Exact localized solutions of two-dimensional field theories of massive fermions with Fermi interactions*

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The classical equations of motion for field theories of massive fermions with Fermi interactions in one space and one time dimension are investigated. It is shown that they all possess exact stationary solutions which are localized, due to the vanishing of the stress tensor, a feature for this type of solutions only in two dimensions. The explicit forms of these solutions are presented. More importantly, exact solutions for the bound state of N localized massive fermions with scalar or vector Fermi interactions are also found.

I. INTRODUCTION

It has long been known that nonlinear field equations possess classical solutions having particle-like properties.[1] Recently, there has been wide interest in such explicit finite-energy solutions which might represent the classical approximation to extended particles. The question of quantizing these solutions[2] is of particular importance in trying to construct quantum theories of extended hadrons.

Here we investigate the classical equations of motion for field theories of massive fermions with Fermi interactions in one space and one time dimension. We find that they all possess exact localized solutions. In particular, we find an exact solution for the bound state of N localized fermions in theories such as the massive Gross-Neveu model. The solution for the phase between upper and lower components is shown to be independent of the particular type of interaction. This and a general method of finding the solution is presented in Sec. II. Exact, explicit solutions for specific models are worked out in Sec. III. Then, in Sec. IV, the solution for the bound state of N localized fermions with scalar or vector Fermi interaction is presented. Finally, in Sec. V, some relevant points are discussed.

II. THE METHOD

We will be concerned here with models of fermions with Fermi interactions in one space and one time dimension. We are interested in obtaining exact solutions of such models, which are localized (or confined) in space and other possible interesting properties that such solutions might possess. We begin by discussing a method toward such a goal. In general, the Lagrangian for such models is

$$L=\overline{\Psi}(i\partial\!\!\!/-m)\Psi+L_I\,, \tag{1}$$

where L_I contains all types of Fermi interactions of the typical form $(g^2/2)(\overline{\Psi}\Gamma\Psi)^2$ so that one has

$$\overline{\Psi}\,\frac{\partial L_I}{\partial\overline{\Psi}}=2L_I,\quad \frac{\partial L_I}{\partial\Psi}\Psi=2L_I\,. \tag{2}$$

We are looking for stationary solutions

$$\Psi(x,t)=e^{-iEt}\,\psi(x)\,.$$

From (1), the equation of motion is

$$(i\partial\!\!\!/-m)\Psi+\frac{\partial L_I}{\partial\overline{\Psi}}=0\,, \tag{3}$$

and the energy-momentum stress tensor is given by

$$T_{\mu\nu}=i\overline{\Psi}\gamma_\mu\partial_\nu\Psi-g_{\mu\nu}L\,. \tag{4}$$

The Hamiltonian density, therefore, is

$$H=m\overline{\Psi}\Psi+T_{11}\,, \tag{5}$$

where

$$T_{11}=L+i\overline{\Psi}\gamma_1\frac{\partial}{\partial x}\Psi\,.$$

In addition

$$T_{10}=i\overline{\Psi}\gamma_1\frac{\partial}{\partial t}\Psi=E\overline{\psi}\gamma_1\psi\,.$$

For the stationary solutions, the conservation law $\partial^\mu T_{\mu\nu}=0$ then becomes

$$\frac{dT_{11}}{dx}=0,\quad \frac{dT_{10}}{dx}=0\,, \tag{6a}$$

leading to

$$T_{11}=\text{const},\quad T_{10}=\text{const}. \tag{6b}$$

We are looking for localized (confined) solutions here and, therefore, with the boundary condition $\psi(\pm\infty)=0$, demanding that the constants in (6b) vanish, we obtain the following constraints:

$$E\overline{\psi}\gamma_1\psi=0\,, \tag{7a}$$

$$L+i\overline{\psi}\gamma_1\frac{\partial}{\partial x}\psi=0\,. \tag{7b}$$

With (7b), the Hamiltonian density for the localized stationary solutions then reads

$$H = m\bar{\psi}\psi\ . \tag{7c}$$

Equation (7a) is easily satisfied by the choice of γ matrices $\gamma_0 = \sigma_3$, $i\gamma_1 = \sigma_1$, and $\gamma_5 = \gamma_0\gamma_1 = \sigma_2$, and by requiring $\psi(x)$ to be real. On the other hand, Eq. (7b) becomes by virtue of (1)–(3)

$$E\psi^\dagger\psi - m\bar{\psi}\psi + L_I = 0\ , \tag{8a}$$

or equivalently

$$E\psi^\dagger\psi - m\bar{\psi}\psi + i\bar{\psi}\gamma_1\frac{\partial}{\partial x}\psi = 0\ . \tag{8b}$$

