+\frac{N\lambda_0}{4!}\left(\theta+\frac{\langle\phi\rangle^2}{N}+\frac{\hbar\Lambda}{2\pi^2}\right)^2+\frac{N\eta_0}{3\times5!}\left(\theta+\frac{\langle\phi\rangle^2}{N}+\frac{\hbar\Lambda}{2\pi^2}\right)^3\\ &-\frac{\eta_0}{2\times5!}\left(\theta+\frac{\langle\phi\rangle^2}{N}+\frac{\hbar\Lambda}{2\pi^2}\right)^2[\pi^2+(\sigma+\langle\phi\rangle)^2]-\frac{\lambda_0}{12}\left(\theta+\frac{\langle\phi\rangle^2}{N}+\frac{\hbar\Lambda}{2\pi^2}\right)[\pi^2+(\sigma+\langle\phi\rangle)^2]\Big]\\ &+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln\left[\Box+m_0{}^2+\frac{\lambda_0}{6}\left(\theta+\frac{\langle\phi\rangle^2}{N}+\frac{\hbar\Lambda}{2\pi^2}\right)+\frac{\eta_0}{5!}\left(\theta+\frac{\langle\phi\rangle^2}{N}+\frac{\hbar\Lambda}{2\pi^2}\right)^2\right]\,.\end{aligned} \tag{3.38}$$

Then using the renormalization prescription of Eq. (3.24) and Eq. (3.35) for the value of the constant $\langle\phi\rangle$, we obtain, with considerable algebraic manipulation,

$$\begin{aligned}\Gamma(\pi,\sigma,\theta)=\int d^3x\,\Big\{&\tfrac{1}{2}(\partial_\mu\pi)^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{12}\left(\lambda+\frac{\eta\langle\phi\rangle^2}{10N}\right)\pi^2\theta-\frac{\eta}{2\times5!}\theta^2\pi^2\\ &+\frac{N}{4!}\left[\lambda+\frac{\eta\langle\phi\rangle^2}{10N}+\eta\left(\frac{\hbar\Lambda}{2\pi^2}\right)\right]\theta^2-\frac{\langle\phi\rangle}{6}\left(\lambda+\frac{\eta}{10}\frac{\langle\phi\rangle^2}{N}\right)\theta\sigma+\frac{N\eta\theta^3}{3\times5!}-\frac{\eta}{2\times5!}\theta^2\sigma^2-\frac{\eta\langle\phi\rangle}{5!}\theta^2\sigma\\ &-\frac{1}{12}\left(\lambda+\frac{\eta\langle\phi\rangle^2}{10N}\right)\theta\sigma^2+\frac{N}{12}\left(\lambda+\frac{\eta\langle\phi\rangle^2}{10N}\right)\left(\frac{\hbar\Lambda}{2\pi^2}\right)\theta\Big\}\\ &+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln\left[\Box+\frac{1}{6}\left(\lambda+\frac{\eta\langle\phi\rangle^2}{10N}\right)\theta+\frac{\eta\theta^2}{5!}\right]\,.\end{aligned} \tag{3.39}$$

There remain two cutoff-dependent terms in this expression, which will be canceled in the expansion of the logarithm, as we shall see shortly.

Notice first that the π fields correspond to the $N-1$ massless Goldstone bosons. Their inverse propagator is simply

$$\frac{\delta^2\Gamma}{\delta\pi(x)\delta\pi(y)}\bigg|_{\pi=\sigma=\theta=0}=-\Box\delta^3(x-y)\,. \tag{3.40}$$

To simplify Eq. (3.39) we will now take $\pi=0$ and introduce the notation

$$\bar\lambda=\lambda+\frac{\eta\langle\phi\rangle^2}{10N}\,. \tag{3.41}$$

Then the effective action for the fields σ and θ alone is

$$\Gamma(0,\sigma,\theta)=\int d^3x\left\{\tfrac{1}{2}(\partial_\mu\sigma)^2+\frac{N}{4!}\left[\bar\lambda+\eta\left(\frac{\hbar\Lambda}{2\pi^2}\right)\right]\theta^2+\frac{N\bar\lambda}{12}\left(\frac{\hbar\Lambda}{2\pi^2}\right)\theta-\frac{\bar\lambda\langle\phi\rangle}{6}\theta\sigma+\frac{N\eta\theta^3}{3\times 5!}-\frac{\eta\theta^2\sigma^2}{2\times 5!}-\frac{\eta\langle\phi\rangle}{5!}\theta^2\sigma-\frac{\bar\lambda\theta\sigma^2}{12}\right\}$$

$$+\frac{N}{2}i\hbar\,\mathrm{Tr}\ln\left(\Box+\frac{\bar\lambda\theta}{6}+\frac{\eta\theta^2}{5!}\right). \tag{3.42}$$

