Estimate of the rate of the rare decay $\pi^0 \to 3\gamma$†

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An estimate is given for the decay rate of the neutral pion into three photons by using a single quark loop in a model with parity violation.

The decay of the neutral π meson into three photons is forbidden by charge-conjugation invariance. Some time ago estimates were given for the rate of this decay if the C violation occurs in the strong interaction.[1] Since there is no evidence for strong C violation the process can probably only proceed through a weak, parity-violating interaction. Therefore it must be very small; we wish to determine how small.

The decay rate for the π^0 into two photons can be calculated quite accurately by simply assuming the process goes through a quark loop with a quark mass of about 1 GeV.[2] Thus, it seems a sensible procedure to calculate the three-photon decay with diagrams involving a quark loop, suitably modified by a weak interaction. The kinds of diagrams we have in mind are shown in Fig. 1, where the theory is assumed to be renormalizable with weak vector mesons, Higgs scalars, etc.[3] Since the branching ratio of the three-photon decay to the two-photon decay will be small it does not seem necessary to actually calculate any diagrams. We will simply write a general amplitude restricted by gauge invariance and Bose symmetry and then use dimensional arguments to estimate the value of the matrix element that would be given by the diagrams of Fig. 1.

Define the matrix element for $\pi^0(p) \to \gamma(k_1) + \gamma(k_2) + \gamma(k_3)$ as

$$\epsilon_\lambda(k_3)\epsilon_\rho(k_2)\epsilon_\sigma(k_1)T^{\lambda\rho\sigma}(k_3,k_2,k_1)\,. \tag{1}$$

This must violate parity, so $T^{\lambda\rho\sigma}$ has the form

$$\begin{aligned}T^{\lambda\rho\sigma} = {}& k_1^\lambda k_1^\rho k_2^\sigma F_1(k_1\cdot k_3, k_2\cdot k_3, k_1\cdot k_2)\\ &+ k_1^\lambda k_1^\rho k_3^\sigma F_2 + k_2^\lambda k_3^\rho k_2^\sigma F_3\\ &+ k_2^\lambda k_1^\rho k_3^\sigma F_4 + k_2^\lambda g^{\rho\sigma} F_5\\ &+ \dots\,,\end{aligned} \tag{2}$$

where there are 14 possible terms if we neglect terms proportional to k_3^λ, k_2^ρ, or k_1^σ. The conditions of gauge invariance

$$k_{3\lambda}T^{\lambda\rho\sigma} = 0\,, \tag{3a}$$

$$k_{2\rho}T^{\lambda\rho\sigma} = 0\,, \tag{3b}$$

$$k_{1\sigma}T^{\lambda\rho\sigma} = 0\,, \tag{3c}$$

restrict $T^{\lambda\rho\sigma}$ to be

$$\begin{aligned}T^{\lambda\rho\sigma}(k_3,k_2,k_1) = {}& \left[k_1^\lambda(k_1^\rho k_2^\sigma - k_1\cdot k_2 g^{\rho\sigma}) - k_2^\lambda(k_1^\rho k_2^\sigma - k_1\cdot k_2 g^{\rho\sigma})\frac{k_1\cdot k_3}{k_2\cdot k_3}\right]F_1\\ &+\left[k_1^\rho(k_3^\sigma k_1^\lambda - k_1\cdot k_3 g^{\sigma\lambda}) - k_3^\rho(k_3^\sigma k_1^\lambda - k_1\cdot k_3 g^{\sigma\lambda})\frac{k_1\cdot k_2}{k_2\cdot k_3}\right]F_2\\ &+\left[k_2^\sigma(k_3^\rho k_2^\lambda - k_2\cdot k_3 g^{\rho\lambda}) - k_3^\sigma(k_3^\rho k_2^\lambda - k_2\cdot k_3 g^{\rho\lambda})\frac{k_1\cdot k_2}{k_1\cdot k_3}\right]F_3\\ &+\left[k_2^\lambda(k_1^\rho k_3^\sigma - k_1\cdot k_3 g^{\rho\sigma}) - k_1^\lambda(k_3^\rho k_2^\sigma - k_2\cdot k_3 g^{\rho\sigma})\right.\\ &\left.+ k_2^\sigma k_1\cdot k_3 g^{\lambda\rho} + k_3^\rho k_1\cdot k_2 g^{\lambda\sigma} - k_1^\rho k_2\cdot k_3 g^{\lambda\sigma} - k_3^\sigma k_1\cdot k_2 g^{\lambda\rho}\right]F_4\,.\end{aligned} \tag{4}$$

