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Unitarity effects in pion-production isobar amplitudes

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In isobar analyses, the amplitude for particle + particle → particle + isobar usually is taken to be independent of the mass of the isobar, but Aaron and Amado have shown this approximation to be inconsistent with unitarity. Here we continue that work by calculating numerically the absorptive part of isobar amplitudes in the case of $\pi N \to \pi\pi N$ using the recent world analysis of 200000 resonance region events.

I. INTRODUCTION

In two-body elastic scattering the data can be fitted with any parametrization desired, but one should always check that the amplitudes thus obtained fall inside the unitarity circle. This step can be omitted if one uses an elastic phase-shift parametrization, since this builds in the unitarity constraint. For three-body analyses there does not exist a *simple* formula (such as the elastic phase-shift one) which builds in unitarity. Hence it is important to check that results obtained with other methods actually satisfy the unitarity constraints. This work is such a check of a recent extensive and important analysis of the reaction $\pi N \to \pi\pi N$ by Herndon *et al.*[1]

Final states with three or more hadrons are an important source of information on the interaction of unstable particles in particle physics. Data analysis of such states, however, has proved a formidable task. An important method in such analysis is the isobar model.[2,3] In this model the amplitude for the reaction $a+b \to 1+2+3$ is written in the form

$$\langle \vec{p}_1, \vec{p}_2, \vec{p}_3 | T_{23} | \vec{p} \rangle = \sum_{\substack{ijk=1 \\ (\text{cyclic})}}^{3} \langle \vec{p}_k | f | \vec{p} \rangle G(\vec{p}_i, \vec{p}_j). \qquad (1)$$

For the case of $\pi N \to \pi\pi N$, in the over-all center-of-mass (c.m.) system, a typical term from the sum in (1) is represented graphically in Fig. 1. In the figure the isobar can be formed by either an $N\pi$ or a $\pi\pi$ system. The quantity $G(\vec{p}_i, \vec{p}_j)$ describes the propagation and decay of the (i, j) subsystem and is written explicitly in Appendix A. An important feature in almost all applications of the isobar model is that the quasi-two-body amplitude $\langle \vec{p}_\alpha | f | \vec{p} \rangle$, which describes the isobar production from the $N\pi$ system, is taken to be independent of the two-body subenergy variable σ_α for fixed over-all center-of-mass energy W. The subenergy variable which is just the square of the isobar mass is defined by $\sigma_\alpha = (P - p_\alpha)^2$ where $P = (\vec{0}, W)$.

In a recent paper,[4] two of us have investigated the subenergy dependence implied by three-body unitarity and showed that, in general, the σ_α dependence of $\langle \vec{p}_\alpha | f | \vec{p} \rangle$ may well not be negligible, particularly when resonance bands are wide and overlap strongly as is the case in intermediate energy π production in πN collisions. The purpose of the present paper is to study the σ_α-dependent part of f in that case; namely, $\pi + N \to \pi + \pi + N$. Single-pion production data at intermediate energies are in themselves an important source of information concerning meson-baryon resonances. One can obtain from their analysis partial widths of known resonances and, perhaps, discover new resonances which might be difficult to identify in an elastic phase-shift analysis. Recent theoretical advances have generated considerable further interest in this process. In particular, a proposed connection between current and constituent quarks[5] can be tested through the

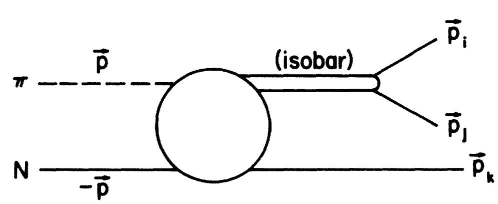

FIG. 1. Graphical representation of isobar production and decay corresponding to Eq. (1).

magnitudes and signs of amplitudes for pionic transitions between hadrons.[6] Equivalently, modified versions of $SU(6)_w$ classify baryon resonances and at the same time predict amplitudes for reactions of the type $\pi N \to \pi\Delta$, $\pi N \to \rho N$, $\pi N \to \epsilon N$, etc.[7]

In order to examine the σ_α dependence of the isobar amplitudes we make use of the recent, extensive analysis of $\pi N \to \pi\pi N$ by Herndon *et al.*[1] In this analysis the three-body final-state distributions in the energy range 1300 MeV $\lesssim W \lesssim$ 2000 MeV are fit using an isobar model which includes the particle-isobar states $\pi\Delta$, ρN, and ϵN. For a given total isospin I, the amplitude $\langle \vec{p}_\alpha | f | \vec{p} \rangle$ is decomposed into partial-wave amplitudes

$$f^J_\alpha(l'l),$$

where J is the total angular momentum, l is the orbital momentum of the initial πN system, and l' is the orbital angular momentum of the final particle-isobar system. In the LBL/SLAC analysis it is assumed that the f_α's are independent of one another and that each f_α is a slowly varying function of σ_α and, therefore, can be approximated by a complex constant at fixed total energy W for fixed J, l, and l'. At each W the best values of the f^J_α's are determined by a maximum-likelihood fit to the final-state distributions. On the other hand, Aaron and Amado[4] have shown that if we write