We are now in a position to find the mixed σ, θ inverse propagator. Take the second functional derivatives of $\Gamma(0,\sigma,\theta)$,

$$\frac{\delta^2\Gamma}{\delta\sigma(x)\delta\sigma(y)}\bigg|_{\sigma=\theta=0}=-\Box\delta^3(x-y)\,,$$

$$\frac{\delta^2\Gamma}{\delta\theta(x)\delta\sigma(y)}\bigg|_{\sigma=\theta=0}=-\frac{\bar\lambda\langle\phi\rangle}{6}\delta^3(x-y)\,, \tag{3.43}$$

$$\frac{\delta^2\Gamma}{\delta\theta(x)\delta\theta(y)}\bigg|_{\sigma=\theta=0}=\frac{N}{12}\left[\bar\lambda+\frac{\eta}{10}\left(\frac{\hbar\Lambda}{2\pi^2}\right)\right]+\frac{N}{2}i\hbar\frac{\delta}{\delta\theta(x)}\frac{\delta}{\delta\theta(y)}\mathrm{Tr}\ln\left(\Box+\frac{\bar\lambda\theta}{6}+\frac{\eta\theta^2}{5!}\right)\bigg|_{\theta=0}.$$

Using the matrix identity

$$\delta\,\mathrm{Tr}\ln A=\mathrm{Tr}A^{-1}\delta A\,, \tag{3.44}$$

note that

$$\frac{\delta}{\delta\theta(x)}\mathrm{Tr}\ln\left(\Box+\frac{\bar\lambda\theta}{6}+\frac{\eta\theta^2}{5!}\right)=\mathrm{Tr}_{u,v}\int\left(\Box+\frac{\bar\lambda\theta}{6}+\frac{\eta\theta^2}{5!}\right)^{-1}_{u,z}\left(\bar\lambda+\frac{\eta\theta(z)}{10}\right)\delta^3(z-x)\delta^3(z-v)\,d^3z\,. \tag{3.45}$$

At $\theta=0$, when multiplied by $\frac{1}{2}Ni\hbar$, this is

$$\frac{N}{2}i\hbar\,\mathrm{Tr}_{u,v}\frac{\bar\lambda}{6}\Box_{uv}^{-1}\delta^3(x-v)=-\frac{N\hbar\bar\lambda}{2}\int_E^\Lambda\frac{d^3p}{(2\pi)^3}\frac{1}{p^2}=\frac{N\bar\lambda}{2}\left(\frac{\hbar\Lambda}{2\pi^2}\right), \tag{3.46}$$

which cancels the Λ-dependent term linear in θ noticed previously.

Taking the second derivative in Eq. (3.45) leads to the result

$$\frac{\delta}{\delta\theta(x)}\frac{\delta}{\delta\theta(y)}\mathrm{Tr}\ln\left(\Box+\frac{\bar\lambda\theta}{6}+\frac{\eta\theta^2}{5!}\right)\bigg|_{\theta=0}=\frac{i}{60}\eta\left(\frac{\Lambda}{2\pi^2}\right)\delta^3(x-y)-\frac{\bar\lambda^2}{36}\int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2(p+k)^2}\,. \tag{3.47}$$

This can also be calculated from the graphs in Fig. 7 and the reader may check Eq. (3.47) graphically using Feynman rules derived from the Lagrangian of Eq. (3.42).

Now substitute Eq. (3.47) into (3.43), take Fourier transforms, and continue all momenta into the Euclidean region,

$$\int \frac{\delta^2\Gamma}{\delta\sigma(x)\delta\sigma(y)}\bigg|_{\sigma=\theta=0} d^3x e^{-ip\cdot(x-y)} = -p^2 ,$$

$$\int \frac{\delta^2\Gamma}{\delta\sigma(x)\delta\theta(y)}\bigg|_{\sigma=\theta=0} d^3x e^{-ip\cdot(x-y)} = -\frac{\bar\lambda\langle\phi\rangle}{6},$$

$$\int \frac{\delta^2\Gamma}{\delta\theta(x)\delta\theta(y)}\bigg|_{\sigma=\theta=0} d^3x e^{-ip\cdot(x-y)} \tag{3.49}$$

$$= \frac{N\bar\lambda}{12} + \frac{N\bar\lambda^2}{72}\hbar\int_E \frac{d^3k}{(2\pi)^3}\frac{1}{k^2(k+p)^2} .$$

The momenta on the right-hand side of these equations are Euclidean, and the value of the integral in Eq. (3.49) is $1/8p$, where $p=\sqrt{p^2}$. Thus the inverse propagator in Euclidean momentum space, $G^{-1}(p)$, is

$$iG^{-1}(p) = \begin{bmatrix} -p^2 & \dfrac{-\bar\lambda\langle\phi\rangle}{6} \\ \dfrac{-\bar\lambda\langle\phi\rangle}{6} & \dfrac{N}{12}\bar\lambda\left(1+\dfrac{\bar\lambda\hbar}{48p}\right)\end{bmatrix}. \tag{3.50}$$

