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Bound-state solutions of the Dirac equation in extended hadron models

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We discuss numerical solutions for the classical boson fields in the four-dimensional SU(2) model of Dashen, Hasslacher, and Neveu. For their separation of the Dirac equation, a fermion in the classical Yang-Mills field has one bound-state solution with zero binding energy; consequently the model will not bind fermions, and is unstable. The Dirac equation for a fermion on an Abelian vortex line likewise has no bound-state solutions except for zero binding energy.

I. INTRODUCTION

Some considerable effort has been invested in the search for bound states of fermions in models with spontaneously broken non-Abelian gauge symmetries. In particular, Dashen, Hasslacher, and Neveu[1] have suggested a four-dimensional model with SU(2) Yang-Mills fields, where the symmetry is spontaneously broken by a scalar isospinor. In this paper we show that for their separation of the Dirac equation, a fermion in the classical Yang-Mills field has only one bound state, with binding energy zero. This means the fermions are not trapped in this model, and the composite system, bosons plus fermions, is unstable. Also we examine the closely analogous case of fermion states on a magnetic vortex line and find a similar spectrum.

For the SU(2) isospinor model the meson piece of the Lagrangian density is taken to be

$$\mathcal{L} = -\tfrac{1}{4}(G^a_{\mu\nu})^2 + \tfrac{1}{2}(D_\mu K)^\dagger (D^\mu K) + \tfrac{1}{2}\mu^2 (K^\dagger K) - \tfrac{1}{4}\lambda (K^\dagger K)^2 ,$$

where

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\epsilon_{abc} W^b_\mu W^c_\nu ,$$

$$D_\mu = \partial_\mu - \tfrac{1}{2} i e W^a_\mu \tau^a ,$$

and the τ^a are the Pauli matrices. Fermions will be added later.

II. SOLUTIONS FOR YANG-MILLS FIELDS

The classical equations of motion are separated by assuming, following Wu and Yang,[2] that

$$W^a_0 = 0, \quad W^a_i = \epsilon_{iak} \frac{x^k}{r} g(r) , \quad r = (x_1{}^2 + x_2{}^2 + x_3{}^2)^{1/2} ,$$

and

$$K = i \frac{\vec{\tau}\cdot\vec{x}}{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix} f(r) .$$

(We consider only static solutions.)

Varying the Lagrangian then yields coupled nonlinear equations for f and g,

$$\begin{aligned} & g'' + \frac{2}{r} g' - \frac{2}{r^2} g - \frac{3}{r} e g^2 - e^2 g^3 - \frac{e}{2r} f^2 - \frac{e^2}{4} f^2 g = 0 , \\ & f'' + \frac{2}{r} f' - \frac{2}{r^2} f - \frac{2e}{r} f g - \frac{e^2}{2} f g^2 + \mu^2 f - \lambda f^3 = 0 , \end{aligned} \tag{1}$$

where primes indicate the operation d/dr. Requiring that the total energy of the solution be finite and that the energy density be finite at the origin furnishes the boundary conditions

$$f(0) = g(0) = 0, \quad f(\infty) = \frac{\mu}{\sqrt{\lambda}}, \quad g(\infty) = 0 .$$

One easily shows that the asymptotic form of the solution is

$$f(r) \sim \frac{\mu}{\sqrt{\lambda}} \left[1 - O\left(\frac{1}{r} e^{-\sqrt{2}\mu r} \right) \right] ,$$

$$g(r) \sim -\frac{2}{er} \left[1 - O\left(\exp\left(-\frac{\mu e r}{2\sqrt{\lambda}} \right) \right) \right] , \quad \text{as } r \to \infty .$$

By the rescaling

$$f \to \frac{\mu}{\sqrt{\lambda}} f, \quad g \to \frac{\mu}{\sqrt{\lambda}} g , \quad r \to \frac{r}{\beta\mu} ,$$

where

$$\beta \equiv \frac{e}{\sqrt{\lambda}} = \frac{m_{\text{YM}}}{\mu} ,$$

it is apparent that the solution to the Yang-Mills equations (1) depends only on the ratio β, equal to the ratio of the Yang-Mills field (after symmetry breaking) to the Higgs mass parameter. β is a measure of the coupling strength.

We obtain an approximate numerical solution to the equations (1) using the Henyey method.[3,4] The solution, shown in Fig. 1, is of the same form as that found by Dashen, Hasslacher, and Neveu. For our solutions (β between 0.1 and 0.5), in terms of the rescaled fields and radial parameter, the form of g seems relatively insensitive to the value of β, while f rises more rapidly near the origin for small β.

By an argument of Mandelstam,[5] the Yang-Mills and scalar fields alone are not stable against small oscillations; consequently one could only hope to stabilize the solution by adding fermions. If the fermions have bound states, it may make the total configuration, fermions plus Yang-Mills and scalar fields, energetically stable against decay. So we turn to the introduction of fermions.

III. SOLUTIONS TO THE DIRAC EQUATION

It is consistent with our classical solution to the Yang-Mills field equations to solve the "classical" Dirac equation (not second-quantized)

$$(i\not{D} - m)\psi = 0\,,$$

where D is defined as before. Each component of the Dirac spinor is an isospinor.