$$f^J_\alpha = \text{Disp} f^J_\alpha + i\, \text{Abs} f^J_\alpha, \qquad (2)$$

where $\text{Abs} f_\alpha$ is the discontinuity of f_α in σ_α, i.e.,

$$\text{Abs} f^J_\alpha(\sigma_\alpha) = \frac{1}{2i}\left[f^J_\alpha(\sigma_\alpha + i\eta) - f^J_\alpha(\sigma_\alpha - i\eta)\right], \qquad (3)$$

then unitarity implies that $\text{Abs} f^J_\alpha$ is a large and a rapidly varying function of σ_α and depends on all the f^J_β's. When resonance bands overlap as they do in the LBL/SLAC case, we expect $\text{Abs} f^J_\alpha$ to be large and rapidly varying. Furthermore, $\text{Abs} f^J_\alpha$ satisfies schematic integral relations of the type (the full equations are given in Appendix A)

$$\text{Abs} f^J_\alpha = \sum_{\beta \neq \alpha} \int_{\substack{\text{finite}\\ \text{limits}}} f^J_\beta G_\beta \qquad (4)$$

and adding analyticity to unitarity one has

$$\text{Disp} f^J_\alpha(\sigma_\alpha) = \frac{P}{\pi} \int_{\text{threshold}}^{\infty} d\sigma'_\alpha \frac{\text{Abs} f^J_\alpha(\sigma'_\alpha)}{\sigma'_\alpha - \sigma_\alpha} + \text{“left-hand” cut contributions.} \qquad (5)$$

Equations (4) and (5) imply a rapid variation of $\text{Disp} f^J_\alpha$ if $\text{Abs} f^J_\alpha$ is large.[8] It should be noted that these variations occur in the physical region of the LBL/SLAC analysis.

II. *P*11 PARTIAL WAVE

In this paper we concentrate on the $P11$ ($l=0, I=\frac{1}{2}, J=\frac{1}{2}$)$\pi N$ partial wave because pion production is large in this wave, resonant-band overlap effects are significant, and the angular momentum algebra is relatively simple. In a subsequent paper we plan to return to a discussion of all of the partial waves employed in the LBL/SLAC analysis. We examine the production amplitudes in this channel in the energy range 1400 MeV $\lesssim W \lesssim$ 1550 MeV. The important isobar amplitudes in this case are the $PS11$(πN in a $P11$ state $\to \epsilon N$ in an S state) and $PP11(\pi N \to \pi\Delta)$ which we shall call f_ϵ and f_Δ, respectively. According to LBL/SLAC, ρ production is negligible in the above energy range and we therefore neglect its effect. Our f^J_α's are *exactly* those dimensionless quantities which LBL/SLAC plot on their Argand diagrams.

To investigate the effects of unitarity we use the LBL/SLAC results for the f_β's (subenergy independent) to find $\text{Abs} f_\alpha$ (subenergy dependent) in Eq. (4). We then compare the calculated $\text{Abs} f_\alpha$ with the LBL/SLAC result for f_α. The LBL/SLAC assumption of constant f_β on the right-hand side in Eq. (4) is consistent if it generates a small left-hand side. Our procedure is somewhat like using ordinary two-body elastic unitarity to check the validity of a purely real amplitude. If that purely real amplitude used on the right-hand side of unitarity generates a small imaginary part, the real amplitude is approximately unitary. If the generated imaginary part is large, however, the full structure of some dynamical principle must be invoked to obtain a unitary amplitude. Our results at $W = 1490$ and 1540 MeV are shown in Figs. 2 and 3 where the real and imaginary parts of $\text{Abs} f_\epsilon$ and $\text{Abs} f_\Delta$, respectively, are plotted as functions of subenergy. They have been calculated using the LBL/SLAC results for f_ϵ and f_Δ in Eq. (4). The latter amplitudes (constant as a function of subenergy) are included in the figures for comparison. The function G used in Eq. (4) carries the two-body information-standard $\pi\pi$, and πN phase shifts have been used here, but the results are qualitatively insensitive to variations in these phase shifts consistent with experimental data. The detailed formulas for carrying out these calculations are given in Appendixes A and B. It should be noted that while $\text{Abs} f_\Delta$ is small and we may conclude that f_Δ = constant is consistent with unitarity and analyticity, $\text{Abs} f_\epsilon$ is large and rapidly varying for $W_{\pi\pi}$ corresponding to the most populated parts of the Dalitz plot, and thus the choice f_ϵ = constant may well violate these principles.