Inverting this

$$G(p) = \frac{i}{-p^2\dfrac{N\bar\lambda}{12}\left(1+\dfrac{\bar\lambda\hbar}{48p}\right)-\dfrac{\bar\lambda^2\langle\phi\rangle^2}{36}} \times\begin{bmatrix}\dfrac{N\bar\lambda}{12}\left(1+\dfrac{\hbar\bar\lambda}{48p}\right) & \frac{1}{6}\bar\lambda\langle\phi\rangle \\ \frac{1}{6}\bar\lambda\langle\phi\rangle & p^2\end{bmatrix}. \tag{3.51}$$

The σ-σ propagator, the one of physical interest, is

$$G_{\sigma\sigma}(p) = \frac{-i\left(p+\dfrac{\hbar\bar\lambda}{48}\right)}{p\left(p^2+\dfrac{\hbar\bar\lambda}{48}p+\dfrac{\bar\lambda\langle\phi\rangle^2}{3N}\right)}. \tag{3.52}$$

We can put this into a more convenient form. Using Eqs. (3.35) and (3.41) we deduce that

$$m^2+\frac{\bar\lambda\langle\phi\rangle^2}{6N}-\frac{\eta\langle\phi\rangle^4}{5!N^2}=0 . \tag{3.53}$$

Thus with the notation

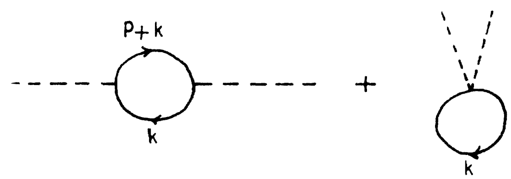


FIG. 7. The π bubble contribution to the inverse $\theta-\theta$ propagator. There is a factor of $\theta(x)$ on external lines, but no propagator.

$$\bar m^2 = m^2 - \frac{\eta\langle\phi\rangle^4}{5!N^2} , \tag{3.54}$$

Eq. (3.53) may be written

$$\frac{\bar\lambda\langle\phi\rangle^2}{3N} = -2\bar m^2 , \tag{3.55}$$

and the σ-σ propagator as

$$G_{\sigma\sigma}(p) = \frac{-i\left(p+\dfrac{\bar\lambda\hbar}{48}\right)}{p\left(p^2+\dfrac{\bar\lambda\hbar}{48}p-2\bar m^2\right)} . \tag{3.56}$$

This completes the derivation of the particle propagators in the presence of a spontaneously broken O(N) symmetry.

Note the resemblance of Eq. (3.55) to the tree approximation for $\lambda\phi^4$ theory, and Eq. (3.56) with the analogous result obtained for $\lambda\phi^4$ theory in three dimensions by Coleman, Jackiw, and Politzer.[2] In Sec. IV we elaborate on this similarity and discuss the σ propagator. In Appendix B we derive this propagator by an independent method.

IV. ANALYSIS OF THE σ PROPAGATOR

From Eq. (3.56) we see that the poles of the σ propagator, $G_{\sigma\sigma}(p)$, are given by

$$p^2+\frac{\bar\lambda\hbar}{48}p-2\bar m^2=0 , \tag{4.1}$$

where the momenta are Euclidean. Solving for p,

$$p=-\frac{\bar\lambda\hbar}{96}\pm\left[\left(\frac{\bar\lambda\hbar}{96}\right)^2+2\bar m^2\right]^{1/2} . \tag{4.2}$$

We wish to write this in terms of our renormalized parameters m^2, λ, η. Eliminating $\langle\phi\rangle$ from Eqs. (3.35) and (3.41) we obtain the relation

$$\bar\lambda = +(\tfrac{6}{5}\eta)^{1/2}\left(\frac{5\lambda^2}{6\eta}-m^2\right)^{1/2} . \tag{4.3}$$