Equation (8b) is very interesting because it does not depend on the particular type of interaction involved. In addition, it assumes a greater significance if one realizes that (8b) solves exactly for the phase between upper and lower components. In particular, if one defines

$$\psi(x) = \begin{bmatrix} u \\ v \end{bmatrix} = R(x)\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

then

$$\bar{\psi}i\gamma_1\frac{\partial}{\partial x}\psi = \psi^\dagger\psi\frac{d\theta}{dx}\ .$$

Equation (8b) turns into a differential equation solely in terms of θ which is easily integrable, i.e.,

$$\frac{d\theta}{dx} = -E + m\cos 2\theta\ .$$

Performing the simple integration $x = \int d\theta/(m\cos 2\theta - E)$ one immediately obtains the solution

$$\theta(x) = \tan^{-1}(\alpha\tanh\beta x)\ , \tag{9}$$

with

$$\alpha = \left(\frac{m-E}{m+E}\right)^{1/2}, \quad \beta = (m^2 - E^2)^{1/2}\ .$$

Here one should realize that $E < m$ corresponds to a confined solution. It is, in fact, remarkable that the phase θ does not depend on the interaction, but is solely determined by the parameters α and β. A large step is, therefore, taken in the determination of the exact classical solutions that have confinement character. It remains now to determine $R(x)$ which obviously depends on the interaction. To do that, one can use the equation of motion or, alternatively, Eq. (8a). This will be dealth with in the next section.

III. EXACT LOCALIZED SOLUTIONS FOR SCALAR, VECTOR, AND PSEUDOSCALAR FERMI INTERACTIONS

A. Scalar interaction

The function $R(x)$ as determined by the equations of motion for $\psi(x) = R(x)\left[\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right]$ will now be calculated. First, we will discuss the scalar Fermi interaction, i.e., choose as the interaction Lagrangian $L_I = \frac{1}{2}g^2(\bar{\Psi}\Psi)^2$. From the equation of motion, (3), we obtain in terms of u and v

$$\frac{dv}{dx} + mu - g^2(u^2 - v^2)u - Eu = 0\ ,$$

$$\frac{du}{dx} + mv - g^2(u^2 - v^2)v + Ev = 0\ .$$

In addition, Eq. (8a) gives

$$ER_S^2 - mR_S^2\cos 2\theta + \tfrac{1}{2}g^2R_S^4\cos^2 2\theta = 0\ , \tag{10}$$

where the subscript S refers to the type of interaction under consideration. Using (9) and (10), solutions other than the trivial one can be easily found to be

$$R_S^2(x) = \frac{2}{g^2}\left(\frac{-E + m\cos 2\theta}{\cos^2 2\theta}\right) = \frac{2(m-E)}{g^2}\,\frac{1 + \alpha^2\tanh^2\beta x}{\cosh^2\beta x(1 - \alpha^2\tanh^2\beta x)^2}\ . \tag{11}$$

The appropriate normalization condition in the fermion-number-equal-one sector is

$$\int \psi^\dagger\psi dx = 1\ ,$$

which determines the energy eigenvalue of E to be

$$E = \frac{m}{[1 + (g^2/2)^2]^{1/2}}\ .$$

In order to see that the exact classical solution describes a localized state, we calculate the expectation value of the classical Hamiltonian:

$$\langle H\rangle = m\int \bar{\psi}\psi\, dx = \frac{2m}{g^2}\sinh^{-1}\frac{g^2}{2}\ . \tag{12}$$

Threefore, for $g^2 \to 0$, $\langle H\rangle \to m$ (free massive fermion). In fact, $\langle H\rangle \leq m$ for all values of g^2, a consequence in agreement with our expectation.

B. Vector interaction

The interaction Lagrangian in this case is $L_I = \frac{1}{2}g^2(\bar{\Psi}\gamma_\mu\Psi)^2$. The equation of motion now becomes

$$\frac{dv}{dx}+mu-g^2(u^2+v^2)u-Eu=0\,,$$

$$\frac{du}{dx}+mv+g^2(u^2+v^2)v+Ev=0\,.$$

The equivalent of Eq. (10) here is

$$ER_V{}^2-mR_V{}^2\cos 2\theta+\tfrac{1}{2}g^2R_V{}^4=0\,. \tag{13}$$

Again, with (9), the solution[3] is easily determined to be

$$R_V{}^2(x)=\frac{2(m-E)}{g^2}\frac{1}{\cosh^2\beta x(1+\alpha^2\tanh^2\beta x)}\,. \tag{14}$$