III. *D*13 PARTIAL WAVE

Although we have not carried out a unitarity analysis based on the LBL/SLAC results for the *D*13 partial wave, using dynamical equations, Aaron and Amado have obtained results for the full *D*13 amplitudes which exhibit considerable subenergy dependence. This dynamical scheme involved solution of coupled-channel integral equations of the Blankenbecler-Sugar[9] type which incorporate both two- and three-body unitarity and analyticity.[10] For the *D*13 case the coupled channels considered were $\pi\Delta$ and ρN (which here can be produced in *S* waves). ϵN (which is produced in *P* waves) was found to be relatively unimportant in the energy range considered. The results obtained for the elastic scattering amplitude were in good agreement with experiment for energies 1400 MeV $\leq W \leq$ 2000 MeV.[11] Within their model, the isobar amplitudes for $\pi N \to \pi\Delta$ and $\pi N \to \rho N$ were also predicted. The results (unpublished) were that $f_\rho(DS13)$ was large and approximately constant while $f_\Delta(DS13)$ was large and rapidly varying. Finally, in contradiction with the LBL/SLAC analysis, Aaron and Amado found $f_\Delta(DD13)$ to be very small.

IV. DISCUSSION

Is the LBL/SLAC analysis valid? The *P*11 partial wave discussed in Sec. II and the *D*13 partial wave discussed briefly in Sec. III together account for about three quarters of the inelasticity in the energy region near 1500 MeV and in both cases there is reason to suspect that the LBL/SLAC results violate unitarity. Possible anomalies in the LBL/SLAC fit could arise from the nonunitary nature of the parametrization. For example, LBL/SLAC obtain the result that $f_\Delta(DS13)$ and $f_\Delta(DD13)$ are roughly of equal magnitude near the $\pi\Delta$ production threshold, i.e., 1450 MeV $\leq W \leq$ 1550 MeV, while the dynamical calculation of Aaron and Amado predicts $f_\Delta(DD13) \approx 0$ in this energy range. The latter result is the more reasonable in view of the ranges of the forces involved, the nearness to the $\pi\Delta$ threshold (~ 1370 MeV), and the fact that all the obvious Feynman diagrams enhance rather than suppress *S*-wave production of $\pi\Delta$. However, to properly answer the original question of the validity of the LBL/SLAC analysis requires that the analysis be repeated with careful attention to enforcing unitarity and analyticity. Our collaboration hopes to perform such an analysis. With such large and rapidly varying Abs*f* it will be necessary to use some other principle such as analyticity to incorporate the variation of Disp*f* as well in this analysis. In the meantime we would advise some caution in using the LBL/SLAC results.[12]

In conclusion, our calculations indicate that the subenergy dependence of the isobar amplitudes could be a major effect in the analysis of $\pi N \to \pi\pi N$, and that future isobar analyses should allow for subenergy dependence when resonance bands overlap so strongly as they do in the $\pi\pi N$ case. We are presently preparing for publication a more exten-

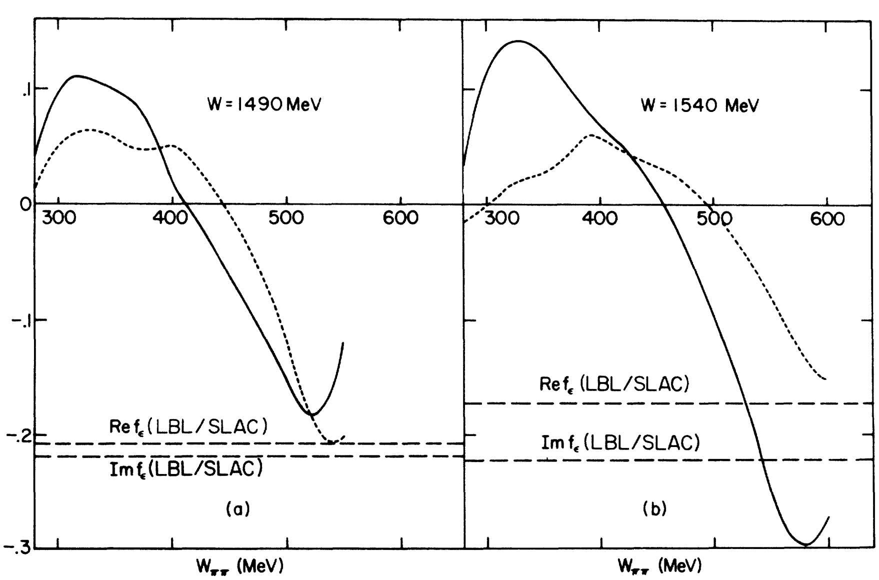

FIG. 2. Real (shown by dotted line) and imaginary (shown by solid line) parts of *i*Abs f_ϵ vs $\pi\pi$ c.m. energy at (a) *W*=1490 MeV and (b) *W*=1540 MeV. Corresponding LBL/SLAC amplitudes are also shown.