So that if the symmetry is broken, $\bar\lambda$ *is always positive*. Then from Eq. (3.53),

$$\bar m^2 = -\frac{1}{6}\frac{\bar\lambda\langle\phi\rangle^2}{N}, \tag{4.4}$$

But for symmetry breaking to occur we must have $\langle\phi\rangle^2 > 0$, so that $\overline{m}^2$ *is always negative*. Thus the allowed region of the parameters $\overline{m}^2$ and $\overline{\lambda}$ for *symmetry breaking* to occur is

$$\overline{m}^2 < 0, \quad \overline{\lambda} > 0\,. \tag{4.5}$$

This is analogous to the result obtained for $\lambda\phi^4$ in three dimensions. In fact, as expected, if $\eta = 0$ the σ propagator is identical to that of $\lambda\phi^4$ theory[2] (a slightly different numerical coefficient is used in the Lagrangian of Ref. 2). As a consequence of this similarity in three dimensions, the σ propagator for ϕ^6 theory has the qualitative behavior of the σ propagator of $\lambda\phi^4$ theory. This behavior is discussed in Ref. 2, to which we refer the reader for details. Here we note that the poles of the σ propagator in the complex p^2 plane are both on the second sheet (the σ particle is unstable), and the real part of p^2 at the poles is negative. This means that the σ particle is physical, not a tachyon, consequently the vacuum for the broken symmetry is expected to be stable. In Appendix A we demonstrate this stability by studying the effective potential directly.

V. ONE AND TWO DIMENSIONS

From Eq. (2.24) and (2.28) we have

$$B = \int_E^{\Lambda} \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 + c^2}\,, \tag{5.1}$$

$$\frac{\partial V}{\partial \phi_a} = c^2 \phi_a\,, \tag{5.2}$$

where we have introduced the notation

$$c^2 = m_0{}^2 + \frac{\lambda_0}{6}\left(\frac{\phi^2}{N} + \hbar B\right) + \frac{\eta_0}{5!}\left(\frac{\phi^2}{N} + \hbar B\right)^2. \tag{5.3}$$

In one dimension the integral of Eq. (5.1) is finite and gives

$$B = 1/4c\,. \tag{5.4}$$

Thus the bare parameters may be taken to be finite. Note that if the symmetry is broken, Eq. (5.2) requires that, at the value of ϕ^2 for which $V(\phi)$ is a minimum,

$$c^2 = 0 \tag{5.5}$$

must be satisfied, which in turn requires, from Eq. (5.4), that $B(\phi)$ diverge at this value of ϕ^2. But inspection of Eq. (5.3) shows that $c^2 = 0$ cannot then be satisfied for any finite value of ϕ^2. Thus symmetry breaking is forbidden.

In two dimensions $B(\phi)$ is logarithmically divergent,

$$B = \frac{1}{4\pi}\left[\ln\left(\frac{\Lambda^2}{M^2}\right) - \ln\left(\frac{c^2}{M^2}\right)\right]\,, \tag{5.6}$$

where M is an arbitrary renormalization mass. Define renormalized quantities by

$$\overline{B} = B - \frac{1}{4\pi}\ln\left(\frac{\Lambda^2}{M^2}\right),$$

$$m^2 = m_0{}^2 + \frac{\lambda_0}{6}\left[\frac{\hbar}{4\pi}\ln\left(\frac{\Lambda^2}{M^2}\right)\right] + \frac{\eta_0}{5!}\left[\frac{\hbar}{4\pi}\ln\left(\frac{\Lambda^2}{M^2}\right)\right]^2, \tag{5.7}$$

$$\lambda = \lambda_0 + \frac{\eta_0}{10}\left[\frac{\hbar}{4\pi}\ln\left(\frac{\Lambda^2}{M^2}\right)\right].$$

With this prescription,

$$c^2 = m^2 + \frac{\lambda}{6}\left(\frac{\phi^2}{N} + \hbar\overline{B}\right) + \frac{\eta}{5!}\left(\frac{\phi^2}{N} + \hbar\overline{B}\right)^2, \tag{5.8}$$

$$\overline{B} = -\frac{1}{4\pi}\ln\left(\frac{c^2}{M^2}\right). \tag{5.9}$$

c^2 and $\overline{B}$ are now cutoff independent, but $\overline{B}$ again diverges at $c^2 = 0$. As before we cannot allow this possibility if the symmetry is spontaneously broken. Thus, symmetry breaking is forbidden, which verifies Coleman's theorem.[17]

VI. CONCLUSIONS

We have found the effective potential and particle propagators for O(N)-symmetric ϕ^6 theory to leading order in the $1/N$ expansion for one, two, and three space-time dimensions. The O(N) symmetry can be spontaneously broken only in three dimensions, and we have found three regions of the renormalized coupling constants for which this can occur.

We have checked all these results by independent methods, and find that the $1/N$ expansion is consistent for ϕ^6 theory to leading order in $1/N$.

Because we are restricted to three dimensions by renormalizability, and because of the rich critical behavior exhibited by this model, it is hoped that these results will be useful in the study of critical phenomena as well as advancing the understanding of symmetry breaking in applications to particle physics.

