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Solutions for gravity coupled to massless gauge fields*

Philip B. Yasskin

Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

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An explicit procedure is presented so that from each solution of the coupled Einstein-Maxwell equations for gravity and source-free electromagnetic fields, one can construct a set of solutions of the coupled Einstein-Yang-Mills equations for gravity and source-free, unquantized, massless, gauge fields for any gauge group which has an invariant metric. These solutions show that the Rainich-Misner-Wheeler "already unified field theory" cannot be extended to massless gauge fields. As an example of the procedure a solution is constructed which describes the exterior of a rotating black hole which has gauge charges such as isospin and hypercharge. A curved-space generalization of the Wu-Yang solution is shown to be a special case except written in a different gauge. Such black-hole solutions with massless gauge fields question Wheeler's "black holes have no hair" conjecture.

I. INTRODUCTION

This paper discusses some solutions of the coupled Einstein-Yang-Mills (EYM) equations for gravity and source-free unquantized massless gauge fields.

It should be emphasized that the gauge fields considered here are not just those for the gauge group SO(3) as originally considered by Yang and Mills.[1] Rather, they can be the gauge fields for any Lie group which has an invariant group metric,[2] including all the gauge groups considered by particle theorists in recent attempts to unify the electromagnetic, weak, and possibly strong interactions.[3,4] The gauge fields are then the vector mesons which mediate the interactions, and the gauge charges are the conserved quantities like isospin and hypercharge derived by applying Noether's theorem to the gauge symmetries. Section II reviews the derivation of the EYM equations and the special case of the Einstein-Maxwell (EM) equations.

Section III presents an explicit procedure so that from each solution of the EM equations one can quite easily construct a set of solutions of the EYM equations. These "linear solutions"[5] have been known to some experts, but to my knowledge have not appeared in the literature. Their physical significance is discussed in Sec. IV. They also demonstrate that the Rainich-Misner-Wheeler "already unified field theory"[6] cannot be extended to massless gauge fields as discussed in Sec. V.

As an example, a class of solutions is presented in Sec. VI which describes the exterior of a rotating black hole which has gauge charges instead of or in addition to electric charge. Special cases of these solutions have been found independently by other researchers.[7] In particular a curved-space generalization of one of the Wu-Yang solutions for the gauge group SO(3) is shown in Sec. VII to be a special case. At first glance, the Wu-Yang solution appears horribly nonlinear because of mixing between the space indices and the gauge group indices. However, a gauge transformation is presented which transforms the Wu-Yang solution into a linear gauge. This same transformation also works in flat space.

The black-hole solutions with massless gauge fields provide an exception to Wheeler's "black holes have no hair" conjecture,[8] as discussed in Sec. VIII. Spontaneous-symmetry-breaking scalar fields[4] are also an exception. It is hoped that it will be possible to generalize the Higgs mechanism[4] to curved space to produce black-hole solutions with massive gauge fields. Finally, the Israel-Carter black-hole uniqueness conjecture[9] is generalized to the EYM equations.

II. FIELD EQUATIONS[10]

Consider an N-parameter Lie group G with structure constants $c^{p}{}_{qr}$. Assume the group has an invariant group metric[2] γ_{pq}. This invariance of the group metric under the action of infinitesimal group elements implies $c_{pqr} = \gamma_{ps} c^{s}{}_{qr}$ is totally antisymmetric. The gauge potentials are $A^{p}{}_{\mu}$ and the gauge fields are

$$F^{p}{}_{\mu\nu} = \partial_\mu A^{p}{}_{\nu} - \partial_\nu A^{p}{}_{\mu} + c^{p}{}_{qr} A^{q}{}_{\mu} A^{r}{}_{\nu}. \tag{1}$$

