Energy-momentum tensor: Field-theoretic approach for coupled gravitational and electromagnetic fields

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The deduction of the energy-momentum tensor from a variational principle is clarified in the field-theoretic approach starting from the unrenormalized flat space-time. The procedure is applied to coupled gravitational and electromagnetic fields obtaining the relevant, total energy-momentum tensor to second order in the gravitational coupling constant f.

I. INTRODUCTION

In the field-theoretic approach[1,2] to gravity, the symmetric stress-energy density $T_{\alpha\beta}$ is required. An elegant method, originally due to Weyl[3] and successively developed by Trautman,[4] obtains $T_{\alpha\beta}$ by a variational principle. There are good proofs[5] of this method. However, some clarifications, essential for practical applications, are missing. The clarifications are emphasized in Sec. II while in Sec. III the total energy-momentum tensor $T^{(\text{tot})}_{\alpha\beta}$ is explicitly obtained for coupled gravitational and electromagnetic fields, since they are not found in the literature. Furthermore, the above stress tensor will be used in the following paper.[6]

II. VARIATIONAL METHOD TO OBTAIN A SYMMETRIC ENERGY-MOMENTUM TENSOR

We start from the well-known variational principle

$$0=\delta\int d^4x\sqrt{-a}(L^{(m)}+L^{(f)})\,, \tag{1}$$

where $a=\det(a_{\alpha\beta})$ is the value of the determinant of the fundamental metric tensor $a_{\alpha\beta}$, $L^{(m)}$ the Lagrangian density for the matter and the interaction of the latter with the fields (more than one, in general), and $L^{(f)}$ the Lagrangian density for the fields and their mutual interactions.

If the matter consists of pointlike particles having z^α coordinates which are particular values of the current coordinates x^α, and the fields are, for example, a vector field A_α and a tensor field $\psi_{\alpha\beta}$, it is

$$L^{(m)}=(-a)^{-1/2}\int_{-\infty}^{+\infty}ds\,\delta^4(x-z)\Lambda(\dot z^\alpha,a_{\alpha\beta},A_\alpha,\psi_{\alpha\beta})\,, \tag{2}$$

where Λ is a scalar function, ds the invariant line element (coinciding with the proper-time element since the light speed is assumed to be unity), and $\dot z^\alpha=dz^\alpha/ds$. If A_α is the electromagnetic field and $\psi_{\alpha\beta}$ the gravitational field, only $\psi_{\alpha\beta}$ couples with the electromagnetic energy-momentum tensor, so that

$$L^{(f)}=L^{(f)}(A_{\alpha,\beta},\psi_{\alpha\beta},\psi_{\alpha\beta,\gamma},a^{\alpha\beta},a^{\alpha\beta}{}_{,\gamma})\,. \tag{3}$$

The explicit dependence of (2) on $a_{\alpha\beta}$ is due to the fact that the quantities considered as "primitive" are z^α (contravariant), A_α, and $\psi_{\alpha\beta}$ (covariant). The exclusion of z_α implies the presence of $a_{\alpha\beta}$ in (2). This presence, neglected by Trautman,[4] is necessary only for the particle part of z_α (while invariant interaction terms of $\dot z^\alpha$ with A_α and $\psi_{\alpha\beta}$ can be obtained without the use of $a_{\alpha\beta}$).

As to the presence of $a^{\alpha\beta}{}_{,\gamma}$ notice that Eq. (3) must be invariant. To obtain such invariance it is convenient (as usual) to construct the Lagrangian by covariant quantities such as $A_{\alpha;\beta}$ and $\psi_{\alpha\beta;\gamma}$ (where the semicolons denote covariant differentiations). But our primitive quantities are $A_{\alpha,\beta}$ and $\psi_{\alpha\beta,\gamma}$ (where commas denote partial differentiations). The passage from $\psi_{\alpha\beta;\gamma}$ to $\psi_{\alpha\beta,\gamma}$ implies the Christoffel symbols containing $a^{\alpha\beta}{}_{,\gamma}$.