Here the normalization condition $\int\psi^\dagger\psi\,dx=1$ yields

$$E=m\cos\tfrac{1}{2}g^2\,, \tag{15}$$

which imposes the requirement on the coupling constant $g^2\leqslant\pi$ so that the spectrum is composed of positive-energy fermion states. The expectation value of the classical Hamiltonian in this case turns out to be

$$\langle H\rangle=\frac{2m}{g^2}\sin\frac{g^2}{2}\,. \tag{16}$$

Here again, one sees that the spectrum is restricted to $\langle H\rangle\leqslant m$ for all possible values of g^2 in the range $0\leqslant g^2\leqslant\pi$, showing that the exact solution exhibits "bound"-state behavior.

C. Pseudoscalar interaction

The choice of the interaction Lagrangian here is $L_I=-\frac{1}{2}g^2(\overline{\Psi}\gamma_5\Psi)^2$. To obtain the desired solution, we proceed following the method established previously. The equations of motion are

$$\frac{dv}{dx}+mu-2g^2uv^2-Eu=0\,,$$

$$\frac{du}{dx}+mv+2g^2u^2v+Ev=0\,.$$

The constraint equation is

$$ER_P{}^2-mR_P{}^2\cos 2\theta+\tfrac{1}{2}g^2R_P{}^4\sin^2 2\theta=0\,. \tag{17}$$

Again, with (9) and (17), the solution for $R_P{}^2(x)$ is easily seen to be

$$R_P{}^2(x)=\frac{2(m-E)}{4\alpha^2g^2}\frac{1+\alpha^2\tanh^2\beta x}{\sinh^2\beta x}\,. \tag{18}$$

If we follow the normalization procedure established, we obtain

$$E=-\frac{m}{[1+(2/g^2)^2]^{1/2}}\,. \tag{19}$$

In contrast to the previously discussed cases, however, the energy eigenvalue for the pseudoscalar Fermi interaction is negative. Therefore, if we restrict ourselves to positive-energy fermion states, we must reject the solution of Eq. (18). It appears that positive-energy localized fermion states do not appear in the solution of the two-dimensional fermion model with pseudoscalar self-interactions. If only for completeness, we record here the expectation value of the classical Hamiltoian:

$$\langle H\rangle=-m[1+(2/g^2)^2]^{1/2}\,. \tag{20}$$

D. Other types of interactions

Here we would like to point out that other types of interaction such as S-P, V-A, or A are all equivalent with V. In addition, an interesting relation between the functions R_S, R_P, and R_V is the following:

$$R_S{}^{-2}+R_P{}^{-2}=R_V{}^{-2}\,.$$

This simply follows from the solutions discussed in (11), (14), and (18).

IV. BOUND STATE OF N LOCALIZED FERMIONS

In this section, we will be concerned with solutions to the massive Gross-Neveu model. In particular, we will show that there exists an exact "bound"-state solution of N localized fermions. The massive Gross-Neveu[4] model is described by the Lagrangian

$$L=\overline{\Psi}(i\partial\!\!\!/-m)\Psi+\tfrac{1}{2}g^2(\overline{\Psi}\Psi)^2\,, \tag{21}$$

where Ψ is the N-component massive fermion field. We seek again stationary solutions of the form $\Psi(x,t)=e^{-iEt}\psi(x)$ within the classical framework established previously. The corresponding equations that must be solved here are much more complicated since they involve quartic interactions among N-fermion fields. Their form, however, remains almost the same. The equations of motion read

$$\left[i\gamma_1\frac{\partial}{\partial x}-m-g^2(\overline{\psi}\psi)\right]\psi_i=E\gamma_0\psi_i\,, \tag{22}$$

where ψ_i stands for the wave function of the ith field. In particular, expressing ψ_i in terms of R_i and θ_i, where the usual definition

$$\psi_i=R_i\begin{bmatrix}\cos\theta_i\\ \sin\theta_i\end{bmatrix}$$

is used, the equations of motion become

$$\frac{d\theta_i}{dx}=E-m\cos 2\theta_i+g^2\left(\sum_{j=1}^{N}R_j{}^2\cos 2\theta_j\right)\cos 2\theta_i\,, \tag{23a}$$

$$\frac{dR_i}{dx}=-\left(m-g^2\sum_{j=1}^{N}R_j{}^2\cos 2\theta_j\right)R_i\sin 2\theta_i\,. \tag{23b}$$