Then the gravity and gauge field action can be chosen as

$$S = \int \frac{\sqrt{-g}}{16\pi}(R - g^{\kappa\mu} g^{\lambda\nu} \gamma_{pq} F^{p}{}_{\kappa\lambda} F^{q}{}_{\mu\nu}) d^4x, \tag{2}$$

where R is the scalar curvature. Variation of the action with respect to the spacetime metric $g_{\mu\nu}$ yields the Einstein equations

$$G^{\mu\nu} = 8\pi T_{A}{}^{\mu\nu}, \tag{3}$$

where the gauge stress-energy tensor is

$$T_A^{\mu\nu} = \gamma_{pq}(F^{p\mu\lambda}F^{q\nu}{}_\lambda - \tfrac{1}{4}g^{\mu\nu}F^{p\kappa\lambda}F^q{}_{\kappa\lambda})/4\pi. \tag{4}$$

Variation of the action with respect to the gauge potentials $A^p{}_\mu$ yields the Yang-Mills equations

$$F_p{}^{\mu\nu}{}_{;\nu} = 4\pi j_p{}^\mu, \tag{5}$$

where the electric gauge currents carried by the gauge fields are

$$j_p{}^\mu = c^q{}_{pr}A^r{}_\nu F_q{}^{\nu\mu}/4\pi. \tag{6}$$

Another interesting identity,[11] which can be derived using the symmetry of the spacetime connection and then the definition (1) of the fields, is

$$(*F_p)^{\mu\nu}{}_{;\nu} = 4\pi j_{Mp}{}^\mu, \tag{7}$$

where $*F_p$ is the spacetime dual of F_p, and where the magnetic gauge currents carried by the gauge fields are

$$j_{Mp}{}^\mu = c^q{}_{pr}A^r{}_\nu(*F_q)^{\nu\mu}/4\pi. \tag{8}$$

Here $j_p{}^\mu$ and $j_{Mp}{}^\mu$ are called the electric and magnetic gauge currents because they appear in the Yang-Mills equations where the electric and magnetic electromagnetic currents would appear in Maxwell's equations.

In the electromagnetic case the gauge group is the 1-parameter group U(1). The only structure constant is $c^0{}_{00} = 0$ and the invariant group metric is $\gamma_{00} = 1$. The electromagnetic potential is $A^0{}_\mu$ and the electromagnetic field is

$$F^0{}_{\mu\nu} = \partial_\mu A^0{}_\nu - \partial_\nu A^0{}_\mu. \tag{9}$$

So the action (2) reduces to

$$S = \int \frac{\sqrt{-g}}{16\pi}(R - g^{\kappa\mu}g^{\lambda\nu}F^0{}_{\kappa\lambda}F^0{}_{\mu\nu})d^4x. \tag{10}$$

Einstein's equations become

$$G^{\mu\nu} = 8\pi T_{\text{em}}{}^{\mu\nu}, \tag{11}$$

where the electromagnetic stress-energy tensor is

$$T_{\text{em}}{}^{\mu\nu} = (F^{0\mu\lambda}F^{0\nu}{}_\lambda - \tfrac{1}{4}g^{\mu\nu}F^{0\kappa\lambda}F^0{}_{\kappa\lambda})/4\pi, \tag{12}$$

and finally, Maxwell's equations are

$$F^{0\mu\nu}{}_{;\nu} = 0, \tag{13}$$

and

$$(*F^0)^{\mu\nu}{}_{;\nu} = 0. \tag{14}$$

These are reasonable because the electromagnetic field itself does not carry electric charge or magnetic charge.

