Relativistic quasipotential calculation of quark-model meson masses*

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Relativistic calculations of the $L = 0$ energy levels of a $q\bar{q}$ pair bound by a linear potential are done both for light quarks (the ρ system) and heavy quarks (charmonium).

The suggestion[1,2] that the $\psi(3105)$ and $\psi(3695)$ are bound states of the charmed quark $\mathcal{P}'$ and its antiquark $\bar{\mathcal{P}}'$ has revived interest in dynamical calculations of the masses of the $q\bar{q}$ bound states. Detailed calculations have been done using the nonrelativistic Schrödinger equation with a linear confinement potential for the heavy ($\mathcal{P}'$) quarks[3,4] and for the light quarks.[5,6] For the heavy-quark calculations *a posteriori* justification of the nonrelativistic calculation was given e.g. by computing the expectation value of p^2/m^2 with the Schrödinger wave functions and finding that it is small, i.e., the system is essentially nonrelativistic. Our relativistic calculation confirms this. The light-quark system is *a priori* expected to be relativistic, so the results of the relativistic calculations are of greater interest in this case.

Our approach is by way of the three-dimensional formulations of the relativistic two-body problem ("quasipotential" approach). In this approach,[7] one starts with the (formally) exact field-theoretic Bethe-Salpeter equation for the two-body system, and by an infinite rearrangement of the iteration solution and a certain projection operation one arrives at a three-dimensional Lippmann-Schwinger type equation with relativistic kinematics. For bound states, an equation like the momentum-space Schrödinger equation, again with relativistic kinematics, is obtained. In the Kadyshevskii version[8] of the quasipotential approach this is

$$(W - 2E_p)\psi_W(\vec{p}) = \int d\omega_k\, \tilde{V}(\vec{p}, \vec{k}; W)\psi_W(\vec{k})\,. \tag{1}$$

In this equation W is the relativistic c.m. energy ($s = W^2$), $\vec{p}$ is the momentum of one of the particles in the c.m. system, and $E_p = (\vec{p}^2 + m^2)^{1/2}$. The relativistic invariant momentum integration is

$$d\omega_k = \frac{m}{(2\pi)^3}\frac{d^3k}{(\vec{k}^2 + m^2)^{1/2}}\,.$$

The quasipotential $\tilde{V}(\vec{p}, \vec{k}; W)$ is related to the exact Bethe-Salpeter kernel in a rather complicated way[7] which will not concern us. If (1) were the nonrelativistic momentum space Schrödinger equation, one could immediately recover the standard coordinate space Schrödinger equation by means of Fourier transformation with the functions $e^{i\vec{p}\cdot\vec{r}}$. To obtain a relativistic configuration space version of (1), it is necessary to find a new set of functions which are relativistic generalizations of the nonrelativistic plane wave functions. These are known from the harmonic analysis[9] of the Lorentz group. They are

$$\xi(\vec{p}, \vec{\rho}) = \left(\frac{p_0 - \vec{p}\cdot\vec{n}}{m}\right)^{-1-im\rho}, \qquad \vec{\rho} = \vec{n}\rho, \quad p_0 = (\vec{p}^2 + m^2)^{1/2}\,. \tag{2}$$

One can directly check that in the nonrelativistic limit, $m \to \infty$, these functions go to nonrelativistic plane-wave functions:

$$\xi(\vec{p}, \vec{\rho}) \underset{m\to\infty}{\longrightarrow} e^{i\vec{p}\cdot\vec{\rho}}\,. \tag{2'}$$

Thus the Lorentz-invariant variable ρ has, in the nonrelativistic limit, the interpretation as the relative coordinate distance. These functions satisfy relativistic orthogonality and completeness relations[8]

$$\int d^3\rho\;\xi^*(\vec{p}', \vec{\rho})\xi(\vec{p}, \vec{\rho}) = \frac{p_0}{m}(2\pi)^3\delta(\vec{p}' - \vec{p})\,, \tag{3a}$$

$$\int d\omega_p\,\xi(\vec{p}, \vec{\rho})\xi^*(\vec{p}, \vec{\rho}') = \delta(\vec{\rho} - \vec{\rho}')\,. \tag{3b}$$