ACKNOWLEDGMENTS

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APPENDIX A: STABILITY OF THE VACUUM

We wish to investigate the behavior, as a function of ϕ^2, of the renormalized effective potential given by Eqs. (2.31) and (2.30). To simplify the algebraic manipulations we introduce the notation,

$$\frac{\hbar^2\eta}{(4\pi)^2 5!}=\zeta\,,\qquad \hbar B+\frac{\phi^2}{N}=y\,,\qquad \frac{\phi^2}{N}=x\,. \tag{A1}$$

Then Eq. (2.30) becomes

$$y-x=-\zeta^{1/2}\left(y^2+\frac{20\lambda}{\eta}y+\frac{5!m^2}{\eta}\right)^{1/2}. \tag{A2}$$

With some algebra,

$$y^2(1-\zeta)-2\left(x+\frac{10\lambda}{\eta}\zeta\right)y+\left(x^2-\frac{5!m^2}{\eta}\zeta\right)=0\,. \tag{A3}$$

Solving for y,

$$(1-\zeta)y=x+\frac{10\lambda}{\eta}\zeta\pm\zeta^{1/2}\left\{x^2+\frac{20\lambda}{\eta}x+\frac{5!m^2}{\eta}+\left[\left(\frac{10\lambda}{\eta}\right)^2-\frac{5!m^2}{\eta}\right]\zeta\right\}^{1/2}. \tag{A4}$$

Now,

$$x^2+\frac{20\lambda}{\eta}x+\frac{5!m^2}{\eta}=\frac{20\eta}{5!}U'(x)\,, \tag{A5}$$

where $U(x)$ is the classical potential function defined in Sec. I, and

$$U'(x)=\frac{\partial U(x)}{\partial x}\,,\qquad U'(x_{\rm vac})=0\,. \tag{A6}$$

For $x>x_{\rm vac}$ we must have $U'(x)>0$ since $x_{\rm vac}$ minimizes $U(x)$. Equation (1.5) which gives the solution of Eq. (2.33) for the value of ϕ^2 at the minimum shows that for symmetry breaking to occur

$$\left(\frac{10\lambda}{\eta}\right)^2-\frac{5!m^2}{\eta}>0\,. \tag{A7}$$

This means that for $x>x_{\rm vac}$, y is real. Furthermore, expanding the right-hand side of Eq. (A4) in powers of $1/x$ we find that y is increasing as $x\to\infty$,

$$(1-\zeta)y=\left(x\pm\frac{10\lambda}{\eta}\zeta^{1/2}\right)(1\pm\zeta)+O\left(\frac{1}{x}\right). \tag{A8}$$

In our notation of this appendix, Eq. (2.31) is

$$\frac{\partial V}{\partial\phi_a}=\left(m^2+\frac{\lambda}{6}y+\frac{\eta}{5!}y^2\right)\phi_a\,. \tag{A9}$$

Thus as $\phi^2\to\infty$ from its value in the vacuum, $\partial V/\partial\phi_a$ is real and increasing. The same applies to $V(\phi)$, confirming the stability noted in Sec. IV.

Note that even to the left of the minimum of $U(x)$, i.e., $x<x_{\rm vac}$, y, and hence $V(\phi)$, is still real for a small region of x. This persistence of the reality of the effective potential into an unphysical region was noticed for $\lambda\phi^4$ theory by Coleman, Jackiw, and Politzer.[2]

APPENDIX B: σ PROPAGATOR

We wish to find the σ propagator by the direct summation of graphs.

Take the Lagrangian of Eq. (1.1) and shift the fields as in equation (3.32). After some algebra we find the new Lagrangian $\mathcal{L}(\pi,\sigma)$ to be

$$\begin{aligned}\mathcal{L}(\pi,\sigma)=&\tfrac{1}{2}(\partial_\mu\pi)^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{2}\left(m_0{}^2+\frac{\lambda_0\langle\phi\rangle^2}{6N}+\frac{\eta_0\langle\phi\rangle^4}{5!N^2}\right)(\pi^2+\sigma^2)-\frac{\langle\phi\rangle^2}{6N}\left(\lambda_0+\frac{\eta_0\langle\phi\rangle^2}{10N}\right)\sigma^2\\&-\left(m_0{}^2+\frac{\lambda_0\langle\phi\rangle^2}{6N}+\frac{\eta_0\langle\phi\rangle^4}{5!N}\right)\langle\phi\rangle\sigma-\frac{8\eta_0\langle\phi\rangle^3}{6!N^2}\sigma^3-\frac{\langle\phi\rangle}{6N}\left(\lambda_0+\frac{\eta_0\langle\phi\rangle^2}{10N}\right)(\pi^2+\sigma^2)\sigma-\frac{\eta_0\langle\phi\rangle}{5!N^2}(\pi^2+\sigma^2)^2\sigma\\&-\frac{2\eta_0}{5!}\frac{\langle\phi\rangle^2}{N}(\pi^2+\sigma^2)\sigma^2-\frac{1}{4!N}\left(\lambda_0+\frac{\eta_0\langle\phi\rangle^2}{10N}\right)(\pi^2+\sigma^2)^2-\frac{\eta_0}{6!N^2}(\pi^2+\sigma^2)^3\,.\end{aligned} \tag{B1}$$