III. SOME SOLUTIONS

In general it is difficult to find solutions to the coupled EYM equations (3) and (5) because of nonlinearities in the gauge fields as well as nonlinearities in the gravitational field. However, the following theorem presents a class of solutions in which the nonlinearities in the gauge fields drop out.[5]

Theorem. Let G be an N-parameter Lie group with an invariant metric γ_{pq}. Then for every solution of the coupled Einstein–source-free-Maxwell (EM) equations there is an (N-1)-parameter set of solutions of the coupled Einstein–source-free-massless-Yang-Mills (EYM) equations for the gauge group G. In particular[10] if $g_{\mu\nu}$, $A^0{}_\mu$, $F^0{}_{\mu\nu}$, $T_{\text{em}}{}^{\mu\nu}$ characterize the EM solution, then $g_{\mu\nu}, A^p{}_\mu = \beta^p A^0{}_\mu$, $F^p{}_{\mu\nu} = \beta^p F^0{}_{\mu\nu}$, $T_A{}^{\mu\nu} = T_{\text{em}}{}^{\mu\nu}$, $j_p{}^\mu = 0$, $j_{Mp}{}^\mu = 0$ characterize the EYM solution, where β^p are N parameters which satisfy the single constraint

$$\gamma_{pq}\beta^p\beta^q = 1. \tag{15}$$

Proof. Let $g_{\mu\nu}$ and $A^0{}_\mu$ be a solution of the EM equations (11) and (13). Pick N arbitrary parameters β^p subject only to condition (15). Define gauge potentials:

$$A^p{}_\mu = \beta^p A^0{}_\mu. \tag{16}$$

I will show that $g_{\mu\nu}$ and $A^p{}_\mu$ are a solution of the EYM equations (3) and (5). To see this, first compute the gauge fields. Substitute (16) into (1) and use (9) and the antisymmetry of $c^p{}_{qr}$ in q and r:

$$\begin{aligned} F^p{}_{\mu\nu} &= \beta^p(\partial_\mu A^0{}_\nu - \partial_\nu A^0{}_\mu) + c^p{}_{qr}\beta^q\beta^r A^0{}_\mu A^0{}_\nu \\ &= \beta^p F^0{}_{\mu\nu}. \end{aligned} \tag{17}$$

Next compute the gauge stress-energy tensor. Substitute (17) into (4) and use (15) and (12):

$$\begin{aligned} T_A{}^{\mu\nu} &= \gamma_{pq}\beta^p\beta^q(F^{0\mu\lambda}F^{0\nu}{}_\lambda - \tfrac{1}{4}g^{\mu\nu}F^{0\kappa\lambda}F^0{}_{\kappa\lambda})/4\pi \\ &= T_{\text{em}}{}^{\mu\nu}. \end{aligned} \tag{18}$$

Constraint (15) was chosen to make result (18) possible. Since the gauge and electromagnetic stress-energy tensors are the same, their Einstein equations (3) and (11) are the same. Since $g_{\mu\nu}$ is a solution of (11), it is also a solution of (3). Now compute the electric gauge currents of the gauge fields. Substitute (16) and (17) into (6) and use the antisymmetry of c_{qpr} in q and r:

$$\begin{aligned} j_p{}^\mu &= c_{qpr}\beta^r\beta^q A^0{}_\nu F^{0\nu\mu}/4\pi \\ &= 0. \end{aligned} \tag{19}$$

On the other hand, to compute the left side of the Yang-Mills equations (5), differentiate (17) and use Maxwell's equations (13):

$$\begin{aligned} F_p{}^{\mu\nu}{}_{;\nu} &= \beta_p F^{0\mu\nu}{}_{;\nu} \\ &= 0. \end{aligned} \tag{20}$$

Equations (19) and (20) show that both sides of the Yang-Mills equations (5) are zero, so that $g_{\mu\nu}$ and

$A^p{}_\mu$ are a solution. Finally, compute the magnetic gauge currents of the gauge fields. Substitute (16) and (17) into (8) and use the antisymmetry of c_{qpr} in q and r.

$$j_{Mp}{}^\mu = c_{qpr}\beta^r\beta^q A^0{}_\nu ({}^*F^0)^{\nu\mu}/4\pi = 0. \tag{21}$$

The theorem is proved.

