Quadratic relations among observables in p-p scattering

R. M. Delaney and J. L. Gammel
Saint Louis University, St. Louis, Missouri 63103
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The 25 observables which describe p-p scattering are related by 16 independent quadratic relations. A method is developed for determining these quadratic relations. Provided a certain condition is satisfied, the particular set so determined and the results of 15 planned experiments at Argonne National Laboratory yield a unique solution for the 25 observables and, up to a common phase factor, for the five amplitudes.

I. INTRODUCTION

Halzen and Thomas[1] have given a complete set of observables for p-p scattering. Since there are five amplitudes, there are 25 observables. The amplitudes are complex numbers, but their overall phase is not observable, and therefore only $2\times 5-1=9$ of the 25 observables can be independent of each other. Thus there must be $25-9=16$ independent quadratic relations among the 25 observables. In this paper we work out one such set of 16 relations. The method applies to any similar problem.

II. THE METHOD

The work begins by considering some pure final state

$$A\alpha\alpha+B\alpha\beta+C\beta\alpha+D\beta\beta\,. \tag{1}$$

The direct product states in (1) are composed of the spin states of the two protons, where α and β are the $m=\pm\frac{1}{2}$ states. One then works out the final-state expectations values for the complete set of 16 operators formed by the direct product of the spin and identity operators for the two proton subspaces

$$\begin{array}{llll}\langle 11\rangle^F & \langle 1\sigma_x\rangle^F & \langle 1\sigma_y\rangle^F & \langle 1\sigma_z\rangle^F\\ \langle\sigma_x 1\rangle^F & \langle\sigma_x\sigma_x\rangle^F & \langle\sigma_x\sigma_y\rangle^F & \langle\sigma_x\sigma_z\rangle^F\\ \langle\sigma_y 1\rangle^F & \langle\sigma_y\sigma_x\rangle^F & \langle\sigma_y\sigma_y\rangle^F & \langle\sigma_y\sigma_z\rangle^F\\ \langle\sigma_z 1\rangle^F & \langle\sigma_z\sigma_x\rangle^F & \langle\sigma_z\sigma_y\rangle^F & \langle\sigma_z\sigma_z\rangle^F\,,\end{array} \tag{2}$$

where, for example,

$$\begin{aligned}&\langle 11\rangle^F=A^*A+B^*B+C^*C+D^*D,\\&\langle\sigma_x 1\rangle\;=A^*C+B^*D+C^*A+D^*B\,,\\&\cdots\\&\langle\sigma_z\sigma_z\rangle^F=A^*A-B^*B-C^*C+D^*D\,,\end{aligned} \tag{3}$$

and solves these 16 linear equations for the 16 quantities

$$\begin{array}{llll}A^*A & B^*A & C^*A & D^*A\\ A^*B & B^*B & C^*B & D^*B\\ A^*C & B^*C & C^*C & D^*C\\ A^*D & B^*D & C^*D & D^*D\end{array} \tag{4}$$

where, for example

$$\begin{aligned}&A^*A=\tfrac{1}{4}(\langle 11\rangle^F+\langle\sigma_z 1\rangle^F+\langle 1\sigma_z\rangle^F+\langle\sigma_z\sigma_z\rangle^F)\,,\\&\cdots\\&D^*D=\tfrac{1}{4}(\langle 11\rangle^F+\langle\sigma_z 1\rangle^F-\langle 1\sigma_z\rangle^F+\langle\sigma_z\sigma_z\rangle^F)\,.\end{aligned} \tag{5}$$

Then, one can form identities such as

$$(A^*A)(B^*B)=(A^*B)(B^*A)\,. \tag{6}$$

There are 120 of these, of which all but 24 are redundant; of the 24 identities, six are of real and 18 of complex quantities. One derives a quadratic relation by substituting the required parts of Eq. (5) into Eq. (6) and applying the equation of Halzen and Thomas:

$$\begin{aligned}\langle\sigma_\alpha\sigma_\beta\rangle^F=&I(00;\alpha\beta)+I(0j;\alpha\beta)+I(i\,0;\alpha\beta)\\&+I(ij;\alpha\beta)\,,\end{aligned} \tag{7}$$

where α and β can be $0,1,2,3,$ while i and j can be 1, 2, or 3. The $I(\alpha'\beta';\alpha\beta)$ are the center-of-mass observables defined by