Define renormalized parameters by,

$$m^2=m_0{}^2+\delta m^2,\qquad\lambda=\lambda_0+\delta\lambda,\qquad\eta=\eta_0\,. \tag{B2}$$

We have anticipated the absence of ϕ^6 coupling renormalization, and δm^2 and $\delta\lambda$ will be specified later. With these substitutions and using Eqs. (3.35) and (3.41) for $\langle\phi\rangle$ and $\bar\lambda$, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}(\pi,\sigma)=&\tfrac{1}{2}(\partial_\mu\pi)^2+\tfrac{1}{2}(\partial_\mu\sigma)^2+\tfrac{1}{2}\left(\delta m^2+\frac{\delta\lambda\langle\phi\rangle^2}{6N}\right)(\pi^2+\sigma^2)-(\bar\lambda-\delta\lambda)\frac{\langle\phi\rangle^2\sigma^2}{6N}+\left(\delta m^2+\frac{\delta\lambda\langle\phi\rangle^2}{6N}\right)\langle\phi\rangle\sigma\\&-\frac{8\eta\langle\phi\rangle^3}{6!N}\sigma^3-(\bar\lambda-\delta\lambda)\frac{\langle\phi\rangle}{6N}(\pi^2+\sigma^2)\sigma-\frac{\eta\langle\phi\rangle}{5!N^2}(\pi^2+\sigma^2)^2\sigma-\frac{2\eta\langle\phi\rangle^2}{5!N^2}(\pi^2+\sigma^2)\sigma^2\\&-\frac{1}{4!N}(\bar\lambda-\delta\lambda)(\pi^2+\sigma^2)^2-\frac{\eta}{6!N^2}(\pi^2+\sigma^2)^3\,.\end{aligned} \tag{B3}$$

Note that the counterterms contribute to the σ one-point function and the π mass. Both of these quantities must vanish. The σ one-point function must vanish because we have shifted the field to ensure just this. So we may ignore graphs with σ transitions to the vacuum in our calculations below. The π mass must vanish because these fields correspond to the Goldstone bosons. We will not ignore the contribution of the counterterms to the π mass because this term is needed to cancel other contributions explicitly.

Those Feynman rules which we shall need from this Lagrangian are shown in Fig. 8 for Minkowski momentum. The substitution $p^2 \to -p^2$ will give the Feynman rules for Euclidean momentum, which we use throughout this appendix.

By the same arguments as in Sec. II B the only graphs which contribute to the σ propagator to leading order in $1/N$ are bubble trees with two external σ lines. As a first step in the summation of these graphs we define a quantity S to be the sum of all vacuum bubble trees with one zero-momentum insertion. We can write S in the form of an implicit expansion, shown graphically in Fig. 9. A perusal of this figure will convince the reader that this expansion includes all graphs in S. With the convention that there is a factor of $1/N$ at the insertion, which makes S of order 1,

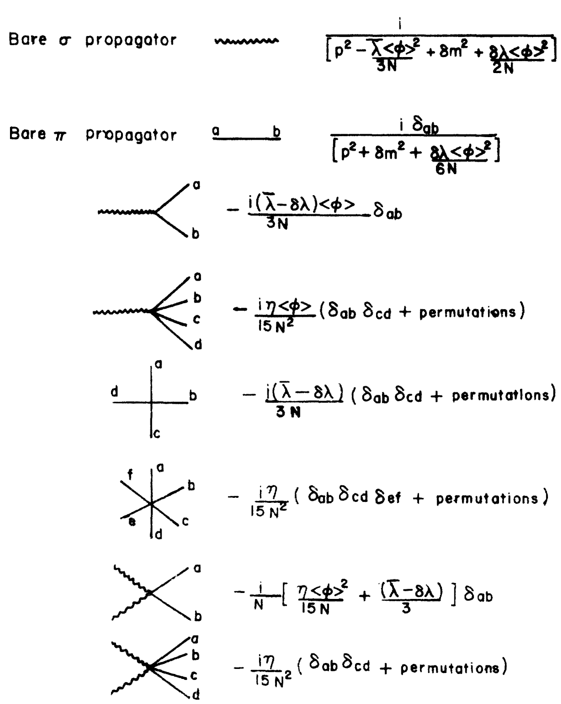

FIG. 8. Relevant Feynman rules for $\mathcal{L}(\pi,\sigma)$ (Minkowski momentum).