IV. PHYSICAL SIGNIFICANCE OF THE SOLUTIONS

A. Solutions are distinct

It might be asked whether these linear[5] EYM solutions are physically distinct from the original EM solution out of which they were derived. After all, if all gauge fields $F^p{}_{\mu\nu}$ appear in the ratios β^p, isn't it possible to pick a new basis in the Lie algebra (gauge space) so that $\beta^p = (1, 0, \ldots, 0)$? Then the only gauge field present would be $F^1{}_{\mu\nu}$ and the EYM solutions would be identical in form with the original EM solution. So aren't the new EYM solutions just the original EM solution described in strange variables, i.e., a case of mistaken identity?

No! First, although the gauge fields in these linear solutions are present in the ratios β^p, a test particle does not need to have gauge charges Q^p in these same ratios. The gauge charges on a test particle are measured in the laboratory before we send the particle into the region of space approximated by the solution. We know whether the particle is a proton or a neutron, whether it has electric charge or not, whether it has isospin $+\frac{1}{2}$ or $-\frac{1}{2}$. Since the test particle feels a generalization of the Lorentz force,

$$m u^\mu{}_{;\nu} u^\nu = \gamma_{pq} Q^p F^{q\mu}{}_\nu u^\nu, \tag{22}$$

it can probe the solution to find out which type of gauge field is present.

Second, although an over-all rotation of the basis in gauge space may be irrelevant, the relative direction between gauge fields and test charges is significant. The basis cannot be rotated so that both the fields and charges are proportional to $(1, 0, \ldots, 0)$. In fact even the over-all rotation was chosen in the laboratory by convention (up to quantum-mechanical limitations due to noncommutativity) before the test charges were measured.

Third, if the symmetry is broken then there is a preferred basis which cannot be rotated at all. Thus, the linear EYM solutions are physically distinct from each other and from the original EM solution.

B. Solutions with sources

Presently under investigation is the problem of incorporating sources into the solutions. I suspect the theorem may be immediately generalized to produce solutions with sources whose gauge charges Q^p are in the same ratios β^p as the potentials $A^p{}_\mu$ and the gauge fields $F^p{}_{\mu\nu}$. (These charge ratios will show up in the black-hole solution (27), where the gauge charges are $Q^p = \beta^p Q^0$.) Such solutions have several physical applications.

One application might be to make the admittedly poor approximation that a neutron star is all neutrons. Then the gauge charges would be proportional to the expectation values of the neutron's I-spin, U-spin, and V-spin:

$$\begin{aligned} &\langle I_1\rangle = 0, \quad \langle I_2\rangle = 0, \quad \langle I_3\rangle = -\tfrac{1}{2}, \\ &\langle U_1\rangle = 0, \quad \langle U_2\rangle = 0, \quad \langle U_3\rangle = +1, \\ &\langle V_1\rangle = 0, \quad \langle V_2\rangle = 0, \quad (\langle V_3\rangle = \langle I_3\rangle + \langle U_3\rangle = +\tfrac{1}{2}), \end{aligned} \tag{23}$$

with electric charge and hypercharge proportional to those of the neutron:

$$\begin{aligned} &\langle Q_{\rm em}\rangle = \tfrac{4}{3}\langle I_3\rangle + \tfrac{2}{3}\langle U_3\rangle = 0, \\ &\langle Y\rangle = \tfrac{2}{3}\langle I_3\rangle + \tfrac{4}{3}\langle U_3\rangle = +1. \end{aligned} \tag{24}$$

It might then be expected that the gauge fields would also be present in these ratios. Protons, electrons, and other particles with different gauge charge ratios could then be incorporated as perturbations or as test particles.

Another application might be to take initial data for two widely separated black holes with different gauge charge ratios, to allow them to approach each other, and to investigate the nonlinear interaction of the gauge fields.

C. Solutions are unquantized

Finally, these solutions are unquantized. This is not a serious objection because eventual quantum solutions will have to have these solutions among their classical limits.