$$I(\alpha'\beta';\alpha\beta)=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{\alpha'}\sigma_{\beta'}M^\dagger\sigma_\alpha\sigma_\beta), \tag{8}$$

where M is the scattering matrix and σ_0 is the unit matrix. Then, with a superscript I denoting initial-state expectation values,

$$\langle\sigma_\alpha\sigma_\beta\rangle^F=\sum_{\alpha'=0}^{4}\sum_{\beta'=0}^{4}\langle\sigma_{\alpha'}\sigma_{\beta'}\rangle^I I(\alpha'\beta';\alpha\beta)\,. \tag{9}$$

The initial beam consists of statistically independent particles, so that

$$\langle\sigma_i\sigma_j\rangle^I=\langle\sigma_i\rangle^I\langle\sigma_j\rangle^I\,; \tag{10}$$

for Eq. (7) we have chosen a pure state, thus $\langle\sigma_i\rangle^I=1$ and $\langle\sigma_j\rangle^I=1$ for some i and j. There are nine possibilities in all, so that finally we derive

$(6+2\times 18)\times 9=378$ quadratic relations by this method. Not all of these can be independent. We apply the Schmidt-Hilbert orthogonalization scheme, as adapted for testing independence of nonlinear equations,[2] to the 378 quadratic relations and verify that there are exactly 16 independent ones. Of course there are many such sets and we have tried to find a "nice" set of 16.

III. RESULT

The result in terms of I's defined in Table I is given by the following independent quadratic relations.

$$(I_1+I_9)^2-(I_5+I_{11})^2-(I_7-I_{10})^2-(I_{21}+I_{23})^2=0, \tag{11}$$

$$(I_1-I_{24})(I_{18}+I_{19})-(I_2+I_{17})(I_4+I_{13})-(I_8-I_{25})(I_{15}+I_{16})=0, \tag{12}$$

$$(I_2+I_{17})(I_6+I_{12})+(I_7-I_9)(I_{15}-I_{16})+(I_8-I_{25})(I_{20}+I_{22})+(I_{10}+I_{24})(I_{15}-I_{16})=0, \tag{13}$$

$$(I_2+I_5)(I_{12}+I_{16})+(I_2-I_9)(I_{14}-I_{16})-(I_3-I_{15})(I_{10}+I_{17})-(I_4-I_{22})(I_8-I_{21})=0, \tag{14}$$

$$(I_2+I_9)(I_3+I_{15})-(I_2+I_{11})(I_6+I_{15})+(I_8-I_{23})(I_{13}-I_{20})+(I_{10}-I_{17})(I_{14}+I_{16})=0, \tag{15}$$

$$(I_2+I_9)(I_4-I_{18})-(I_2+I_{11})(I_4+I_{22})+(I_8-I_{23})(I_{12}-I_{16})+(I_{10}-I_{17})(I_{13}-I_{19})=0, \tag{16}$$

$$(I_2+I_5)(I_{13}+I_{20})-(I_2-I_9)(I_{13}+I_{19})+(I_4+I_{18})(I_{10}+I_{17})-(I_6-I_{15})(I_8-I_{21})=0, \tag{17}$$

$$(I_2-I_9)(I_3-I_{15})-(I_2-I_{11})(I_6-I_{15})-(I_8+I_{23})(I_{13}+I_{20})-(I_{10}+I_{17})(I_{14}-I_{16})=0, \tag{18}$$

$$(I_2-I_5)(I_{12}-I_{16})+(I_2+I_9)(I_{14}+I_{16})+(I_3+I_{15})(I_{10}-I_{17})+(I_4+I_{22})(I_8+I_{21})=0, \tag{19}$$

$$(I_2-I_5)(I_{13}-I_{20})-(I_2+I_9)(I_{13}-I_{19})-(I_4-I_{18})(I_{10}-I_{17})+(I_6+I_{15})(I_8+I_{21})=0, \tag{20}$$

$$(I_2-I_9)(I_4+I_{18})-(I_2-I_{11})(I_4-I_{22})-(I_8+I_{23})(I_{12}+I_{16})-(I_{10}+I_{17})(I_{13}+I_{19})=0, \tag{21}$$