Bose statistics requires

$$T^{\lambda\rho\sigma}(k_3,k_2,k_1) = T^{\rho\lambda\sigma}(k_2,k_3,k_1) \tag{5a}$$

$$= T^{\sigma\rho\lambda}(k_1,k_2,k_3) \tag{5b}$$

$$= T^{\lambda\sigma\rho}(k_3,k_1,k_2)\,, \tag{5c}$$

and these conditions restrict the form of $F_1, \dots, F_4$. If we define $x = k_1\cdot k_3$, $y = k_2\cdot k_3$, $z = k_1\cdot k_2$ then we must have

$$F_1(x,y,z) = -\frac{y}{x}F_1(y,x,z)\,, \tag{6a}$$

$$F_2(x,y,z) = F_3(y,x,z)\,, \tag{6b}$$

$$F_1(x,y,z) = F_2(z,y,x)\,, \tag{6c}$$

$$F_3(x,y,z) = -\frac{x}{z}F_3(z,y,x) , \quad (6d)$$

$$F_1(x,y,z) = -\frac{y}{x}F_3(x,z,y) , \quad (6e)$$

$$F_2(x,y,z) = -\frac{y}{z}F_2(x,z,y) , \quad (6f)$$

and

$$F_4(x,y,z) = -F_4(y,x,z) \quad (6g)$$

$$= -F_4(x,z,y) \quad (6h)$$

$$= -F_4(z,y,x) . \quad (6i)$$

The F_i must be free of kinematic singularities. The most general such solution of these equations is

$$F_1(x,y,z) = ya(x,y)f(z) , \quad (7a)$$

$$F_2(x,y,z) = -ya(y,z)f(x) , \quad (7b)$$

$$F_3(x,y,z) = -xa(x,z)f(y) , \quad (7c)$$

where $f(x)$ is any function and $a(x,y)$ is any odd function,[4]

$$a(x,y) = -a(y,x) . \quad (8)$$

Similarly F_4 must have the form

$$F_4(x,y,z) = b(x,y)b(y,z)b(x,z) , \quad (9)$$

where $b(x,y)$ is another function that is odd under $x \leftrightarrow y$.

Thus far our discussion is completely general. The crucial point is that gauge invariance and Bose symmetry force the matrix element to contain a large number of powers of k. Each of these powers of k will end up in the decay rate going as some fraction of the pion mass divided by some mass (e.g., the quark mass) that is equal to, or more likely larger than, the pion mass. Therefore, the decay rate will be dominated by the terms in the matrix element with the fewest powers of k and it should be a very good approximation to take the simplest possible form for $a(x,y)$ and $b(x,y)$. Thus we will now specialize to

$$a(x,y) = (x-y) , \quad (10a)$$

$$f(x) = A , \quad (10b)$$

$$b(x,y) = (x-y)B^{1/3} , \quad (10c)$$

where A and B are constants with dimensions of inverse mass to the seventh and the ninth power, respectively.

The square of the matrix element is

$$\sum_{\text{pol}} |\epsilon_\lambda(k_3)\epsilon_\rho(k_2)\epsilon_\sigma(k_1)T^{\lambda\rho\sigma}(k_3,k_2,k_1)|^2 = 12A^2(k_1\cdot k_2)(k_1\cdot k_3)(k_2\cdot k_3) \times\{(k_1\cdot k_3)^2(k_1\cdot k_2)^2+(k_2\cdot k_3)^2(k_1\cdot k_2)^2+(k_1\cdot k_3)^2(k_2\cdot k_3)^2 - (k_1\cdot k_3)(k_2\cdot k_3)(k_1\cdot k_2)^2-(k_1\cdot k_3)^2(k_2\cdot k_3)(k_1\cdot k_2)-(k_1\cdot k_3)(k_2\cdot k_3)^2(k_1\cdot k_2)\} + B^2\{\cdots\} . \quad (11)$$

The AB cross term is zero, and we drop the B^2 term since it is higher order in k. [If we did not drop the B term we would also have to keep higher-order terms in $a(x,y)$ and/or $f(x)$.]