V. RAINICH-MISNER-WHEELER "ALREADY UNIFIED FIELD THEORY"

In 1925 Rainich[6] showed that the electromagnetic field is completely determined (up to a constant called the "complexion") either by its stress tensor or, via Einstein's equations, by the curvature of a spacetime filled only by electromagnetic radiation. Further, he gave necessary and sufficient, algebraic and differential conditions on the curvature so that it could describe a spacetime filled only by electromagnetic radiation. In 1957 Misner and Wheeler[6] extended this "already unified field theory" by pointing out that unquantized mass could be described as either wormholes or geons in a multiply connected spacetime which satisfies Rainich's conditions. Unquantized charge could then be described as field lines which thread through the wormholes. Thus, all of classical unquantized physics could be described in terms of the curvature of spacetime.

It had been hoped that the remainder of physics could also be described in terms of the geometry.[12] This requires that physically distinct situations have distinct geometries. However, the theorem proved here provides EYM solutions that are geometrically indistinguishable from each other and from an EM solution. They have the same metrics, the same stress tensors, the same singularity structures, and the same gravitational fields. A test mass which carries no electromagnetic or gauge charges cannot distinguish whether the geometry was produced by an electromagnetic field or a gauge field. However, as shown in Sec. IV, the solutions are physically distinct. Thus, the theorem produces many physically distinct solutions which all have the same geometry. The already unified field theory cannot be extended to massless gauge fields.[13]

However, the solutions considered here are only for massless gauge fields, and the only observed or theorized massless gauge meson is the photon. All others have distinct masses. These distinct masses may leave distinct footprints on the geometry permitting a new extension of the already unified field theory.[13]

VI. EXTERIOR SOLUTIONS FOR A ROTATING BLACK HOLE WITH GAUGE CHARGE

As an application of the theorem, begin with the Kerr-Newman solution in Boyer-Lindquist coordinates[14]:

$$\begin{aligned} ds^2 &= -\rho^{-2}\Delta[dt - a\sin^2\theta\, d\phi]^2 \\ &\quad + \rho^{-2}\sin^2\theta[(r^2+a^2)d\phi - a\,dt]^2 \\ &\quad + \rho^2\Delta^{-1}dr^2 + \rho^2 d\theta^2, \\ A^0 &= -Q^0\cos\alpha\,\rho^{-2}r[dt - a\sin^2\theta\, d\phi] \\ &\quad - Q^0\sin\alpha\,\rho^{-2}\cos\theta[(r^2+a^2)d\phi - a\,dt], \end{aligned} \tag{25}$$

where

$$\Delta = r^2 - 2Mr + a^2 + (Q^0)^2,$$

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

and M, a, Q^0, and α are constants which may be interpreted as the mass, angular momentum per unit mass, electromagnetic charge, and complexion of the black hole. Thus, $Q^0\cos\alpha$ is the electric charge and $Q^0\sin\alpha$ is the magnetic charge. The complexion α is usually set equal to zero. This is a four-parameter solution of the EM equations.

The theorem prescribes how to construct an $(N+3)$-parameter solution of the EYM equations (four parameters from the Kerr-Newman solution and N-1 from the theorem). Thus, pick parameters β^p subject to

$$\gamma_{pq}\beta^p\beta^q = 1. \tag{15}$$

Then ds^2 and $A^p = \beta^p A^0$ are a solution of the EYM equations with parameters M, a, Q^0, α, and (N-1) of the β^p's. Replace these parameters by M, a, α, and $Q^p = \beta^p Q^0$. Then constraint (15) yields

$$\gamma_{pq}Q^pQ^q = (Q^0)^2. \tag{26}$$

So the new solution can be written as

$$\begin{aligned} ds^2 &= -\rho^{-2}\Delta[dt - a\sin^2\theta\, d\phi]^2 \\ &\quad + \rho^{-2}\sin^2\theta[(r^2+a^2)d\phi - a\,dt]^2 \\ &\quad + \rho^2\Delta^{-1}dr^2 + \rho^2 d\theta^2 \\ A^p &= -Q^p\cos\alpha\,\rho^{-2}r[dt - a\sin^2\theta d\phi] \\ &\quad - Q^p\sin\alpha\,\rho^{-2}\cos\theta[(r^2+a^2)d\phi - a\,dt], \end{aligned} \tag{27}$$