$$(I_2+I_5)(I_8-I_{21})-(I_2-I_{11})(I_8+I_{23})-(I_4-I_{22})(I_{12}+I_{16})-(I_6-I_{15})(I_{13}+I_{20})=0, \tag{22}$$

$$(I_2-I_5)(I_8+I_{21})-(I_2+I_{11})(I_8-I_{23})+(I_4+I_{22})(I_{12}-I_{16})+(I_6+I_{15})(I_{13}-I_{20})=0, \tag{23}$$

$$(I_2+I_{17})^2-(I_6+I_{12})^2-(I_7-I_9)^2+(I_8-I_{25})^2+(I_{10}+I_{24})^2-(I_{20}+I_{22})^2=0, \tag{24}$$

$$(I_1-I_{24})^2+(I_2+I_{17})^2-(I_4+I_{13})^2+(I_8-I_{25})^2-(I_{15}+I_{16})^2+2(I_1-I_{24})(I_9-I_{10})=0, \tag{25}$$

$$(I_2+I_5)^2-(I_2-I_9)^2-(I_3-I_{15})^2-(I_4-I_{22})^2+(I_8-I_{21})^2+(I_{10}+I_{17})^2-(I_{12}+I_{16})^2+(I_{14}-I_{16})^2=0. \tag{26}$$

IV. APPLICATION

The polarized proton beam at Argonne National Laboratory will be used to examine p-p elastic scattering with these constraints:

1. The incident beam will not be polarized in its direction of motion.
2. Any direction of polarization is possible for the target.
3. The polarization of the scattered beam (the high-energy beam) will not be observed.
4. The polarization of the recoil proton will be measured, but not in the direction of its motion.

There will then be 15 possible experiments. These correspond to the laboratory observables of Halzen and Thomas, which are listed in Table II together with their relationships to the center-of-mass observables. θ_R is the recoil angle.

There are nine center-of-mass observables which are determined directly from the measured

TABLE I. Notation for center-of-mass observables of Halzen and Thomas.

$I_1=I(0,0;0,0)$	$I_{14}=I(0,z;z,0)$
$I_2=I(0,y;0,0)$	$I_{15}=I(y,x;0,x)$
$I_3=I(0,z;0,z)$	$I_{16}=I(x,y;0,x)$
$I_4=I(0,z;0,x)$	$I_{17}=I(x,x;0,y)$
$I_5=I(0,y;0,y)$	$I_{18}=I(y,x;0,z)$
$I_6=I(0,x;0,x)$	$I_{19}=I(x,y;0,z)$
$I_7=I(z,z;0,0)$	$I_{20}=I(z,y;0,x)$
$I_8=I(x,z;0,0)$	$I_{21}=I(z,x;0,y)$
$I_9=I(y,y;0,0)$	$I_{22}=I(y,z;0,x)$
$I_{10}=I(x,x;0,0)$	$I_{23}=I(x,z;0,y)$
$I_{11}=I(y,0;0,y)$	$I_{24}=I(x,x;x,x)$
$I_{12}=I(x,0;0,x)$	$I_{25}=I(x,x;x,z)$
$I_{13}=I(x,0;0,z)$	

quantities:

$$I_1,\ I_2,\ I_5,\ I_8,\ I_9,\ I_{10},\ I_{11},\ I_{17},\ \text{and } I_{23}\,. \quad (27)$$

The six remaining measured laboratory observables are linearly related to 10 center-of-mass observables:

$$I_3,\ I_4,\ I_6,\ I_{12},\ I_{13},\ I_{15},\ I_{16},\ I_{18},\ I_{19},\ \text{and } I_{22}. \quad (28)$$

The six linear equations, alone, are not sufficient for the unique determination of these 10 observables.

We now show that if a certain condition is satisfied, the use of the quadratic relations allows the unique determination of all 25 center-of-mass observables. The five amplitudes can then be calculated up to a common phase factor.

The eight quadratic relations (14) through (21) may be viewed as eight equations linear in the 12 unknowns:

$$I_3,\ I_4,\ I_6,\ I_{12},\ I_{13},\ I_{14},\ I_{15},\ I_{16},\ I_{18},\ I_{19},\ I_{20},\ \text{and } I_{22}. \quad (29)$$

The coefficients of these unknowns are linear combinations of the directly determined center-of-mass observables listed in Eq. (27) and of one additional unknown, I_{21}.