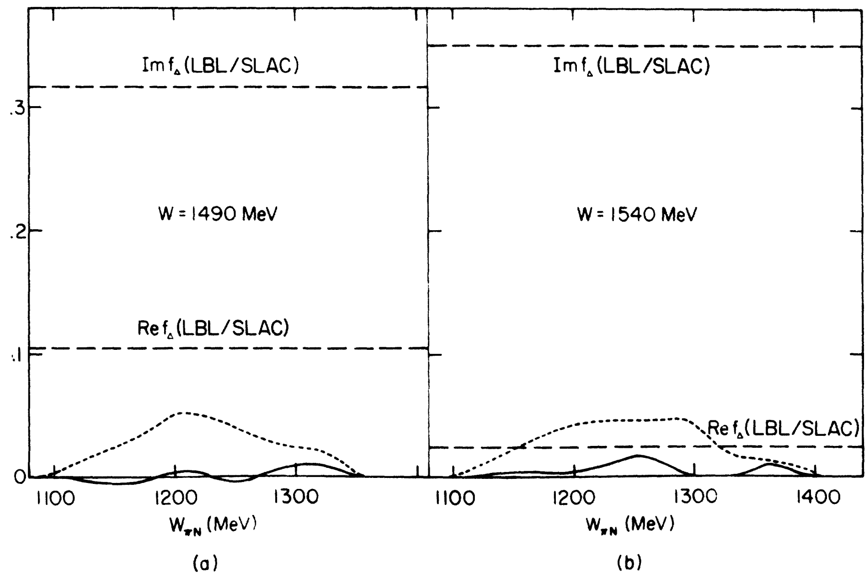

FIG. 3. Real (shown by dotted line) and imaginary (shown by solid line) parts of iAbs f_Δ vs πN c.m. energy at (a) W =1490 MeV and (b) W =1540 MeV. Corresponding LBL/SLAC amplitudes are also shown.

sive paper in which we shall derive in detail and motivate the equations used in this paper, and in addition shall check the f_α = constant approximation for all partial waves in the $\pi N \to \pi\pi N$ problem.[13] Clearly, much work is needed to gather experience on the problems of implementing unitarity and analyticity as discussed above and we are addressing ourselves to these questions.

Note added in proof. Since completion of the manuscript we have realized that the very rapid variation of Absf_ϵ comes from singularities on the "wrong" Riemann sheet (studied extensively in the past and related to what was then called the Peierls mechanism). Therefore these singularities are not in the physical amplitude. To ensure this result one must exploit analyticity and disperse the absorptive part [see Eq. (5) of the text]. This point has been made independently by Aitchison and Golding[17] and by Badalyan *et al.*[18] in recent notes. However, the relations of Eq. (4) are still useful. When the absorptive parts are small as in the case of Absf_Δ, calculated above, one can conclude that unitarity is not important and take f_Δ to be constant in phenomenological analyses with the understanding that there is *still* the possibility of nonsingular subenergy variation due to dynamical mechanisms.

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We are grateful to the members of the LBL/SLAC collaboration for helpful conversations and correspondence. The VPI group particularly enjoyed a productive visit from R. Cashmore.

APPENDIX A

In this Appendix we express in some detail the content of Eq. (4) as applied in Sec. II. We first rewrite Eq. (1) of the text more explicitly. In the three-body c.m. system the amplitude T_{23} for the P-wave $I=\frac{1}{2}$, $J=\frac{1}{2}$ transition

$$\pi(\vec{p}) + N(-\vec{p}, \lambda) \to \pi(\vec{p}_1) + \pi(\vec{p}_2) + N(\vec{p}_3, r),$$

where λ and r are the z components of the nucleon spin in the initial and final states, respectively, is written

$$\langle \vec{p}_1, \vec{p}_2, \vec{p}_3 r | T_{23} | \vec{p}\lambda\rangle = \sum_{\substack{i,j=1\\ \text{(cyclic)}\\ \mu}}^{2} \frac{1}{R_\Delta\sqrt{p_i}} \langle \vec{p}_i \mu | f_\Delta | \vec{p}\lambda\rangle \frac{e^{i\delta_{j3}}\sin\delta_{j3}}{q_{j3}^2} \Delta_\mu(\vec{p}_j, \vec{p}_3 r) + \frac{1}{R_\epsilon\sqrt{p_3}} \langle \vec{p}_3 r | f_\epsilon | \vec{p}\lambda\rangle \frac{e^{i\delta_{12}}\sin\delta_{12}}{q_{12}}, \tag{A1}$$

where we have suppressed isospin indices,[14] and we have adopted the convention that, unless stated otherwise, all momenta labels represent three-momenta. In the above equation

$$q_{12}^2 = \sigma_3/4 - \mu^2,$$
$$q_{i3}^2 = [\sigma_j - (M-\mu)^2][\sigma_j - (M+\mu)^2]/(4\sigma_j), \tag{A2}$$