and using the Feynman rules of Fig. 8, we obtain

$$S=\frac{\hbar}{2}\int_E \frac{d^3k}{(2\pi)^3}\left\{\frac{-i}{p^2-\delta m^2-\frac{\delta\lambda\langle\phi\rangle^2}{6N}} + \frac{-i}{p^2-\delta m^2-\frac{\delta\lambda\langle\phi\rangle^2}{6N}}\left[\left(\frac{\bar\lambda-\delta\lambda}{3}\right)S-\frac{\eta S^2}{30}\right] \times \frac{-i}{p^2-\delta m^2-\frac{\delta\lambda\langle\phi\rangle^2}{6N}}+\cdots\right\}. \tag{B4}$$

Summing this series and inserting a momentum cutoff,

$$S=\frac{\hbar}{2}\int_E^\Lambda \frac{d^3k}{(2\pi)^2}\left(p^2+\frac{\bar\lambda S}{3}+\frac{\eta S^2}{30}-\frac{\delta\lambda S}{3}-\frac{\delta\lambda\langle\phi\rangle^2}{6N}-\delta m^2\right)^{-1}. \tag{B5}$$

This integral is familiar from Sec. II

$$S=\frac{\hbar}{2}\left[\frac{\Lambda}{2\pi^2}-\frac{1}{4\pi}\left(\frac{\bar\lambda S}{3}+\frac{\eta S^2}{30}-\frac{\delta\lambda S}{3}-\frac{\delta\lambda\langle\phi\rangle^2}{6N}-\delta m^2\right)^{1/2}\right]. \tag{B6}$$

We now define a renormalized quantity $\bar S$ which we will show to be finite,

$$\bar S = S-\frac{\hbar\Lambda}{4\pi^2}. \tag{B7}$$

Choose the counterterms to be the same as in Sec Secs. II and III, but expressed in terms of renormalized parameters (hence the minus sign),

$$\delta m^2=\frac{\lambda}{6}\left(\frac{\hbar\Lambda}{2\pi^2}\right)-\frac{\eta}{5!}\left(\frac{\hbar\Lambda}{2\pi^2}\right)^2, \qquad \delta\lambda=\frac{\eta}{10}\left(\frac{\hbar\Lambda}{2\pi^2}\right). \tag{B8}$$

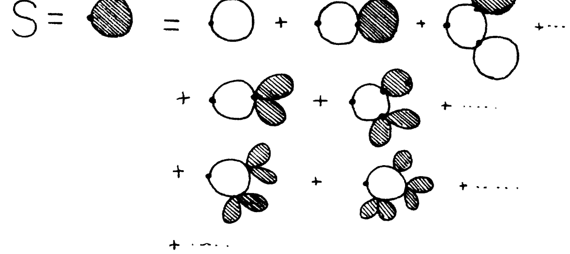

FIG. 9. The expansion of the vacuum bubble trees of $\mathcal{L}(\pi,\sigma)$ with one zero-momentum insertion.

With these substitutions Eq. (B6) becomes

$$\bar{S} = -\frac{\hbar}{8\pi}\left(\frac{\bar{\lambda}\bar{S}}{3} + \frac{\eta\bar{S}^2}{30}\right)^{1/2} \tag{B9}$$

which shows that $\bar{S}$ is finite. Now the π propagator to leading order in $1/N$ is

$$\frac{-i}{p^2 - \delta m^2 - \dfrac{\delta\lambda\langle\phi\rangle^2}{6N} - \Sigma_\pi^*}, \tag{B10}$$

where Σ_π^* in the proper π two-point function, shown graphically in Fig. 10. Algebraically,

$$i\Sigma_\pi^* = -\frac{i(\bar{\lambda} - \delta\lambda)}{3} S - \frac{i\eta S^2}{30}. \tag{B11}$$

It is easy to see from these equations that the π mass is simply the quantity under the square root in Eq. (B9). This must vanish as we have already remarked, which implies that $\bar{S} = 0$. It is sufficient to note that this is a solution of Eq. (B9). This means that in summing the chains of bubbles which which contribute to the proper σ two-point function in Fig. 11(a), we can use the massless π propagator,

$$-i/p^2 . \tag{B12}$$

The proper σ two-point function, $\Sigma_\sigma^*(p)$, is shown graphically in Figs. 11(a) and 11(b). Along the chain of bubbles in Fig. 11(a) the two subgraphs of Fig. 12(a) sum to an effective vertex,

$$-i\left(\frac{\bar{\lambda} - \delta\lambda}{3} + \frac{1}{2}\,\frac{\eta\hbar}{15}\int_E^\Lambda \frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2}\right). \tag{B13}$$