Using Eqs. (28) and (29) it is seen that the eight quadratic relations (14) through (21) and the six linear equations in Table II which involve the measured L_i ($i=1,\ldots,6$) form 14 equations linear in the 12 unknowns of Eq. (29). The coefficients, apart from those depending on I_{21}, are all known from the experiments.

A more convenient set of unknowns is

$$\begin{aligned} x_1 &= I_3 + I_{15}\,, \\ x_2 &= I_3 - I_{15}\,, \\ x_3 &= I_6 + I_{15}\,, \\ x_4 &= I_{13} - I_{20}\,, \\ x_5 &= I_{13} + I_{20}\,, \\ x_6 &= I_{14} - I_{16}\,, \\ x_7 &= I_{14} + I_{16}\,, \\ x_8 &= I_4 - I_{18}\,, \\ x_9 &= I_4 + I_{18}\,, \\ x_{10} &= I_4 - I_{22}\,, \\ x_{11} &= I_{12} + I_{16}\,, \\ x_{12} &= I_{13} - I_{19}\,. \end{aligned} \quad (30)$$

The 14 equations that result are given in the Appendix as Eqs. (A1)–(A14).

I_{21} is determined by the condition that any 13 of these equations must be linearly dependent. Thus, choosing a particular set, the determinant of the 13×13 matrix of coefficients and constant terms must vanish. If one chooses the set as all equations except Eq. (A7), a cubic equation for I_{21} results:

$$a_3 I_{21}{}^3 + a_2 I_{21}{}^2 + a_1 I_{21} + a_0 = 0\,. \quad (31)$$

Choosing another set as all equations except Eq. (A6) a second cubic equation for I_{21} results:

$$a_3' I_{21}{}^3 + a_2' I_{21}{}^2 + a_1' I_{21} + a_0' = 0\,. \quad (32)$$

If one assumes that Eqs. (31) and (32) have a single solution in common, then with

$$A_{ij} = a_i a_j' - a_j a_i'\,, \quad i,j = 0,1,2,3 \quad (33)$$

one finds

$$I_{21} = \frac{A_{20}A_{23} + A_{30}A_{31}}{A_{23}(A_{12} - A_{30}) - A_{31}{}^2}\,. \quad (34)$$

The condition for a unique solution for I_{21} is thus

$$A_{23}(A_{12} - A_{30}) - A_{31}{}^2 \neq 0\,. \quad (35)$$

If this condition is not satisfied then either

$$A_{20}A_{23} + A_{30}A_{31} = 0\,,$$

implying more than one solution for I_{21}, or

$$A_{20}A_{23} + A_{30}A_{31} \neq 0\,,$$

implying that the experimental data are inconsistent. All numerical examples have led to a unique

TABLE II. The laboratory observables to be measured (θ_R is the recoil angle).

$I(0,0;0,0) = I_1$
$I(n,0;0,0) = I(0,n;0,0) = I(0,0;0,n) = I(n,n;0,n) = -I_2$
$I(0,n;0,n) = I_5$
$I(s,l;0,0) = -I_8$
$I(n,n;0,0) = -I_9$
$I(s,s;0,0) = I_{10}$
$I(n,0;0,n) = I_{11}$
$I(s,s;0,n) = -I_{17}$
$I(s,l;0,n) = I_{23}$
$L_1 = I(0,s;0,s) = I_4 \sin\theta_R - I_6 \cos\theta_R$
$L_2 = I(0,l;0,s) = I_3 \sin\theta_R + I_4 \cos\theta_R$
$L_3 = I(n,s;0,s) = -I_{15} \cos\theta_R - I_{18} \sin\theta_R$
$L_4 = I(n,l;0,s) = I_{15} \sin\theta_R + I_{22} \cos\theta_R$
$L_5 = I(s,0;0,s) = -I_{12} \cos\theta_R - I_{13} \sin\theta_R$
$L_6 = I(s,n;0,s) = I_{16} \cos\theta_R + I_{19} \sin\theta_R$

TABLE III. The matrix M ($\theta=\theta_R$).

