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Light scattering properties of naked singularities*†

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The astrophysical implications of the light scattering and capture cross sections of a Curzon naked singularity are discussed. It is found that a Curzon naked singularity possesses a neatly circular capture cross section for photons of approximately the same size as that of a Schwarzschild black hole of equal mass. This suggests that a black hole and a naked singularity can produce similar astrophysical effects.

In the past few years considerable effort has been expended in the study of astrophysical phenomena associated with black holes. This effort seems finally to have been vindicated by the probable identification of these improbable objects in compact x-ray sources. Naked singularities—spacetime singularities not hidden behind an event horizon—represent solutions to the field equations of general relativity for compact gravitational objects that are as exotic as black holes. Although no argument can presently be given for the astrophysical evolution[1] of naked singularities, they should not be *a priori* dismissed as unphysical; some consideration of their observational properties is worthwhile.

Of particular interest are those effects which could distinguish a naked singularity from other compact objects (e.g., a black hole or a neutron star). Astrophysical processes involving only the weak-field region of the geometry would not seem to be capable of such a distinction because of the ambiguity of the source (e.g., a Newtonian multipole field could in principle be constructed to produce any weak static field). The large-angle scattering of photons offers a greater potential for observational distinction.

In general the cross sections for large-angle scattering or capture for a naked singularity will be very different from that for a black hole. It will be shown, for instance, that the addition of a quadrupole term can enlarge the capture cross section without bound. Sufficiently contrived naked-singularity geometries can be constructed to have almost arbitrarily peculiar properties in this regard. Perhaps a more astrophysically relevant question is: Can naked singularities mimic the light deflection characteristics of black holes?

To investigate this question we consider the Weyl-metric,[2] static axially symmetric vacuum geometries of the form

$$(ds)^2 = -e^{2\psi}dt^2 + e^{2\gamma-2\psi}(dr^2 + r^2 d\theta^2) + e^{-2\psi}r^2\sin^2\theta\, d\phi^2 . \quad (1)$$

The function ψ, which plays somewhat the role of a Newtonian potential, satisfies

$$\nabla^2\psi = r^{-2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + r^{-2}\sin^{-2}\theta\,\frac{\partial}{\partial\theta}\left(\sin^2\theta\frac{\partial\psi}{\partial\theta}\right) = 0 , \quad (2)$$

and the function γ is generated from ψ by

$$\frac{\partial\gamma}{\partial r} = -r^{-1}\sin^2\theta\left(\frac{\partial\psi}{\partial\theta}\right)^2 + 2\sin\theta\cos\theta\left(\frac{\partial\psi}{\partial\theta}\right)\left(\frac{\partial\psi}{\partial r}\right) + r\sin^2\theta\left(\frac{\partial\psi}{\partial r}\right)^2 , \quad (3)$$

$$\frac{\partial\gamma}{\partial\theta} = \sin\theta\cos\theta\left(\frac{\partial\psi}{\partial\theta}\right)^2 + 2r\sin^2\theta\left(\frac{\partial\psi}{\partial\theta}\right)\left(\frac{\partial\psi}{\partial r}\right) - r^2\sin\theta\cos\theta\left(\frac{\partial\psi}{\partial r}\right)^2 . \quad (4)$$

The solution for ψ that could represent the exterior field of a compact object is

$$\psi(r,\theta) = -\sum_{n=0}^{\infty} r^{-(n+1)} a_n P_n(\cos\theta) , \quad (5)$$

where $P_n(x)$ is a Legendre polynomial and the a_n are arbitrary coefficients. The a_0 and a_2 terms correspond respectively to Newtonian monopoles (i.e., "mass" M as measured by Keplerian orbits) and quadrupoles. For a finite number of nonzero a_n's these geometries have singularities which are not shielded by a nonsingular event horizon and hence are naked.