where $i, j = 1, 2$ $(i \neq j)$ and

$$\sigma_i = (p_j + p_k)^2, \quad i,j,k \text{ cyclic}, \tag{A3}$$

where p_j and p_k are four-momenta. In Eq. (A1) δ_{i3} is the πN phase shift in the Δ channel and δ_{12} is the $\pi\pi$ phase shift in the ϵ channel; $\Delta_\mu(\vec{p}_j, \vec{p}_3 r)$ is the $\pi N\Delta$ vertex function given by

$$\Delta_\mu(\vec{p}_j, \vec{p}_3 r) = \sum_{\lambda,t} \langle \tfrac{1}{2} l t \lambda | \tfrac{3}{2}\mu\rangle \bar{u}_t(-\vec{p}_j, W_i) \frac{V_\lambda}{q_{j3}} u_r(\vec{p}_3, M), \tag{A4}$$

where $W_i = \sqrt{\sigma_i}$ and $u(\vec{p}, M)$ is the usual Dirac spinor of three-momentum $\vec{p}$ and mass M,[15] and the vector $\vec{V}$ ($|V| = q_{j3}$) is defined in Aaron, Amado, and Young.[16] We shall discuss the construction of the $\pi N\Delta$ vertex function Δ_μ in detail elsewhere.[13] Finally, the normalization factors R_ϵ and R_Δ are given by

$$R_\epsilon{}^2 = h(W) \int_{2\mu}^{W-M} dW_3 \frac{\sin^2\delta_{12}}{q_{12}}, \tag{A5}$$

$$R_\Delta{}^2 = \frac{h(W)}{3} \int_{M+\mu}^{W-\mu} dW_1 \frac{\sin^2\delta_{23}}{q_{23}^3}, \tag{A6}$$

$$h(W) = \frac{1}{4}\left(\frac{M}{W}\right)^2 \frac{p}{(2\pi)^6}. \tag{A7}$$

These normalization factors have been chosen so that our amplitudes f_α^J correspond precisely to those amplitudes which LBL/SLAC plot on their Argand diagrams. If in Eq. (A1) we now decompose the matrix elements into partial waves using

$$\langle \vec{p} S m_s | p l S J M\rangle = \langle l S m_l m_s | J M\rangle Y_{l m_l}(\hat{p}) \tag{A8}$$

and substitute the resultant expression into the equation for the total 2 to 3 cross section,

$$\sigma_{23} = \frac{M}{2pW} \int d\rho_3 \tfrac{1}{2} \sum_{\lambda, r} |\langle \vec{p}_1 \vec{p}_2 \vec{p}_3 r | T_{23} | \vec{p}\lambda\rangle|^2, \tag{A9}$$

where the density of states factor is given by

$$\int d\rho_3 = \frac{2M}{(2\pi)^5} \int d^4p_1 \int d^4p_2 \int d^4p_3 \delta^+(p_1^2 - \mu^2)\delta^+(p_2{}^2 - \mu^2)\delta^+(p_3{}^2 - M^2)\delta^4(P - p_1 - p_2 - p_3), \tag{A10}$$

we obtain

$$\sigma_{23} = \frac{\pi}{p^2} \sum_{J,l,l'} (J + \tfrac{1}{2})(2|f_\Delta^J(l, l')|^2 + |f_\epsilon^J(l, l')|^2 + \text{cross terms}), \tag{A11}$$

where f_Δ^J and f_ϵ^J are the partial-wave projections defined in the text. From Eq. (A11) it is clear that these amplitudes are exactly those which LBL/SLAC plot in their Argand diagrams.

Finally, after partial-wave analysis, we substitute Eq. (A1) into the unitarity relation

$$\langle \alpha | T_{23} | \beta\rangle - \langle \alpha | T_{23}^+ | \beta\rangle = i \int \langle \alpha | T_{23'} | 3'\rangle \rho_{3'} \langle 3' | T_{3',3} | \beta\rangle \tag{A12}$$

with the density of states given by Eq. (A10) and the S matrix defined by

$$\langle \alpha | S_{23} | \beta\rangle = \langle \alpha | 1 | \beta\rangle + (2\pi)^4 i \delta^4(P_\alpha - P_\beta)\langle \alpha | T_{23} | \beta\rangle. \tag{A13}$$

Using the methods outlined by Aaron and Amado,[4] one obtains the absorptive parts of the quasiparticle amplitudes, f_Δ^J and f_ϵ^J, given by the following set of equations:

$$\text{Abs} f_\Delta^J(p_1, l', l) = \frac{3W_1\sqrt{p_1}}{\pi(E_{23} + M)}\left(\frac{M}{W}\right) \sum_{l''} [I_{\Delta\epsilon}^J(p_1, l', l'') f_\epsilon^J(l'', l) + I_{\Delta\Delta}^J(p_1, l', l'') f_\Delta^J(l'', l)], \tag{A14}$$