In addition, the constraint equation similar to that in (8b) gives

$$T_{11}=E\psi^\dagger\psi - m\bar\psi\psi + \frac{g^2}{2}(\bar\psi\psi)^2$$
$$=E\sum_{j=1}^N R_j^2 - m\sum_{j=1}^N R_j^2\cos 2\theta_j + \frac{g^2}{2}\left(\sum_{j=1}^N R_j^2\cos 2\theta_j\right)^2$$
$$=0\,. \tag{24}$$

It is easy to see by simply substituting into Eqs. (23) and (24) that

$$R_i = R_S/\sqrt{N}\,,\qquad \theta_i=\theta \tag{25}$$

for all i, where R_S and θ are given in Eqs. (10) and (11), respectively, is a solution. This by no means implies that it is a unique solution, but it is the only solution obtained so far.

It is very interesting to know that this solution exhibits properties of a "bound" state. In order to see that we first calculate the energy eigenvalue by imposing the normalization condition

$$\int \psi^\dagger\psi\, dx = N$$

to obtain

$$E=\frac{m}{[1+(\frac{1}{2}g^2N)^2]^{1/2}}\,. \tag{26}$$

Following the discussion of Sec. III, the expectation value of the classical Hamiltonian is easily calculated to be

$$\langle H\rangle = \frac{2m}{g^2}\sinh^{-1}(\tfrac{1}{2}Ng^2)\,. \tag{27}$$

It is obvious that $\langle H\rangle \le Nm$. Recall from the discussion in Sec. III A that the expectation value of the classical Hamiltonian for a single localized fermion is $\langle H\rangle = (2m/g^2)\sinh^{-1}(\frac{1}{2}g^2)$. So, for any assembly of N such localized fermions noninteracting with each other, the expectation value of the Hamiltonian is

$$\langle H\rangle_N = \frac{2mN}{g^2}\sinh^{-1}(\tfrac{1}{2}g^2)\,. \tag{28}$$

From the inequality

$$\sinh^{-1}(\tfrac{1}{2}g^2N) < N\sinh^{-1}(\tfrac{1}{2}g^2)\,,$$

valid for finite g^2, we immediately see that

$$\langle H\rangle < \langle H\rangle_N\,.$$

Thus, the wave function so obtained in Eq. (25) describes a bound state of N localized fermions.

The model can also be solved if instead of scalar interactions as in the Gross-Neveu model one assumes a vector interaction, i.e., $L_I=(\bar\Psi\gamma_\mu\Psi)^2$ where again, Ψ is an N-component field. The solution outlined then still persists, having the same features as before, except that the function R_S now becomes R_V. That is, the solution and expectation value of the classical Hamiltonian appropriate here are, respectively,

$$R_i^V=\frac{R_V}{\sqrt{N}}\,,\qquad \theta_i=\theta \text{ for all } i\,,\qquad \langle H\rangle = \frac{2m}{g^2}\sin\tfrac{1}{2}Ng^2\,. \tag{29}$$

In a similar manner, the expectation value of the classical total Hamiltonian of a collection of N fermions each of which interacts only with itself through a vector Fermi interaction is simply

$$\langle H\rangle_N = \frac{2Nm}{g^2}\sin\tfrac{1}{2}g^2\,. \tag{30}$$

With the inequality

$$\sin\tfrac{1}{2}Ng^2 \le N\sin\tfrac{1}{2}g^2 \text{ for } g^2\le\pi$$

we again obtain

$$\langle H\rangle < \langle H\rangle_N\,.$$

Thus the solution in (29) also describes a "bound" state of N localized fermions.

V. CONCLUDING REMARKS

It should be pointed out that the massless limit of the localized solutions obtained here does not exist. We have shown that the phase between upper and lower components is independent of interaction. This is particularly important in view of the fact that for $E=0$ the phase obeys the sine-Gordon equation as can be easily demonstrated. It is very interesting to note that the classical solution for $E=0$ found in Sec. III for the massive Thirring model and the solution $\Phi=4\theta$ of the sine-Gordon equation satisfy the correspondence equations recently discussed by Coleman.[5] We also found localized solutions of a bound state composed of N fermions for the massive Gross-Neveu model. This is the first time, to our knowledge, that such solutions have been discussed.

Recently Dashen, Hasslacher, and Neveu[6] have developed a functional method of quantizing classical particle-like solutions for theories containing fermion fields. Their method can be applied to attack the problem of quantizing the solutions presented here. We plan to deal with this matter and related problems in a future publication.

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