To illustrate the possible differences in the scattering of photons by these naked singularities and black holes consider the equatorial ($\theta = \pi/2$) trajectories in a "monopole plus quadrupole" field

$$\psi(r, \theta = \pi/2) = -Mr^{-1} - \alpha M r^{-3} . \quad (6)$$

The two integrals

$$\frac{dt}{d\lambda} = \mathcal{E}e^{-2\psi} , \quad (7)$$

$$\frac{d\phi}{d\lambda} = \mathcal{L}r^{-2}\sin^{-2}\theta\, e^{2\psi} \quad (8)$$

(where $\mathcal{E}$ and $\mathcal{L}$ are constants and λ is an affine parameter) follow from the static and axial symmetries of the Weyl geometry. When these are substituted into the relation

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0 \tag{9}$$

for photon trajectories, the radial equation

$$e^{2\gamma}\left[\frac{dr}{d(\mathcal{E}\lambda)}\right]^2=b^{-2}-r^{-2}\exp(-4Mr^{-1}-4\alpha Mr^{-3}) \tag{10}$$

results, where $b=\mathcal{L}/\mathcal{E}$ is the impact parameter. The minimum radius to which the photon penetrates is given by the vanishing of the right side of Eq. (10). If the right side is positive-definite $dr/d\lambda$ cannot vanish and hence there is no turning point; the photon is captured. The maximum impact parameter for capture is therefore given by the minimum of the function $r\exp(2M/r+2\alpha M^3/r^3)$. For arbitrarily large α this critical impact parameter is arbitrarily large and hence the equatorial width of the capture cross section is arbitrarily large in contrast to, e.g., a Schwarzschild black hole which has a critical value for $b=3\sqrt{3}M$.[3]

As a candidate for a Weyl geometry which may more closely resemble a black hole, a choice is the Curzon geometry.[4] This field has only a "monopole" term ($\alpha=0$) and from the analysis above has a critical impact parameter in the equatorial plane of $b=2\exp(1)M$, which is only 4.6% larger than the critical value for a Schwarzschild black hole. The full Curzon geometry is given by choosing

$$\psi(r)=-\frac{M}{r}\ , \tag{11}$$

and hence

$$\gamma(r,\theta)=-\frac{M^2}{2r^2}\sin^2\theta\ . \tag{12}$$

The geometry is regular everywhere with the exception of the "point" $r=0$. The "point" $r=0$, $t=$ const corresponds to a 2-surface with infinite area on which the static Killing vector $(\partial/\partial t)$ is null and represents the event horizon for the Curzon geometry.[5] The curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges exponentially as r goes to 0 (except at the poles $\theta=0,\pi$ where it goes to 0) and hence the Curzon singularity is naked.

For the special case of photon trajectories in the equatorial plane, the calculation of deflection can be reduced to quadrature. From Eq. (10) we have that the turning point $r_{\min}$ is given by

$$b=r_{\min}\exp(2Mr_{\min}^{-1})\ , \tag{13}$$

and the deflection (change in ϕ) is given by integrating Eq. (8) and Eq. (10),

$$\delta(b)=\pi-2\int_\infty^{r_{\min}}dr\,r^{-1}\exp\left(-\frac{M^2}{2r^2}\right)\times\left[\frac{r^2}{b^2}\exp\left(\frac{4M}{r}\right)-1\right]^{-1/2}. \tag{14}$$

For large impact parameters the integral of course gives the familiar result

$$\delta\sim\frac{4M}{b}\ . \tag{15}$$

For a general null orbit the first integrals given in Eqs. (7), (8), and (9) are not sufficient. Either a fourth integral must be found (cf. Carter's integral for the Kerr geometry) or the geodesic equations must be used to determine the full orbit. With the usual change of dependent variable $r=Mu^{-1}$, the elimination of the t variable through Eq. (7), and the change of the independent variable from λ to ϕ, Eq. (8), the relevant equations for the trajectory are

$$\frac{d^2u}{d\phi^2}=(2u^2-u)\sin^2\theta\,e^{u^2\sin^2\theta}+(2+u^2\sin^2\theta)\left(\frac{du}{d\phi}\right)^2+(2\cot\theta+u^2\sin2\theta)\left(\frac{du}{d\phi}\right)\left(\frac{d\theta}{d\phi}\right)+(2u^2-u-u^3\sin^2\theta)\left(\frac{d\theta}{d\phi}\right)^2\ , \tag{16}$$