where

$$\Delta = r^2 - 2Mr + a^2 + \gamma_{pq}Q^pQ^q,$$

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

and M, a, Q^p, and α are constants which may be interpreted as the mass, angular momentum per unit mass, gauge charges, and complexion of the black hole. Thus, $Q^p\cos\alpha$ are the electric gauge charges and $Q^p\sin\alpha$ are the magnetic gauge charges. Of course an arbitrary position-dependent gauge transformation as well as an arbitrary coordinate transformation may still be made without producing a new solution. The singularity structure of the new solutions is the same as that for the Kerr-Newman solution with the same inequalities between M, a, and $(Q^0)^2 = \gamma_{pq}Q^pQ^q$ to prevent or permit naked singularities.[14]

VII. WU-YANG SOLUTION

In 1969 Wu and Yang[7] found three static spherically symmetric solutions of the Yang-Mills equations in flat space for the gauge group SO(3). In my units and conventions[10] the structure constants are $c^p{}_{qr} = \epsilon^p{}_{qr}e'/\hbar$, and the invariant group metric is $\gamma_{pq} = \delta_{pq}$, where e' is a coupling constant and the gauge group indices p, q, r take the values 1, 2, 3. The first Wu-Yang solution is

$$A^p = r^{-2}\epsilon^p{}_{\eta\zeta}x^\zeta dx^\eta\hbar/e',$$

where the space indices η, ζ are summed over 1, 2, 3. A curved-space generalization of this solution has been obtained by several authors.[7] At first glance this solution appears nonlinear[5] because $\epsilon^p{}_{\eta\zeta}$ mixes gauge indices with space indices and because quantities such as $j_p{}^\mu$ are nonzero. However, it is in fact linear, but expressed in a nonlinear gauge.

To obtain the curved-space Wu-Yang solution from solution (27), choose parameters $\alpha = \frac{1}{2}\pi$ (purely magnetic gauge charges); $a = 0$ (no rotation, spherical symmetry); $Q^1 = Q^2 = 0$, $Q^3 = -\hbar/e'$ [third component of isospin is $-\hbar/e'$, i.e., the isospin of $274(e/e')^2$ protons]. Then perform a position-dependent gauge transformation by the SO(3) matrix

$$R = \begin{bmatrix} \sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ -\cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}. \tag{28}$$

The result is

$$\begin{aligned} ds^2 &= -r^{-2}\Delta dt^2 + r^2\sin^2\theta d\phi^2 + r^2\Delta^{-1}dr^2 + r^2 d\theta^2, \\ A^1 &= (\sin\theta\cos\theta\cos\phi\, d\phi + \sin\phi\, d\theta)\hbar/e', \\ A^2 &= (\sin\theta\cos\theta\sin\phi\, d\phi - \cos\phi\, d\theta)\hbar/e', \\ A^3 &= -\sin^2\theta\, d\phi\hbar/e', \end{aligned} \tag{29}$$

where

$$\Delta = r^2 - 2Mr + (\hbar/e')^2,$$

and M is the mass of the black hole. Finally, transform to rectangular coordinates producing the curved-space Wu-Yang solution. This same transformation also works in flat space.

A word of caution: It is not easy to tell when two solutions of the EYM equations are identical or when one solution is really linear. The fields $F^p{}_{\mu\nu}$ are not gauge invariant; they are only gauge covariant. Further, the potentials $A^p{}_\mu$ and the currents $j_p{}^\mu$ and $j_{Mp}{}^\mu$ are not even gauge covariant. For example, for the Wu-Yang solution in their choice of gauge $j_p{}^\mu \neq 0$ and $j_{Mp}{}^\mu \neq 0$, but in the linear gauge $j_p{}^\mu = j_{Mp}{}^\mu = 0$. Thus, one must be careful.