Thus one can define the relativistic generalization of the three-dimensional Fourier transform as

$$\psi_W(\vec{\rho}) = \int d\omega_p\,\xi(\vec{p}, \vec{\rho})\psi_W(\vec{p})\,, \tag{4a}$$

$$\psi_W(\vec{p}) = \int d^3\rho\;\xi^*(\vec{p}, \vec{\rho})\psi_W(\vec{\rho})\,, \tag{4b}$$

$$\tilde{V}(\vec{p}, \vec{k}; W) = \int d^3\rho' d^3\rho\;\xi^*(\vec{p}, \vec{\rho}')\tilde{V}(\vec{\rho}', \vec{\rho}; W)\xi(\vec{k}, \vec{\rho})\,. \tag{4c}$$

Taking these transforms in Eq. (1), and assuming that the quasipotential is a local potential (function of $\rho - \rho'$), one obtains[8] the relativistic configuration-space equation

$$\left[W - H_0\left(\vec{\rho}, \frac{\partial}{\partial \vec{\rho}}\right) - V(\vec{\rho}; W)\right]\psi_W(\vec{\rho}) = 0\,, \tag{5}$$

where H_0 is constructed such that

$$H_0\left(\vec{\rho}, \frac{\partial}{\partial \vec{\rho}}\right)\xi(\vec{p}, \vec{\rho}) = 2E_p\,\xi(\vec{p}, \vec{\rho})\,. \tag{6}$$

The required H_0 is

$$H_0\left(\vec{\rho}, \frac{\partial}{\partial \vec{\rho}}\right) = 2m\cosh\left(\frac{i}{m}\frac{\partial}{\partial \rho}\right) + \frac{2i}{\rho}\sinh\left(\frac{i}{m}\frac{\partial}{\partial \rho}\right) - \frac{\Delta(\theta, \phi)}{m\rho^2}\exp\left(\frac{i}{m}\frac{\partial}{\partial \rho}\right), \tag{7}$$

where $\Delta(\theta, \phi)$ is the usual angular differential operator

$$\Delta(\theta, \phi)Y_{LM}(\theta, \phi) = -L(L+1)Y_{LM}(\theta, \phi)\,. \tag{8}$$

In principle, the quasipotential $V(\vec{\rho}, W)$ is determined by the field theory (exact Bethe-Salpeter kernel) from which one started, but in practice the only approach available is the perturbation series, and all of the arguments for the quark-confining potential are definitely nonperturbative; thus we simply make the same ansatz for the quasipotential as has previously been made for the potential used in the Schrödinger-equation calculations,[3-5]

$$V(\vec{\rho}; W) = k\rho\,. \tag{9}$$

Of course the idea of quark confinement does not restrict the potential at small separations, say $r \lesssim 1/m$, and one knows from the usual meson spectroscopy that there must be short-range spin-dependent ($\vec{L}\cdot\vec{S}$, $\vec{s}_1\cdot\vec{s}_2$) terms. Since the purpose of this paper is not to do detailed phenomenology[5,6] but rather to investigate the role of relativistic corrections to any nonrelativistic quark-model calculation with a confining potential, we will not consider these additional terms here.

We turn now to the solution of Eq. (5) with the potential (9):

$$\psi_W(\vec{\rho}) = \frac{1}{\rho}\phi_{WL}(\rho)Y_{LM}(\theta, \phi)\,. \tag{10}$$

The radial equation, for $L = 0$, can be put in the form

$$\frac{k}{m}\rho e^{-(i/m)\partial/\partial\rho}\phi(\rho) + \left[1 - \left(\frac{W}{m} + \frac{ik}{m^2}\right)e^{-(i/m)\partial/\partial\rho} + e^{-(2i/m)\partial/\partial\rho}\right]\phi(\rho) = 0\,. \tag{11}$$

Using the Laplace method we obtain an integral representation for the solutions of (11). Different choices of contour produce different integral representations (of the same function):

$$\phi(\rho) = \frac{c}{m}\int_{-\infty}^{\infty} dt\,\exp\left[i\left(\rho - \frac{W}{k}\right)t + i\frac{2m^2}{k}\sinh\frac{t}{m}\right], \tag{12a}$$

$$\phi(\rho) = c\,\exp\left[-\frac{\pi}{2}\left(m\rho - \frac{mW}{k}\right)\right] \times \int_{-\infty}^{\infty} dy\,\exp\left[i\left(m\rho - \frac{mW}{k}\right)y - \frac{2m^2}{k}\cosh y\right]. \tag{12b}$$