$$\frac{d^2\theta}{d\phi^2}=\tfrac{1}{2}\sin2\theta\,e^{u^2\sin^2\theta}-\tfrac{1}{2}\sin2\theta\left(\frac{du}{d\phi}\right)^2+2u\sin^2\theta\left(\frac{du}{d\phi}\right)\left(\frac{d\theta}{d\phi}\right)+(2\cot\theta+\tfrac{1}{2}u^2\sin2\theta)\left(\frac{d\theta}{d\phi}\right)^2\ . \tag{17}$$

[For circumpolar orbits $\phi=$ const, and Eqs. (16) and (17) are invalid. In this case Eqs. (7) and (9) and the geodesic equations can be used to derive equations similar to Eqs. (16) and (17).]

To start the integration of Eqs. (16) and (17) requires a specification of the initial coordinates (θ_0, ϕ_0) and the first derivatives $(du/d\phi)_0$ and $(d\theta/d\phi)_0$, and these must be related to the incident angle and impact parameters $(b_{\hat\theta}, b_{\hat\phi})$ of the incoming photon (see Fig. 1). At $u=u_0=0$ $(r=\infty)$ the geometry is flat and a simple geometric analysis reveals

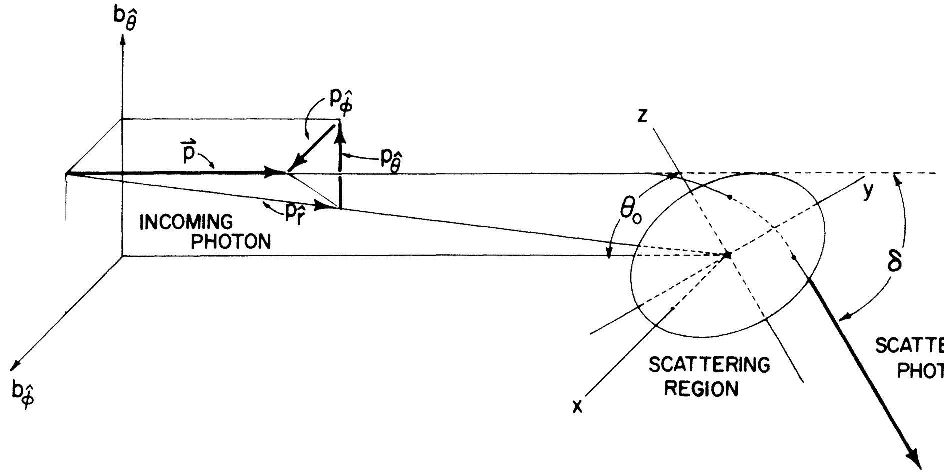

FIG. 1. Geometry for the scattering of light by a Curzon naked singularity. Photons at the left impinge with impact parameters $b_{\hat\theta}$ and $b_{\hat\phi}$ and the naked singularity is inclined at an angle θ_0.

$$b_{\hat\theta} = \lim_{r\to\infty} r\frac{p^{\hat\theta}}{p^{\hat t}} = \left(\frac{d\theta}{d\phi}\right)_0 \left(\frac{du}{d\phi}\right)_0^{-1} , \qquad (18)$$

$$b_{\hat\phi} = \lim_{r\to\infty} r\frac{p^{\hat\phi}}{p^{\hat t}} = \sin\theta_0 \left(\frac{du}{d\phi}\right)_0^{-1} . \qquad (19$$

Since the geometry is axially symmetric we take $\phi_0 = -\pi/2$ without any loss of generality. Given a θ_0, $b_{\hat\theta}$, and $b_{\hat\phi}$ Eqs. (16) and (17) are integrated to find the (θ_f, ϕ_f) for the outgoing photon at $r=\infty$ and the deflection is computed from

$$\delta = \pi - \cos^{-1}(\cos\theta_0 \cos\theta_f - \sin\theta_0 \sin\theta_f \sin\phi_f) \qquad (20)$$

(see Fig. 2). At the critical values of $b_{\hat\theta}$ and $b_{\hat\phi}$ the deflection is infinite and for values smaller than these critical values the singularity captures the photon. [The fate of the few photons which might hit the "$r=0$ point" along the paths of constant $(\sin\theta)/r$ is ignored.] For a given θ_0, the locus of critical values in b space outline the capture cross section (see Fig. 3).