with

$$I_{\Delta\epsilon}^J(p_1, l, l'') = \frac{R_\Delta}{R_\epsilon} \int_{\text{Dalitz plot}} dW_3'' W_3'' \sqrt{p_3''} \frac{e^{i\delta_{12}} \sin\delta_{12}}{q_{12}''} \langle l' p_1 | B^J(\pi\Delta; \epsilon N) | l'' p_3''\rangle \tag{A15}$$

and

$$I_{\Delta\Delta}^J(p_1 l' l'') = \frac{1}{2M} \int_{\text{Dalitz plot}} dW_2'' W_2'' \sqrt{p_2''} \frac{e^{i\delta_{13}} \sin\delta_{13}}{q_{13}''^3} \langle l' p_1 | B^J(\pi\Delta; \pi\Delta) | l'' p_2''\rangle, \tag{A16}$$

where $f_\epsilon^J(l'', l)$ and $f_\Delta^J(l'', l)$ are the amplitudes given by LBL/SLAC, and are by definition constant over the Dalitz plot. $\langle B^J(\pi\Delta; \epsilon N)\rangle$ and $\langle B^J(\pi\Delta; \pi\Delta)\rangle$ are coupling-matrix elements which are defined in Appendix B. The limits of integration in Eq. (A15) and Eq. (A16) are the boundaries of the Dalitz plots. The corresponding equation for $\text{Abs} f_\epsilon^J(p_3, l', l)$ is given by

$$\text{Abs} f_\epsilon^J(p_3, l', l) = \frac{W_3\sqrt{p_3}}{\pi W} \sum_{l''} I_{\epsilon\Delta}^J(p_3, l', l'') f_\Delta^J(l'', l), \tag{A17}$$

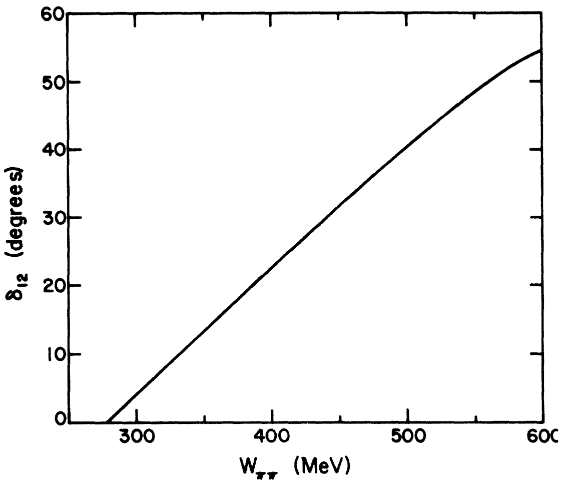

FIG. 4. δ_{12} vs $\pi\pi$ c.m. energy $W_{\pi\pi}$. The phase shift lies within presently accepted experimental values [ref., P. Nath, in *Experimental Meson Spectroscopy—1974*, proceedings of the Boston conference, edited by D. A. Garelick (A.I.P., New York, 1974), p. 224].

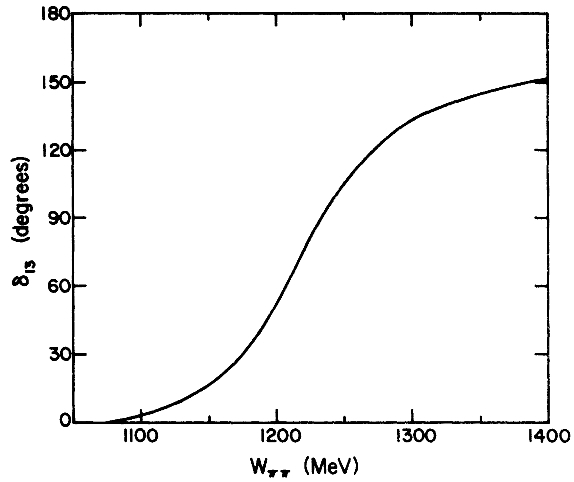

FIG. 5. δ_{13} vs πN c.m. energy $W_{\pi N}$ [ref., D. J. Herndon, A. Barbaro-Galtieri, and A. H. Rosenfeld, UCRL Report No. UCRL-20030 πN, 1970 (unpublished)].

where

$$I^J_{\epsilon\Delta}(p_3 l' l'') = \frac{R_\epsilon}{R_\Delta}\int dW_1'' W_1'' \sqrt{p_1''}\,\frac{e^{i\delta_{23}}\sin\delta_{23}}{q_{23}''^{\,3}} \times \langle l' p_3|B^J(\epsilon N;\pi\Delta)|l'' p_1''\rangle, \quad \text{(A18)}$$

with

$$\langle l' p_3|B^J(\epsilon N;\pi\Delta)|l'' p_1''\rangle = \langle l'' p_1''|B^J(\pi\Delta;\epsilon N)|l' p_3\rangle. \quad \text{(A19)}$$

The above equations represent the detailed versions of Eq. (4) of the test and were used to obtain Figs. 2 and 3. In Figs. 4 and 5 we plot the phase shifts δ_{13} and δ_{12} which we use in evaluating Eqs. (A15), (A16), and (A18). The results of Figs. 2 and 3 are qualitatively insensitive to variations in these phase shifts consistent with present experimental data.