The phase-space integral

$$\int \frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\frac{d^3k_3}{2\omega_3}\sum_{\text{pol}} |\epsilon_\lambda(k_3)\epsilon_\rho(k_2)\epsilon_\sigma(k_1)T^{\lambda\rho\sigma}|^2 \times\delta^4(p-k_1-k_2-k_3) \quad (12)$$

is tedious but trivial and turns out to be of order $10^{-5}\,m_\pi^{16}$, where m_π is the pion mass. The total decay rate is

$$\Gamma = \frac{1}{(2\pi)^5}\frac{1}{2m_\pi}\frac{\pi^2}{3!}\frac{3}{32}A^2 m_\pi^{16}\times 2.4\times10^{-5} . \quad (13)$$

The only thing that remains is to estimate A. If we consider the diagrams of Fig. 1 then from naive dimensional arguments A must go as

$$A \sim \frac{e^3g^2}{(2\pi)^4 m_Z^2}\frac{f}{m^5} , \quad (14)$$

where g is the coupling of the weak vector particle to the quarks and is of order e in gauge theories and m_Z is the mass of the weak vector particle. We will set g^2/m_Z^2 equal to the weak Fermi coupling constant. f is the strong pion-quark coupling constant and m is the quark mass. We have also put in $(2\pi)^4$, since such a factor is generally left over from the loop integral.

Diagrams with more than one vector-meson propagator could give contributions to A with more factors of the weak meson mass in the denominator. However, these are negligible since $m/m_Z \ll 1$. To complete this argument we must show that the diagrams given in Fig. 2 do not contribute since they could presumably bring factors of lepton masses into A. In fact these two diagrams are each zero, as we show in the appendix.

In the one-loop approximation the two-photon decay rate is given by[2]

$$\Gamma_{\pi^0\to2\gamma} = \frac{\alpha^2}{64\pi^2}f^2 m_\pi\left(\frac{m_\pi}{m}\right)^2 . \quad (15)$$

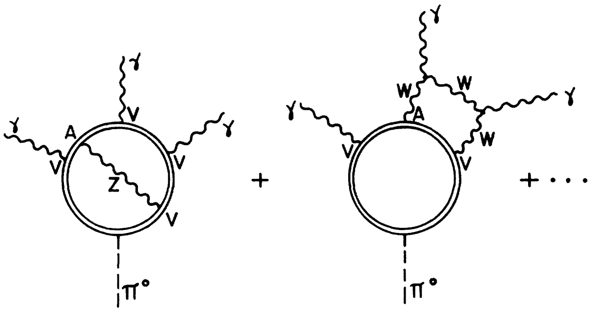

FIG. 1. Diagrams which contribute to the decay $\pi^0 \to 3\gamma$. The double line is a quark loop. The Z and W are neutral and charged weak vector mesons. The symbols V or A around the loop indicate the coupling is vector or axial-vector. Since parity must be violated the neutral current must couple with A at one end and V at the other end.

If we use A as given by (14) in the decay rate (13) and, to avoid specifying the pion-quark coupling, determine the branching ratio with the two-photon process, we find

$$R \equiv \frac{\Gamma_{\pi^0 \to 2\gamma}}{\Gamma_{\pi^0 \to 2\gamma}} = \frac{\alpha}{(2\pi)^5} G^2 m_\pi{}^4 \left(\frac{m_\pi}{m}\right)^8 \times 1.2 \times 10^{-5}. \tag{16}$$

Setting the quark mass equal to the nucleon mass and guessing that our estimates of A may be in error by as much as $(2\pi)^4$ ($\sim 10^3$) gives

$$R \sim 10^{-31 \pm 6}, \tag{17}$$

or a lifetime for the three-photon decay of $\sim 10^{9 \pm 6}$ years.

It is a pleasure to thank Charles Chiu, Bill Hurley, and Ed Kolb for valuable discussions.

APPENDIX

It is essential in the estimate of the three-photon decay rate that there be no contribution inversely proportional to the lepton mass. The only diagrams which could give such a contribution are shown in Fig. 2.