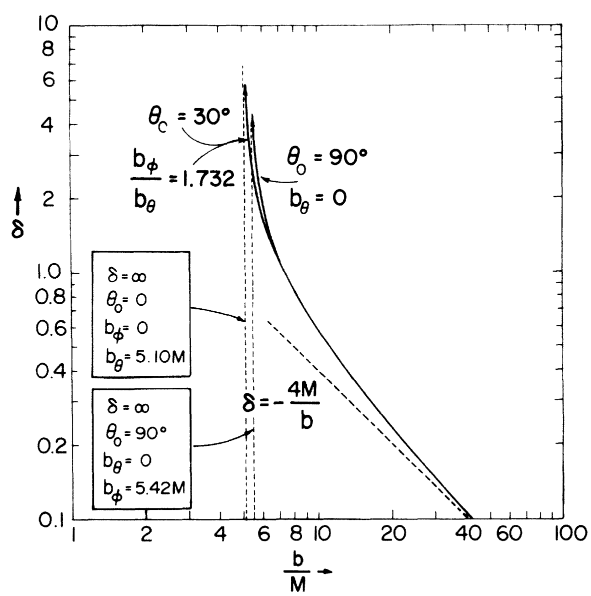

FIG. 2. Deflection angle δ for photons, as a function of impact parameters $(b_{\hat\theta}, b_{\hat\phi})$ [with $b = (b_{\hat\theta}^2 + b_{\hat\phi}^2)^{1/2}$] and inclination angle θ_0 for the Curzon naked singularity. The deflection is infinite for a critical value of the impact parameter and follows $\delta = 4M/b$ for large values of the impact parameters.

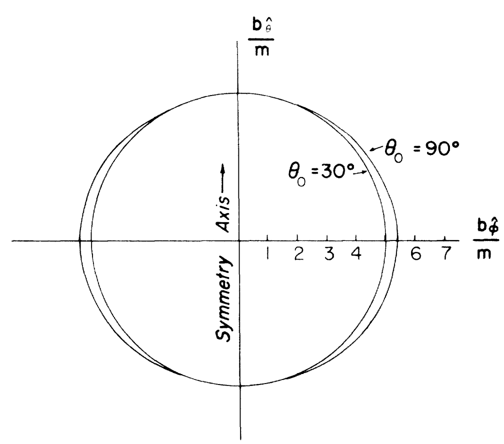

FIG. 3. Critical values of the impact parameters as a function of inclination angle θ_0 for the Curzon naked singularity. The largest critical value is $5.42M$ for equatorial $\theta_0 = \pi/2$ orbits, and the smallest critical value is $5.10M$ for circumpolar $\theta_0 = 0$ orbits; these values bracket the Schwarzschild black-hole value $5.2M$.

The dependence of the deflection angle on the impact parameters shown in Fig. 2 illustrates that for $(b_\theta{}^2 + b_\phi{}^2)^{1/2} \gtrsim 10M$ the asymptotic formula Eq. (15) is a good approximation for a general null orbit. The (weak field) scattering of light by a Curzon naked singularity is very similar to scattering by a Schwarzschild black hole. The interesting result is the size and shape of the capture cross section (a strong field effect). The largest critical impact parameter is for an equatorial orbit with $b_{\rm crit} = 5.42M$, the smallest value is for a circumpolar orbit with a $b_{\rm crit} = 5.10M$, a 6% difference that brackets the Schwarzschild black-hole value $5.2M$. The similarity with regard to scattering and capture of light by the Curzon naked singularity and the Schwarzschild black hole suggests that at least certain types of naked singularities if they can be formed may be difficult to distinguish observationally from black holes.

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