Letting z^α vary in (1) the equations of motion are obtained.[7] Taking into account that dz^α are not free being related by

$$ds^2=a_{\alpha\beta}dz^\alpha dz^\beta\,, \tag{4}$$

the Khalatnikov[8]-Infeld[9]-Kalman[10] equations are obtained,[7]

$$\frac{d}{ds}\frac{\partial L}{\partial\dot z^\alpha}-\frac{\partial L}{\partial z^\alpha}=\frac{D}{ds}\left[\left(\frac{\partial L}{\partial\dot z^\beta}\dot z^\beta-L\right)\dot z_\alpha\right], \tag{5}$$

where D/ds denotes covariant differentiation.

By varying $\psi_{\alpha\beta}$ (considered as the gravitational field) in (1) we get[11]

$$\frac{\partial L}{\partial \psi_{\alpha\beta}} - \left(\frac{\partial L}{\partial \psi_{\alpha\beta;\gamma}}\right)_{;\gamma} + \left(\frac{\partial L}{\partial \psi_{\alpha\beta;\gamma\lambda}}\right)_{;\gamma\lambda} = 0 . \tag{6}$$

Here also second-order differentiations are included since gravitation is nonlinear[2] and, to second order in the coupling constants f, $\psi_{\alpha\beta;\gamma\lambda}$ appear in L.

Also the coordinates x^α can be varied in (1). This implies variations for z^α, A_α, $\psi_{\alpha\beta}$, $a_{\alpha\beta}$, and $a^{\alpha\beta}$. The corresponding variations for L induced by $\delta z^\alpha, \delta A_\alpha, \delta\psi_{\alpha\beta}$ are zero because of (5), (6), and the field equations for the electromagnetic (em) field [obtainable by varying A_α in (1)]. From the remnant one can obtain[5] a symmetric tensor[12]

$$T_{\alpha\beta} = \frac{\delta L}{\delta a^{\alpha\beta}} = \frac{2}{\sqrt{-a}}\left[\frac{\partial(L\sqrt{-a})}{\partial a^{\alpha\beta}} - \left(\frac{\partial(L\sqrt{-a})}{\partial a^{\alpha\beta}{}_{,\gamma}}\right)_{;\gamma}\right], \tag{7}$$

such that $T_{\alpha\beta}{}^{;\beta}=0$ gives (5).

Note that when x^α is varied, $a_{\alpha\beta}$ undergoes a double variation. One is due to the change of z^α and affects $a_{\alpha\beta}$ appearing in (2) only: Its effect is taken into account in (5). Such a variation does not affect (3) since the fields are present everywhere, independently of the test-particle position. The other variation of $a_{\alpha\beta}$ is due to the direct change of x^α, and should be present even if z^α were artificially maintained as fixed. This variation affects both (2) and (3), and is accounted for in (7).

Equations (2) and (3) contain both covariant and contravariant components of the metric tensor. It can be shown that both such quantities are to be assumed, however, with some arbitrariness.[13]

III. APPLICATION TO COUPLED ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

Let us apply the above procedure to find the energy-momentum tensor (to second order in the gravitational coupling constant f) for pointlike particles in gravitational and electromagnetic fields.

The matter-plus-interaction Lagrangian density (2) for point particles in (coupled) electromagnetic and gravitational fields becomes, to first order in the coupling constant f,

$$L^{(m)} = L^{(p)} + L^{(\mathrm{int})} = \frac{1}{\sqrt{-a}}\sum_{(n)}\int_{-\infty}^{+\infty} dz^\alpha \delta^4(x-z_{(n)})\left[(-a_{\alpha\beta}+\psi_{\alpha\beta})\frac{m_{(n)}dz^\beta_{(n)}}{(a_{\lambda\rho}dz^\lambda_{(n)}dz^\rho_{(n)})^{1/2}} + e_{(n)}A_\alpha\right], \tag{8}$$

where $m_{(n)}$ is the proper mass of the nth particle, including the electromagnetic self-reaction mass. Consequently A_α is the electromagnetic (vector) potential due to all the other particles but for the nth considered.