APPENDIX B

The coupling-matrix elements arise naturally from the disconnected part of the amplitude for $1+2+3 \to 1'+2'+3'$ when one combines unitarity with the isobar model. They arise from the matrix elements of the three-particle propagator in the unitarity diagram in Fig. 6 when all three particles in the intermediate state are placed on their mass shells. In this Appendix we give the results for $\epsilon N \to \pi\Delta$ and $\pi\Delta \to \pi\Delta$. In these results we have made an approximation in evaluating the $\pi N\Delta$ vertex function; namely, we rewrite Eq. (A4) as

$$\Delta_\mu(\vec{p}_j, \vec{p}_3 r) = \sum_\lambda \langle \tfrac{1}{2} 1 r\lambda|\tfrac{3}{2}\mu\rangle V_\lambda + O\left(\frac{p_3^{\,2}}{4M^2}\right) \quad \text{(B1)}$$

and neglect the terms of $O(p_3^{\,2}/4M^2)$—this neglect introduces errors of less than 5% at the energies under consideration in the text. Our results are given below.

$\epsilon N \to \pi\Delta$:

$$\langle l' p_1|B^J(\pi\Delta;\epsilon N)|l p_3\rangle = -2(2l'+1)^{1/2}\langle l' 100|l0\rangle W(l' 1J\tfrac{1}{2};\, l\tfrac{3}{2}) \times (C_{1_l} p_1 + C_{2_{l'}} p_3), \quad \text{(B2)}$$

where

$$\begin{aligned} C_{1_l} &= \frac{\pi}{p_1 p_3} a(z_0) P_l(z_0), \\ C_{2_l} &= \frac{\pi}{p_1 p_3} P_l(z_0), \\ z_0 &= [(W - E_3 - \omega_1)^2 - \mu^2 - p_1^{\,2} - p_3^{\,2}]/2p_1 p_3, \\ a(z_0) &= \frac{1}{2}\left[1 + \frac{(M^2-\mu^2)}{(E_3+\omega_{13})^2 - p_1^{\,2}}\right], \\ \omega_{13} &= (p_3^{\,2} + 2p_3 p_1 z_0 + p_1^{\,2} + \mu^2)^{1/2}, \end{aligned} \quad \text{(B3)}$$

$P_l(z_0)$ is a Legendre polynomial of order l, $\langle l' 100|l0\rangle$ is a Clebsch-Gordan coefficient and $W(l' 1J\tfrac{1}{2};\, l\tfrac{3}{2})$ is a Racah W coefficient.

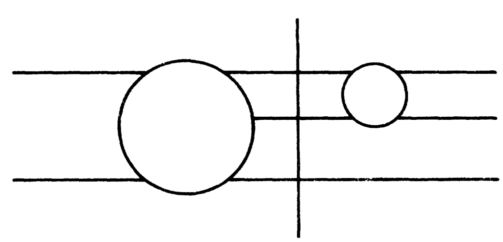

FIG. 6. Unitarity diagram.

$\pi\Delta \rightarrow \pi\Delta$:

$$\langle l'p_1|B^J(\pi\Delta;\pi\Delta)|lp_2\rangle = 4\sum_\Lambda\left[(2\Lambda+1)\langle 1\Lambda 00|l'0\rangle\langle 1\Lambda 00|l0\rangle W(\tfrac{1}{2}1Jl;\tfrac{3}{2}\Lambda)W(\tfrac{1}{2}1Jl';\tfrac{3}{2}\Lambda)(p_2{}^2C_{2_{l'}}+p_1{}^2C_{3_l}+p_1p_2C_{4_\Lambda})+C_{1_\Lambda}p_1p_2\begin{pmatrix}\Lambda & 1 & l\\ 1 & \tfrac{1}{2} & \tfrac{3}{2}\\ l' & \tfrac{3}{2} & J\end{pmatrix}\right], \tag{B4}$$

where

$$\begin{aligned}
C_{1_l} &= \frac{2\pi M}{p_1p_2}P_l(z_0),\\
C_{2_l} &= a(z_0)C_{1_l},\\
C_{3_l} &= b(z_0)C_{1_l},\\
C_{4_l} &= a(z_0)b(z_0)C_{1_l},\\
z_0 &= [(W-\omega_1-\omega_2)^2-\mu^2-p_1{}^2-p_2{}^2]/2p_1p_2,\\
E_{12} &= (p_1{}^2+2p_1p_2z_0+p_2{}^2+M^2)^{1/2},\\
a(z_0) &= \frac{1}{2}\left[1-\frac{(M^2-\mu^2)}{(\omega_1+E_{12})^2-p_2{}^2}\right],\\
b(z_0) &= \frac{1}{2}\left[1-\frac{(M^2-\mu^2)}{(\omega_2+E_{12})^2-p_1{}^2}\right],
\end{aligned} \tag{B5}$$

and

$$\begin{pmatrix}\Lambda & 1 & l\\ 1 & \tfrac{1}{2} & \tfrac{3}{2}\\ l' & \tfrac{3}{2} & J\end{pmatrix}$$

is a Wigner 9-j symbol.