1	2	3	4	5	6	7	8	9	10	11	12	13
I_8-I_{21}			$-I_2+I_9$	I_5+I_9				$I_{10}+I_{17}$			I_2-I_9	
	I_8+I_{21}		I_2-I_5				$-I_{10}+I_{17}$				$-I_2-I_9$	
		$-I_8+I_{21}$			I_2-I_9				$-I_{10}-I_{17}$	I_2+I_5		
		$-I_2+I_{11}$	$-I_{10}-I_{17}$	$-I_{10}-I_{17}$				I_2-I_9		$-I_8-I_{23}$	$I_{10}+I_{17}$	
I_2-I_{11}				$-I_8-I_{23}$	$-I_{10}-I_{17}$				I_2-I_9			
I_2+I_9	I_9-I_{11}		I_8-I_{23}			$I_{10}-I_{17}$			I_2+I_9			
		I_2+I_{11}			I_8-I_{23}	$-I_8+I_{23}$	I_9-I_{11}	$-I_2-I_{11}$		I_8-I_{23}	$I_{10}-I_{17}$	
$\frac{1}{2}\cos\theta$	$-\frac{1}{2}\cos\theta$						$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$				$-L_1$
$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$						$\frac{1}{2}\cos\theta$	$\frac{1}{2}\cos\theta$	$\sin\theta$			$-L_2$
$-\frac{1}{2}\cos\theta$	$-\frac{1}{2}\cos\theta$						$\frac{1}{2}\sin\theta$	$-\frac{1}{2}\sin\theta$				$-L_3$
$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$	$-\cos\theta$					$\frac{1}{2}\cos\theta$	$\frac{1}{2}\cos\theta$				$-L_4$
			$-\frac{1}{2}\sin\theta$	$-\frac{1}{2}\sin\theta$	$-\frac{1}{2}\cos\theta$	$\frac{1}{2}\cos\theta$				$-\cos\theta$		$-L_5$
			$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$	$-\frac{1}{2}\cos\theta$	$\frac{1}{2}\cos\theta$					$-\sin\theta$	$-L_6$

TABLE IV. The matrix M' ($\theta=\theta_R$).

1	2	3	4	5	6	7	8	9	10	11	12	13
I_8+I_{21}					I_2-I_5	I_5+I_9	$I_{10}-I_{17}$			I_2-I_5		
$-I_{10}+I_{17}$	I_8+I_{21}	$-I_{10}+I_{17}$	I_2-I_5					$I_{10}-I_{17}$			$-I_2-I_9$	
		$-I_8+I_{21}$			I_2-I_9				$-I_{10}-I_{17}$	I_2+I_5		
		$-I_2+I_{11}$	$-I_{10}-I_{17}$	$-I_{10}-I_{17}$				I_2-I_9		$-I_8-I_{23}$	$I_{10}+I_{17}$	
	$-I_2+I_{11}$			$-I_8-I_{23}$	$-I_{10}-I_{17}$		I_2-I_{11}		$-I_9+I_{11}$			
	$-I_2-I_{11}$		I_8-I_{23}			$I_{10}-I_{17}$	I_2+I_9					
I_9-I_{11}		I_2+I_9			I_8-I_{23}	$-I_8+I_{23}$		$-I_2-I_9$		I_8-I_{23}	$I_{10}-I_{17}$	
$\frac{1}{2}\sin\theta$	$-\cos\theta$	$\frac{1}{2}\sin\theta$					$\frac{1}{2}\cos\theta$		$-\frac{1}{2}\cos\theta$			$-L_1$
$\frac{1}{2}\cos\theta$		$\frac{1}{2}\cos\theta$					$\frac{1}{2}\sin\theta$		$\frac{1}{2}\sin\theta$			$-L_2$
$\frac{1}{2}\sin\theta$		$\frac{1}{2}\sin\theta$					$-\frac{1}{2}\cos\theta$	$-\sin\theta$	$\frac{1}{2}\cos\theta$			$-L_3$
$\frac{1}{2}\cos\theta$		$-\frac{1}{2}\cos\theta$					$\frac{1}{2}\sin\theta$		$-\frac{1}{2}\sin\theta$			$-L_4$
			$-\frac{1}{2}\sin\theta$	$-\frac{1}{2}\sin\theta$	$-\frac{1}{2}\cos\theta$	$\frac{1}{2}\cos\theta$				$-\cos\theta$		$-L_5$
			$\frac{1}{2}\sin\theta$	$\frac{1}{2}\sin\theta$	$-\frac{1}{2}\cos\theta$	$\frac{1}{2}\cos\theta$					$-\sin\theta$	$-L_6$

determination of I_{21}.