Having in mind the variation to be made, we write (8) as a function of the "primitive" quantities z^α, $\psi_{\alpha\beta}$, and A_α. For example, this prevents us from writing $\dot z^\alpha \dot z_\alpha$, i.e., the choice of just contravariant coordinates z^α involves the introduction of the covariant components of the fundamental metric tensor $a_{\alpha\beta}$. In this spirit, in (8) ds is explicitly written as $(a_{\alpha\beta}dz^\alpha dz^\beta)^{1/2}$ both when it appears as the integration variable and in $\dot z^\alpha = dz^\alpha/ds$.

The Lagrangian density (3) for the gravitational and electromagnetic fields and their interaction is explicitly given (to first order in f) by[1]

$$L^{(f)} = \tfrac{1}{2}\psi_{\alpha\beta;\gamma}\psi^{\alpha\beta;\gamma} - \psi_{\alpha\beta;\gamma}\psi^{\alpha\gamma;\beta} + \psi_{\alpha\beta}{}^{;\beta}\psi^{;\alpha} - \tfrac{1}{2}\psi_{;\alpha}\psi^{;\alpha} + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + f\psi_{\alpha\beta}T^{(\mathrm{em})\alpha\beta}, \tag{9}$$

where $\psi \equiv \psi^\alpha_\alpha$, and, as usual, $T^{(\mathrm{em})}_{\alpha\beta} = F_{\alpha\gamma}F_\beta{}^\gamma - \tfrac{1}{4}a_{\alpha\beta}F_{\gamma\lambda}F^{\gamma\lambda}$ with $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}$. Using (5) and (8) and integrating over a volume containing the nth particle only,[14] we have

$$m_{(n)}[(1+f\psi_{\mu\nu}\dot z^\mu_{(n)}\dot z^\nu_{(n)})\dot z_{(n)\alpha} - 2f\psi_{\alpha\lambda}\dot z^\lambda]_{;\gamma}\dot z^\gamma = -m_{(n)}\psi_{\gamma\lambda;\alpha}\dot z^\gamma \dot z^\lambda + e_{(n)}F_{\alpha\beta}\dot z^\beta . \tag{10}$$

By (6), (8), and (9) the gravitational field equations are obtained,

$$\Box\psi^{\alpha\beta} - \psi_\sigma{}^{(\alpha;\beta)\sigma} + \psi^{;\alpha\beta} + a^{\alpha\beta}(\psi^{\sigma\lambda}{}_{;\sigma\lambda} - \Box\psi) = f(T^{(p)\alpha\beta} + T^{(\mathrm{em})\alpha\beta}), \tag{11}$$

where $\Box\psi^{\alpha\beta} \equiv \psi^{\alpha\beta;\lambda}{}_\lambda$ and $T^{(p)\alpha\beta} = T^{(p)\alpha\beta\gamma}{}_\gamma$ with

$$T^{(p)\alpha\beta\gamma\lambda} = \frac{1}{\sqrt{-a}}\sum_{(n)} m_{(n)}\int_{-\infty}^{+\infty} ds_{(n)}\delta^4(x-z_{(n)})\dot z^\alpha_{(n)}\dot z^\beta_{(n)}\dot z^\gamma_{(n)}\dot z^\delta_{(n)} . \tag{12}$$

To obtain the total energy-momentum tensor $T^{(\mathrm{tot})}_{\alpha\beta}$ (for particles, fields, and their interactions), use is to be made, in (8) and (9), only of the chosen "primitive" quantities z^α, A_α, $A_{\alpha,\beta}$, $\psi_{\alpha\beta}$, and $\psi_{\alpha\beta,\gamma}$. This has already been done in (8), while in (9) the Christoffel symbols must be explicitly obtained. Using Eqs. (7), (8), and (9), and taking into account that $\partial a_{\gamma\lambda}/\partial a^{\alpha\beta} = -a_{\alpha\gamma}a_{\beta\lambda}$ and $\partial a_{\gamma\lambda,\rho}/\partial a^{\alpha\beta}{}_{,\sigma} = -a^\sigma_\rho a_{\alpha\gamma}a_{\beta\lambda}$, we finally obtain