Figure 2(a) is given by

$$\langle \pi^0(p) | A_Z^\mu(0) | 0 \rangle \frac{g_{\mu\nu} - p_\mu p_\nu / m_Z{}^2}{m_Z{}^2 - m_\pi{}^2} \epsilon_\lambda(k_3) \epsilon_\rho(k_2) \epsilon_\sigma(k_1)$$
$$\times \int d^4x\, d^4y\, d^4z\, d^4w\, e^{ik_3 \cdot x} e^{ik_2 \cdot y} e^{ik_1 \cdot z} e^{-ip \cdot w}$$
$$\times \langle 0 | [V_Z^\nu(w) j_{\mathrm{em}}^\lambda(x) j_{\mathrm{em}}^\rho(y) j_{\mathrm{em}}^\sigma(z)]_+ | 0 \rangle ,$$

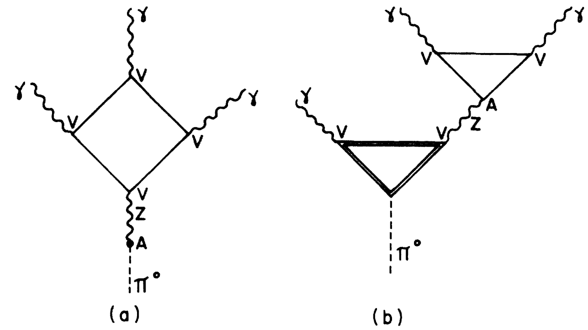

FIG. 2. Diagrams which might give contributions inversely proportional to the lepton mass. The single line is a lepton; the double line is a quark.

where

$$V_Z^\nu(w) = g \bar{\psi}_e(w) \gamma^\nu \psi_e(w) ,$$
$$j_{\mathrm{em}}^\lambda(x) = e \bar{\psi}_e(x) \gamma^\lambda \psi_e(x)$$

are the leptonic neutral and electromagnetic currents and A_Z^μ is the neutral axial-vector current. But

$$\langle \pi^0(p) | A_Z^\mu(0) | 0 \rangle \sim p^\mu :$$

and p^ν times the time ordered product is zero since the neutral leptonic current is conserved and since there are no anomalies in the four-point function.[5]

Parity requires the quark loop in Fig. 2(b) to be proportional to

$$\epsilon_\lambda(k) \epsilon^{\lambda\alpha\beta\mu} p_\alpha q_\beta ,$$

where $q = p - k$ and μ is the index at the neutral-current vertex. Parity, gauge invariance, and Bose symmetry require the lepton loop to go as[6]

$$q^\nu \epsilon_\rho(k') \epsilon_\sigma(k'') \epsilon^{\rho\sigma\tau\eta} k'_\tau k''_\eta ,$$

where ν is the index at the neutral-current vertex. But when we connect the μ and ν indices of these with the propagator for the neutral vector meson the result is zero.

Our argument that Fig. 2(b) is zero depends on the gauge invariance of that diagram alone. This seems justified since this is the only diagram that has a lepton loop [once we have shown Fig. 2(a) is zero] but would not be justified if the lepton loop were replaced by a quark loop. Also notice that a scalar neutral current cannot contribute to either diagram because of Furry's theorem.

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[1]F. A. Berends, Phys. Lett. 16, 178 (1965); A. V. Tarasov, Yad. Fiz. 5, 626 (1967) [Sov. J. Nucl. Phys. 5, 445 (1967)]; L. Galfi and G. Marx, Acta Phys. 22, 99 (1967).

[2]K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969), pp. 171–183, and references therein.

[3]This procedure is reasonable only if the weak inter-

action is renormalizable. We have in mind a theory like the unified gauge theory of Weinberg and Salam [S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Physics: Relativistic Groups and Analyticity* (Nobel Symposium No. 8), edited by N. Svartholm (Almqvist and Wiksells, Stockholm, 1968), p. 367].

[4]This asymmetry in $x \leftrightarrow y$ is a manifestation of the fact that the amplitude must violate parity and therefore the photons must have odd orbital angular momentum.

[5]R. Jackiw and I. Gerstein, Phys. Rev. 181, 1955 (1969); W. A. Bardeen, *ibid.* 184, 1848 (1969).

[6]L. Rosenberg, Phys. Rev. 129, 2786 (1963).