The coefficients a_i and a_i' can be found by a suitable expansion of the determinants in terms of those elements involving I_{21}. That is, after some row and column manipulation, the matrix, M, whose vanishing determinant yields Eq. (31) is given in Table III. With the first three diagonal elements denoted by m_{11}, m_{22}, m_{33}, $\det M$ can be written as

$$\det M = A_0 + B_1 m_{11} + B_2 m_{22} + B_3 m_{33} + C_1 m_{22} m_{33} + C_2 m_{33} m_{11} + C_3 m_{11} m_{22} + D m_{11} m_{22} m_{33}, \quad (36)$$

where

A_0 = determinant of M with $m_{11} = m_{22} = m_{33} = 0$,

B_i = determinant of 12×12 matrix found by deleting the row and column to which a_{ii} belongs, and setting $m_{jj} = m_{kk} = 0$ (i, j, k = cyclic forms of 1, 2, 3),

C_i = determinant of 11×11 matrix found by deleting the rows and columns to which m_{jj} and m_{kk} belong, and setting $m_{ii} = 0$,

D = determinant of 10×10 matrix found by deleting the rows and columns to which m_{11}, m_{22}, and m_{33} belong.

Substituting $m_{11} = I_8 - I_{21}$, $m_{22} = I_8 + I_{21}$, and $m_{33} = -I_8 + I_{21}$ into Eq. (36) gives

$$\begin{aligned} a_0 &= -A_0 - (B_1 + B_2 - B_3) I_8 + (C_1 + C_2 - C_3) I_8^2 + D I_8^3, \\ a_1 &= B_1 - B_2 - B_3 - 2C_2 I_8 - D I_8^2, \\ a_2 &= -C_1 + C_2 + C_3 - D I_8, \\ a_3 &= D. \end{aligned} \quad (37)$$

The a' coefficients of Eq. (32) are found by a similar procedure. The matrix M' is given in Table IV. The a_i' can be expressed in terms of determinants analogous to those of Eq. (36):

$$\begin{aligned} a_0' &= A_0' + (B_1' + B_2' - B_3') I_8 - (C_1' + C_2' - C_3') I_8^2 - D' I_8^3, \\ a_1' &= B_1' + B_2' + B_3' + 2C_3' I_8 - D' I_8^2, \\ a_2' &= C_1' + C_2' + C_3' + D' I_8, \\ a_3' &= D'. \end{aligned} \quad (38)$$

Having found the value of I_{21}, we can solve any 12 of Eqs. (A1) through (A14) as linear equations for the 12 unknowns of Eq. (29). This leaves only I_7, I_{24}, and I_{25} undetermined. Quadratic relations (12) and (13) are linear in these three unknowns, while (11) − (24) + (25) provides a third linear equation. Thus I_7, I_{24}, and I_{25} are uniquely determined.

All 25 center-of-mass observables being determined, the five quadratic relations (22) through (26) will provide a consistency check on the experimental data.

The t-channel amplitudes of Halzen and Thomas are N_0, N_1, N_2, A, and π. Assuming $N_1 \neq 0$ and choosing N_1 real and positive allows one to express the amplitudes in terms of the center-of-mass observables as

$$\begin{aligned} N_0 &= \frac{1}{4N_1}[-(I_4 - I_{25}) + i(I_2 - I_{15})], \\ N_2 &= \frac{-1}{4N_1}[(I_4 + I_{25}) + i(I_2 + I_{15})], \\ A &= \frac{-1}{4N_1}[(I_8 + I_{13}) + i(I_{16} - I_{17})], \\ \pi &= \frac{1}{4N_1}[-(I_8 - I_{13}) + i(I_{16} + I_{17})], \end{aligned} \quad (39)$$

where

$$N_1 = \left[\frac{I_8 I_{13} - I_{16} I_{17}}{2(I_3 - I_6)}\right]^{1/2}. \quad (40)$$

V. CONCLUSION

Sixteen independent quadratic relations among the observables in p-p elastic scattering have been derived. If the condition of Eq. (35) is met, then the 15 experiments of Table II together with the quadratic relations yield a unique solution for the 25 observables and for the five complex amplitudes, up to a common phase factor.