Here, as elsewhere, the factor of $\frac{1}{2}$ is due to the additional twofold symmetry of the two-bubble graph. The integral is just $\Lambda/2\pi^2$, so using the value of $\delta\lambda$ from Eq. (B8), the effective vertex is

$$-i\bar{\lambda}/3 . \tag{B14}$$

Similarly, at the ends of the chain the two subgraphs of Fig. 12(b) sum to an effective vertex,

$$-i\left[\left(\frac{\bar{\lambda} - \delta\lambda}{3}\right)\langle\phi\rangle + \frac{1}{2}\,\frac{\eta\langle\phi\rangle}{15}\hbar\left(\frac{\Lambda}{2\pi^2}\right)\right] = \frac{-i\bar{\lambda}\langle\phi\rangle}{3}. \tag{B15}$$

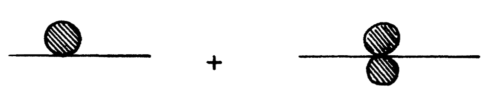

FIG. 10. The proper π two-point function, Σ_π^*. The bare inverse π propagator is not included and there are no propagators on external lines.

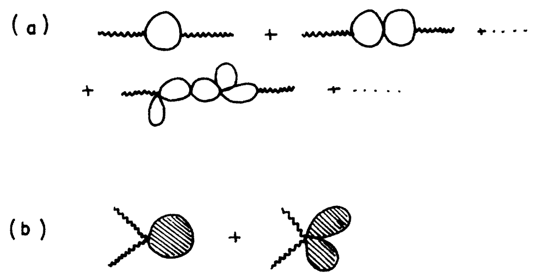

FIG. 11. The proper σ two-point function, $\Sigma_\sigma^*(p)$. (a) The chain graphs. (b) The S insertions.

Therefore, the graphs of Fig. 11(a) give

$$-\left(\frac{\bar{\lambda}\langle\phi\rangle}{3}\right)^2 \sum_{n=1}^{\infty}\left(\frac{-i\bar{\lambda}}{3}\right)^{n-1}\left(\frac{-i}{2}\right)^n \times\left(\hbar\int_E \frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2(k+p)^2}\right)^n . \tag{B16}$$

The integral appeared in Sec. III and its value is $1/8p$, so the above series sums to

$$\frac{i\bar{\lambda}\langle\phi\rangle^2}{3}\,\frac{\lambda\hbar}{48p}\left(1 + \frac{\bar{\lambda}\hbar}{48p}\right)^{-1}. \tag{B17}$$

Recalling the definition of $\bar{m}^2$, in Eq. (3.55), we conclude that the chain graphs give

$$i\Sigma_\sigma^*(p) = -\frac{2i\bar{m}^2}{48p}\bar{\lambda}\hbar\left(1 + \frac{\bar{\lambda}\hbar}{48p}\right)^{-1} + \text{other terms}. \tag{B18}$$

The remaining graphs in $\Sigma_\sigma^*(p)$, of Fig. 11(b), give us the "other terms,"

$$\text{other terms} = -i\left(\frac{\eta\langle\phi\rangle^2}{15N} + \frac{\bar{\lambda} - \delta\lambda}{3}\right)S - \frac{i\eta}{15}S^2\frac{1}{2}. \tag{B19}$$

Now, since $\bar{S} = 0$, we have $S = \Lambda\hbar/4\pi^2$; and using Eq. (B2) the "other terms" become

$$\text{other terms} = -i\left(\delta m^2 + \frac{\delta\lambda\langle\phi\rangle^2}{2N}\right). \tag{B20}$$

Therefore, the proper σ two-point function to lead-

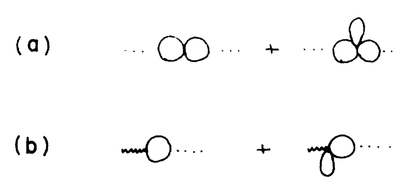

FIG. 12. For calculation of effective vertices. (a) In the chain. (b) At the ends of the chain.

ing order in $1/N$ is

$$\Sigma_\sigma^*(p) = -\frac{\overline{m}^2\overline{\lambda}\hbar}{24p}\left(1+\frac{\overline{\lambda}\hbar}{48p}\right)^{-1} - \delta m^2 - \frac{\delta\lambda\langle\phi\rangle^2}{2N}. \tag{B21}$$

And the σ propagator to leading order in $1/N$ is

$$G_{\sigma\sigma}(p) = \frac{-i}{p^2 + \dfrac{\overline{\lambda}\langle\phi\rangle^2}{3N} - \delta m^2 - \dfrac{\delta\lambda\langle\phi\rangle^2}{2N} - \Sigma_\sigma^*(p)}. \tag{B22}$$

The δm^2, $\delta\lambda$ terms cancel, and using Eq. (3.55) again,

$$G_{\sigma\sigma}(p) = \frac{-i\left(p+\dfrac{\overline{\lambda}\hbar}{48p}\right)}{p\left(p^2+\dfrac{\overline{\lambda}\hbar}{48}p - 2\overline{m}^2\right)}. \tag{B23}$$

This is the result of Sec. III, which completes the independent derivation of the σ propagator.

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