Finally, it should be remembered that because of isospin coupling,[14] each coupling term should carry an additional factor

$$\chi_I = (-1)^{\tau_2+\tau_3-\tau''}(-1)^{\tau''+\tau_1-I} \times[(2\tau'+1)(2\tau''+1)]^{1/2}W(\tau_1\tau_2I\tau_3;\tau'\tau''),$$

where τ' and τ'' are the quasiparticle isospins; τ' breaks up into τ_1 and τ_2, τ'' into τ_2 and τ_3, and I is the total isospin. In Eq. (B2) $\chi_I=\sqrt{\tfrac{2}{3}}$ and in Eq. (B4) $\chi_I=\tfrac{1}{3}$.

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[1] D. J. Herndon, R. Longacre, L. R. Miller, A. H. Rosenfeld, G. Smadja, P. Söding, R. J. Cashmore, and D. W. G. S. Leith, Phys. Rev. D 11, 3183 (1975). We refer to this analysis as LBL/SLAC.

[2] See, for example, B. H. Bransden and R. G. Moorhouse, *The Pion-Nucleon System* (Princeton Univ. Press, Princeton, 1973). These authors suggest the following earlier references: R. M. Sternheimer and S. J. Lindenbaum, Phys. Rev. 123, 333 (1961); D. Z. Freedman, C. Lovelace, and J. M. Namylslovski, Nuovo Cimento A 43A, 218 (1966); M. C. Olsson and G. B. Yodh, Phys. Rev. 145, 1309 (1966); D. Morgan, *ibid.* 166, 1731 (1968); J. M. Namylslovski, M. S. K. Raznii, and R. G. Roberts, *ibid.* 157, 1328 (1967); R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. I.

[3] D. J. Herndon, P. Söding, and R. J. Cashmore, Phys. Rev. D 11, 3165 (1975). This paper contains a detailed description of the formalism used in the analysis of Ref. 1.

[4] R. Aaron and R. D. Amado, Phys. Rev. Lett. 31, 1157 (1973).

[5] H. J. Melosh, Phys. Rev. D 9, 1095 (1974).

[6] F. J. Gilman, M. Kugler, and S. Meshkov, Phys. Rev. D 9, 715 (1974).

[7] D. Faiman and J. Rosner, Phys. Lett. 45B, 357 (1973). A useful review of both the Melosh and $SU(6)_W$ material is given by A. H. Rosenfeld, Lectures presented at the School of Subnuclear Physics, Erice, Italy, 1973, LBL Report No. LBL-2098, 1974 (unpublished).

[8] R D. Amado, Phys. Rev. Lett. 33, 333 (1974); Phys. Rev. C 11, 719 (1975).

[9] R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).

[10] R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. 174, 2022 (1968); R. Aaron, D. C. Teplitz, R. D. Amado, and J. E. Young, *ibid.* 187, 2047 (1969).

[11] R. Aaron and R. D. Amado, Phys. Rev. Lett. 27, 1316 (1971).

[12] Recent calculations have included three-body unitarity in isobar analyses of the 3π system produced in the reaction $\pi N \rightarrow \pi\pi\pi N$ [ref., *Experimental Meson Spectroscopy—1974*, proceedings of the Boston conference, edited by D. A. Garelick (A.I.P., New York, 1974), p. 59]. The results indicate that one can make a fit

which is consistent with unitarity without improving our understanding of the physics over that in the simple isobar model. However, it should be noted that in these analyses Abs f [for reactions $A_1(A_2) \to \pi\rho$, $\pi\epsilon$, πf] was assumed to satisfy Eq. (4) of the text while Disp f was held constant, a procedure which, in general, will violate analyticity.

[13] R. Aaron, R. H. Thompson, R. D. Amado, R. A. Arndt, D. C. Teplitz, and V. L. Teplitz (unpublished).

[14] In our quasiparticle formalism, one includes isospin by multiplying any quasiparticle vertex by $\langle t_1 t_2 t_{1z} t_{2z} | t_3 t_{3z} \rangle$ where t_3 is the isospin of the quasiparticle and t_1 and t_2 are the isospins of the decay products.

[15] We are using the spinor normalization and metric of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[16] See Ref. 10, pp. 2026–2027.

[17] I. J. R. Aitchison and R. J. A. Golding, University of Oxford Report No. 47/75 (unpublished).

[18] A. M. Badalyan, M. I. Polikarpov, and Yu. A. Simonov, ITEP Report No. 38, Moscow, 1975 (unpublished).