The first representation is displayed because taking the nonrelativistic limit ($m\to\infty$) reproduces a standard integral representation of the Airy function, the known solution of the nonrelativistic $L = 0$ radial Schrödinger equation with linear potential. The second representation is used for numerical computation because of the rapid convergence of the integral. Imposition of the boundary condition $\phi(0) = 0$ determines the energies W from the zeros of the function:

$$0 = \text{const}\times\int_0^{\infty} dy\,\cos\left(\frac{mW}{k}y\right) \times\exp\left[-\frac{2m^2}{k}(\cosh y - 1)\right]. \tag{13}$$

We first do the heavy-quark ($\mathcal{P}'\bar{\mathcal{P}}'$) calculations. The two parameters m and k are determined so that the first two energies will be the 3.1 and 3.7 GeV masses of ψ_1 and ψ_2, respectively. The results (obtained by numerical integration) are

$$m = 1.16\text{ GeV},\quad k = 0.205\text{ GeV}^2 \tag{14a}$$

$$W = 3.1^*, 3.7^*, 4.2, 4.7, \ldots\text{ GeV}\,. \tag{14b}$$

These results are almost exactly the same as those obtained from a nonrelativistic-Schrödinger-equation calculation[3] with a linear potential

$$m = 1.16\text{ GeV},\quad k = 0.211\text{ GeV}^2, \tag{14a$'$}$$

$$W = 3.10^*, 3.70^*, 4.18, 4.61, \ldots\text{ GeV}\,. \tag{14b$'$}$$

In each case, the 3.10 and 3.70 GeV masses are input, so that agreement has no significance, but one observes that the effective quark mass is unchanged and the effective force constant is changed only by 3% to reproduce these same energy levels, i.e., the system is essentially nonrelativistic. And the predicted third level at 4.2 GeV is the same in either calculation. At higher energies the relativistic and nonrelativistic energy levels do begin to diverge, but for higher levels, coupling to various decay channels is likely to become important, so the potential approach will break down anyway.

For an example of a light-quark calculation we will consider the spectrum of ρ, ρ' mesons. The two parameters m and k are determined so that the first two energies are the masses of $\rho(0.77)$ and $\rho'(1.25)$. The results are

$$m = 0.0918\ \text{GeV}, \quad k = 0.0336\ \text{GeV}^2, \tag{15a}$$

$$W = 0.77^*, 1.25^*, 1.66, 2.03, 2.40 \ldots \text{GeV}. \tag{15b}$$

The corresponding results from the nonrelativistic Schrödinger equation are

$$m = 0.0644\ \text{GeV}, \quad k = 0.0364, \tag{15a$'$}$$

$$W = 0.77^*, 1.25^*, 1.64, 1.99, 2.30, \ldots \text{GeV}. \tag{15b$'$}$$

Because the first two energy levels are treated as fixed input, while m and k are treated as variable phenomenological parameters, the energy spectra are again very much the same. The relativistic nature of the system shows up in the substantially different values of m and somewhat different values of k required to fit the same energy levels in a relativistic calculation and a nonrelativistic calculation.[10]

Additional questions of interest to be investigated in the relativistic quasipotential approach are the energies of the $L \neq 0$ states, and the behavior of the wave function, particularly at the origin which determines e.g. the e^-e^+ decay widths in the nonrelativistic approximation. Neither of these questions can be answered simply in the context of the calculations presented here. For the $L \neq 0$ energy levels a different technique for numerical solution is required. More important than this purely technical problem is the recognition that the position of the $L \neq 0$ levels relative to the $L = 0$ levels is dependent on the short-distance details of the potential,[5] so more physical input is required. With regard to the wave function, recall that $\rho = 0$ is not the same as $r = 0$ (ρ and r are the same only for $\rho \gg 1/m$; in fact $\rho = 0$ corresponds[8] to $r = 1/m$), so a question of interpretation is involved. These questions are under consideration.

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[10]No significance should be attached to the values of m, k determined in these sample calculations since these values will be changed when one includes additional constant and short-range terms in the potential.