VIII. BLACK-HOLE CONJECTURES

Wheeler[8] has conjectured that "black holes have no hair" other than mass, angular momentum, and electric charge. Bekenstein[15] has proved several theorems in support of this conjecture, but the proofs depend on specific Lagrangians. In his thesis Bekenstein recognizes that the proof does not work for massless Yang-Mills fields so that black holes can have gauge charges, but he does not give an exact solution such as (27). The proof also does not work for spontaneous-symmetry-breaking scalar fields which have a Lagrangian like

$$L = \hbar\sqrt{-g}\,(-\tfrac{1}{2}\phi^{;\mu}\phi_{;\mu} - \tfrac{1}{2}\kappa\phi^2 - \tfrac{1}{4}\lambda\phi^4 - \tau), \tag{30}$$

where $\kappa < 0$ and $\lambda > 0$. There exists solutions for this Lagrangian coupled to gravity in which the geometry is Kerr (except possibly for a cosmological constant depending on the value of τ) and the scalar field has the constant value $\pm(-\kappa/\lambda)^{1/2}$. There may also exist solutions in which the scalar field is nonconstant.

Massless gauge fields and spontaneous-symmetry-breaking scalar fields are unphysical when considered independently. However, at least in flat space, they may be combined via the Higgs mechanism[4] to produce massive gauge fields which are physical. The problem of generalizing the Higgs mechanism to curved space is presently under investigation. In the process it may be possible to bypass Bekenstein's theorem and produce a black-hole solution with massive gauge fields whose gauge charges will be real isospin. When and if this can be done, it will be possible to distinguish between black holes which were formed out of neutrons and those which were formed out of Λ^0 particles because neutrons and Λ^0 particles have different isospin. This says nothing about baryon number because none of the present gauge theories have baryon number as a gauge charge.

A separate question is that of uniqueness. I extend the Israel-Carter conjecture[9] to say that the black-hole solutions (27) are the only solutions of the EYM equations which are stationary, asymptotically flat, and possess nonsingular event horizons of two-sphere topology and whose gauge fields fall off as $1/r$ at infinity and are nonsingular on the horizon. At present, no counterexample is known. Support for the conjecture comes from a theorem by Loos[16] in which he proves that in flat space "all spherically symmetric gauge fields can by a gauge transformation be thrown in Coulomb form, for any gauge group."

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[1]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[2]A group metric γ_{pq} is a metric for the adjoint representation space, i.e., the Lie algebra itself. A group metric is invariant if it transforms into itself under the adjoint representation: $\gamma_{pq} = Ad(g)^r{}_p Ad(g)^s{}_q \gamma_{rs}$ for all group elements g in G.

[3]For example, see S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D 2, 1285 (1970); J. C. Pati and A. Salam, *ibid.* 8, 1240 (1973).

[4]For an instructive review of gauge theories including spontaneous symmetry breaking and the Higgs mechanism, see E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[5]A linear solution of an N-parameter non-Abelian gauge theory is a solution of the nonlinear Yang-Mills equations for which there exists a gauge in which the nonlinearities drop out, i.e., in this gauge the solution also satisfies the Yang-Mills equations for an N-parameter Abelian gauge theory. In an Abelian gauge theory the Yang-Mills equations are just N copies of Maxwell's equations, which are linear. Hence, a linear solution is also called Abelian or Maxwellian. The special gauge, in which the nonlinearities drop out, is then called a linear (or Abelian or Maxwellian) gauge. The theorem proved in Sec. III merely produces linear solutions in their linear gauge. An example of a linear solution in a nonlinear gauge is given in Sec. VII. Such linear solutions are usually considered mathematically trivial because they avoid the inherent nonlinearities of the theory. However, they become significant if they are in some sense unique as conjectured in Sec. VIII. In any case they are physically significant because they describe physically distinct situations as pointed out in Sec. IV.