APPENDIX

The 14 equations linear in the unknowns $x_1, \ldots, x_{12}$ are the following.

$$(I_2 + I_9)x_1 - (I_2 + I_{11})x_3 + (I_8 - I_{23})x_4 + (I_{10} - I_{17})x_7 = 0, \quad (A1)$$

$$(I_8 - I_{23})x_6 - (I_8 - I_{23})x_7 + (I_9 - I_{11})x_8 - (I_2 + I_{11})x_9 + (I_2 + I_{11})x_{10} + (I_8 - I_{23})x_{11} + (I_{10} - I_{17})x_{12} = 0, \quad (A2)$$

$$(I_2 - I_{11})x_1 - (I_9 - I_{11})x_2 - (I_2 - I_{11})x_3 - (I_8 + I_{23})x_5 - (I_{10} + I_{17})x_6 = 0, \quad (A3)$$

$$-(I_{10}+I_{17})x_4-(I_{10}+I_{17})x_5+(I_2-I_9)x_9-(I_2-I_{11})x_{10}-(I_8+I_{23})x_{11}+(I_{10}+I_{17})x_{12}=0\,, \tag{A4}$$

$$-(I_{10}+I_{17})x_2+(I_2-I_9)x_6-(I_8-I_{21})x_{10}+(I_2+I_5)x_{11}=0\,, \tag{A5}$$

$$(I_8-I_{21})x_1-(I_8-I_{21})x_2-(I_8-I_{21})x_3-(I_2-I_9)x_4+(I_5+I_9)x_5+(I_{10}+I_{17})x_9+(I_2-I_9)x_{12}=0\,, \tag{A6}$$

$$(I_{10}-I_{17})x_1+(I_2-I_5)x_6+(I_5+I_9)x_7+(I_8+I_{21})x_8+(I_8+I_{21})x_9-(I_8+I_{21})x_{10}+(I_2-I_5)x_{11}=0\,, \tag{A7}$$

$$(I_8+I_{21})x_3+(I_2-I_5)x_4-(I_{10}-I_{17})x_8-(I_2+I_9)x_{12}=0\,, \tag{A8}$$

$$\tfrac{1}{2}\cos\theta_R\, x_1-\tfrac{1}{2}\cos\theta_R\, x_2-\cos\theta_R\, x_3+\tfrac{1}{2}\sin\theta_R\, x_8+\tfrac{1}{2}\sin\theta_R\, x_9-L_1=0\,, \tag{A9}$$

$$\tfrac{1}{2}\sin\theta_R\, x_1+\tfrac{1}{2}\sin\theta_R\, x_2+\tfrac{1}{2}\cos\theta_R\, x_8+\tfrac{1}{2}\cos\theta_R\, x_9-L_2=0\,, \tag{A10}$$

$$-\tfrac{1}{2}\cos\theta_R\, x_1+\tfrac{1}{2}\cos\theta_R\, x_2+\tfrac{1}{2}\sin\theta_R\, x_8-\tfrac{1}{2}\sin\theta_R\, x_9-L_3=0\,, \tag{A11}$$

$$\tfrac{1}{2}\sin\theta_R\, x_1-\tfrac{1}{2}\sin\theta_R\, x_2+\tfrac{1}{2}\cos\theta_R\, x_8+\tfrac{1}{2}\cos\theta_R\, x_9-\cos\theta_R\, x_{10}-L_4=0\,, \tag{A12}$$

$$-\tfrac{1}{2}\sin\theta_R\, x_4-\tfrac{1}{2}\sin\theta_R\, x_5-\tfrac{1}{2}\cos\theta_R\, x_6+\tfrac{1}{2}\cos\theta_R\, x_7-\cos\theta_R\, x_{11}-L_5=0\,, \tag{A13}$$

$$\tfrac{1}{2}\sin\theta_R\, x_4+\tfrac{1}{2}\sin\theta_R\, x_5-\tfrac{1}{2}\cos\theta_R\, x_6+\tfrac{1}{2}\cos\theta_R\, x_7-\sin\theta_R\, x_{12}-L_6=0\,. \tag{A14}$$

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[1] F. Halzen and G. H. Thomas, Phys. Rev. D 10, 344 (1974).

[2] P. W. Keaton, Jr., J. L. Gammel, and G. G. Ohlsen, Ann. Phys. (N.Y.) 85, 152 (1974).