$$
\begin{aligned}
\tilde{T}^{(\mathrm{tot})}_{\alpha\beta} = {} & T^{(p)}_{\alpha\beta} + f\psi_{\rho\nu}T^{(p)\rho\nu}_{\alpha\beta} + \psi_{\rho\sigma;\alpha}\psi^{\rho\sigma}{}_{;\beta} - 2\psi_{\rho\gamma;(\alpha}\psi_{\beta)}{}^{\rho;\gamma} + 2\psi_{\rho(\alpha}\psi_{\beta)\lambda}{}^{;\rho\lambda} \\
& + 2\psi_{\alpha\rho;\lambda}\psi_{\beta}{}^{\lambda;\rho} - \psi_{\rho(\alpha}\psi_{;\beta)}{}^{\rho} - 2\psi_{\alpha\beta;\gamma\lambda}\psi^{\gamma\lambda} + 2\psi_{\alpha\gamma;\rho}\psi_{\beta}{}^{\gamma;\rho} - 2\psi_{\alpha\beta;\gamma}\psi_{\mu}{}^{\gamma;\mu} \\
& - \psi_{\alpha\beta}\psi_{\sigma\lambda}{}^{;\sigma\lambda} - \psi_{\alpha\beta;\gamma}\psi^{;\gamma} + \psi_{;(\alpha}\psi_{\beta)\mu}{}^{;\mu} - \psi_{;\alpha}\psi_{;\beta} \\
& + a_{\alpha\beta}\left(\tfrac{1}{2}\psi_{;\rho}\psi^{;\rho} - \tfrac{1}{2}\psi_{\rho\sigma;\tau}\psi^{\rho\sigma;\tau} + \psi_{\mu\nu;\lambda}\psi^{\lambda\nu;\mu} + \psi_{;\mu\rho}\psi^{\mu\rho}\right) - \tfrac{1}{2}f\psi_{\alpha\beta}T^{(p)} + F_{\alpha\gamma}F_{\beta}{}^{\gamma} \\
& - \tfrac{1}{4}a_{\alpha\beta}F_{\rho\nu}F^{\rho\nu} + 2f\psi_{\lambda(\alpha}F_{\beta)\gamma}F^{\lambda\gamma} + 2f\psi^{\nu\rho}F_{\nu\alpha}F_{\rho\beta} - \tfrac{1}{2}f\psi_{\alpha\beta}F_{\rho\nu}F^{\rho\nu} \\
& - f\psi F_{\alpha\rho}F_{\beta}{}^{\rho} - f a_{\alpha\beta}\psi_{\lambda\tau}F_{\gamma}{}^{\lambda}F^{\tau\gamma} + \tfrac{1}{4}f\psi a_{\alpha\beta}F_{\rho\nu}F^{\rho\nu}\,.
\end{aligned}
\tag{13}
$$

The symmetric tensor, given by (13), is the desired total energy-momentum tensor since, by equating its divergence to zero and using Eqs. (11) and (12), one obtains Eq. (10).

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[11] The Lagrangian, being an invariant, can be constructed by covariant quantities only, such as $A_{\alpha;\beta}$ and $\psi_{\alpha\beta;\gamma}$ while the primitive quantities are $A_{\alpha,\beta}$ and $\psi_{\alpha\beta,\gamma}$. The result is still covariant and can also be obtained by covariant variations: See B. F. Plybon, J. Math. Phys. 12, 57 (1971).

[12] When $a_{\alpha\beta}$ is varied in (3), it is worth considering that $\psi_{\alpha\beta,\gamma}$ (and not $\psi_{\alpha\beta;\gamma}$) are the primitive quantities, so that also the variations of the Christoffel symbols are to be made. Here as well the procedure by B. F. Plybon (quoted in Ref. 11) is applied.

[13] In this way some terms come either from the interaction Lagrangian or from the field Lagrangian according to the choice of primitive quantities.

[14] Some authors such as Thirring and Sexl in Ref. 1, and C. W. Misner, K. S. Thorne, and J. A. Wheeler [*Gravitation* (Freeman, San Francisco, 1973), Box 7.1A] used the Euler-Lagrange equations and obtained equations of motion where the second term on the left-hand side of (10) is missing. Such equations show their inconsistency when saturated by $\dot{z}^{\alpha}$. Moreover, in order to comply with the Newtonian limit, they are forced to multiply the first term of (8) by $\frac{1}{2}$, thus obtaining a Lagrangian which is no longer the linearization of Einstein's Lagrangian.