[6]See G. Y. Rainich, Trans. Am. Math. Soc. 27, 106 (1925); C. W. Misner and J. A. Wheeler, Ann. Phys. (N.Y.) 2, 525 (1957).

[7]The special case of spherical symmetry for any gauge group has been found independently by E. Beall (private communication). The special case of spherical symmetry for the gauge group SO(3) has been found independently by M. Y. Wang, Phys. Rev. D (to be published); by F. A. Bais and R. J. Russell, *ibid.* 11, 2692 (1975); and by Y. M. Cho and P. G. O. Freund, *ibid.* 12, 1588 (1975). These three articles generalize to curved space the first solution of T. T. Wu and C. N. Yang, in *Properties of Matter Under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York, 1969). The solutions of Bais and Russell and of Cho and Freund include spontaneous-symmetry-breaking scalar fields and so generalize to curved space the solution of G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[8]See J. A. Wheeler, Trans. N. Y. Acad. Sci. 33, 745 (1971); R. Ruffini and J. A. Wheeler, Phys. Today 24, 30 (1971).

[9]For an extensive review see *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), and more recently D. C. Robinson, Phys. Rev. D 10, 458 (1974); Phys. Rev. Lett. 34, 905 (1975).

[10]Space-time conventions and units are those of C. W. Misner, K. S. Thorne, and A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), i.e., the metric has signature $(-+++)$, Greek indices are spacetime indices in a coordinate basis with the summation convention, and $c = G = 1$. Electromagnetic and gauge units are Gaussian so that Coulomb's constant is $K = 1$. Latin indices are gauge group indices which run from 1 to N with the summation convention, except that for the electromagnetic case the only index is 0. Coupling constants have been absorbed into the structure constants.

[11]Let $(t_p)^a{}_b$ be a representation of the generators of the group and consider this representation acting on some fiber bundle over spacetime. Then the gauge potentials $(A^p{}_\mu t_p)^a{}_b$ are the connection in the bundle; the gauge fields $(F^p{}_{\mu\nu} t_p)^a{}_b$ are the curvature of the bundle, and this "interesting identity" is the Bianchi identity.

[12]Some of these hopes were expressed in the articles by Rainich (Ref. 6); Misner and Wheeler (Ref. 6); J. G. Fletcher, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962); L. Witten (in the same book); J. A. Wheeler (1) (Ref. 8); and J. A. Wheeler (2), Ann. Phys. (N.Y.) 2, 604 (1957). The difficulties with the already unified field theory began to appear in those articles, but were not taken seriously. These difficulties include the inability to incorporate spin $\frac{1}{2}$, in Wheeler (2); the nonuniqueness of solutions, in D. R. Brill and J. M. Cohen, J. Math. Phys. 7, 238 (1966); and the impossibility of an initial-value formulation, in L. Witten, *op. cit.*, and C. W. Misner, in *Proceedings of the Philosophy of Science Association, 1972*, edited by K. F. Schaffner and R. S. Cohen (D. Reidel, Dordrecht, The Netherlands, 1974).

[13]Brill and Cohen (Ref. 12) have found several physically distinct solutions with the same geometry, but the spacelike surfaces have a compact simply connected topology. My solutions have spacelike surfaces which are not necessarily compact nor simply connected so that the invariant integrals of the geometry do not determine all the conserved charges of the physics. However, the solutions of Brill and Cohen are more significant because the degeneracy in the geometry cannot be eliminated by giving mass to some fields since the neutrino and photon are both observed massless fields, while the gauge fields should be massive.

[14]See B. Carter, Phys. Rev. 174, 1559 (1968) and his references.

[15]J. D. Bekenstein, Phys. Rev. D 5, 1239 (1972); 5, 2403 (1972); Ph.D. thesis, Princeton University, 1972 (unpublished).

[16]H. G. Loos, Nucl. Phys. 72, 677 (1965).