Once again we require an ansatz to separate the equation; Dashen, Hasslacher, and Neveu take the "large" components of ψ to be

$$e^{-i\omega t}\begin{bmatrix}\begin{pmatrix}0\\ i\end{pmatrix}\\ \\ \begin{pmatrix}-i\\ 0\end{pmatrix}\end{bmatrix} u(r)\,,\quad u\ \text{real} \tag{2}$$

and the small components to be

$$e^{-i\omega t}\, i\,\frac{\vec{\tau}\cdot\vec{x}}{r}\begin{bmatrix}\begin{pmatrix}0\\ i\end{pmatrix}\\ \\ \begin{pmatrix}-i\\ 0\end{pmatrix}\end{bmatrix} d(r)\,,\quad d\ \text{real}\,. \tag{3}$$

With this ansatz, the Dirac equation is equivalent

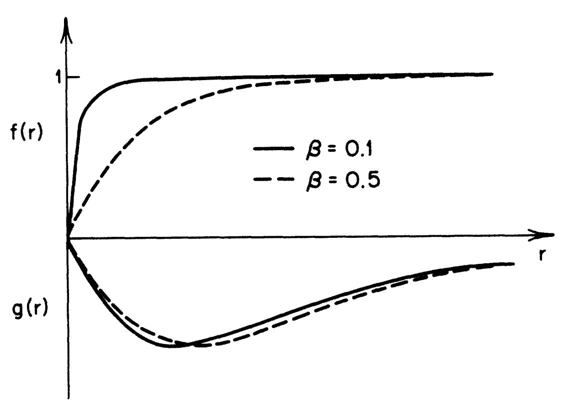

FIG. 1. Qualitative form of solutions for $f(r)$ and $g(r)$.

to the system

$$(m+\omega)d = -u' + egu\,,$$

$$(m-\omega)u = -d' - \frac{2}{r}d - egd\,,$$

with boundary conditions

$$u(\infty) = d(\infty) = d(0) = 0\,,$$

$$u(0) = \text{constant},\quad u'(0) = 0\,.$$

A variational calculation placed an upper bound on the ground-state energy which was only slightly above zero binding energy. This suggested that the system has a bound state at exactly zero binding energy $|\omega| = m$... bound in the sense that the fermion is localized. The suspicion was confirmed by the following exact solution due to Neveu.[6]

Suppose the binding energy of the fermion is exactly zero; then $m = \omega$, and it is consistent to choose $d(r) \equiv 0$. We are left with

$$-u' + egu = 0\,,$$

and integrate to find

$$u(r) = \text{const}\times\exp\left(e\int_0^r g(r')dr'\right)\,. \tag{4}$$

Notice that this solution is valid whenever (1) $g(r)$ vanishes at the origin, and (2) $e\int_0^r g(r')dr'$ diverges faster than $-\frac{3}{2}\ln r$ as $r\to\infty$. The first condition ensures that $u'(0)=0$. The second condition follows from the requirement that the fermion wave function be normalizable: $\int_0^\infty dr\, r^2(u^2+d^2)<\infty$. Thus $u(r)$ must go to zero faster than $r^{-3/2}$ as $r\to\infty$, and consequently $e\int_0^r g(r')dr'$ must diverge faster than $-\frac{3}{2}\ln r$ as $r\to\infty$. Condition (1) is identical to the previous boundary condition on $g(r)$ at $r=0$. Condition (2) is satisfied as well, since as $r\to\infty$ the asymptotic form is $g(r)\sim -2/er$, so $e\int_0^r g(r')dr' \sim -2\ln r$ as $r\to\infty$.

The solution (4) with $g(r)$ defined by Eq. (1) is an exact solution to the coupled fermion and Yang-Mills equations. The coupling is of the form

$$\mathcal{L}_{\text{int}} = -2egdu\,,$$

so the fermion current vanishes for $d(r)\equiv 0$.

Thus there is an analytic solution to the Dirac equation with zero binding energy. It will exist for any $g(r)$ satisfying the boundary conditions at $r=0$ and $r\to\infty$. The wave function for this solution has no nodes and it must therefore be the lowest bound state for the ansatz (2) and (3). Hence there can be no bound state with finite binding energy. We emphasize that this conclusion depends in no way on the numerical solution for $g(r)$ or on the variational calculation of the ground-state energy.

Thus adding fermions will not stabilize the Yang-Mills and scalar fields, in the classical approximation. It seems doubtful that quantum-mechanical corrections will change this situation, and we have not studied them.

IV. FERMION STATES ON AN ABELIAN VORTEX LINE

The SU(2) model of Dashen, Hasslacher, and Neveu has a close analog in the theory of an Abelian gauge field coupled to a charged scalar field. This theory exhibits vortex solutions of the Landau-Ginzburg type, which, according to Nielsen and Olesen,[7] may be identified with the Nambu string.

From the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}|(\partial_\mu - ieA_\mu)K|^2 + \tfrac{1}{2}\mu^2 K^*K - \tfrac{1}{4}\lambda(K^*K)^2,$$

with the ansatz

$$K = if(r)e^{ip\theta}, \quad r = (x_1^{\,2} + x_2^{\,2})^{1/2},$$

$$A_0 = 0, \quad A_3 = 0,$$

$$A_i = \epsilon_{ij}\frac{x^j}{r}g(r), \quad i, j = 1, 2$$

one obtains the equations of motion[8]

$$g'' + \frac{1}{r}g' - \frac{1}{r^2}g - ef^2\left(eg - \frac{p}{r}\right) = 0,$$

$$f'' + \frac{1}{r}f' - \left(eg - \frac{p}{r}\right)^2 + \mu f - \lambda f^3 = 0,$$

where p is the number of flux quanta in the magnetic vortex line. The field $g(r)$ has the asymptotic forms

$$g(r) \sim r^{2p+1}, \quad r \to 0$$

$$g(r) \sim -\frac{p}{er} + O\left(\frac{1}{\sqrt{r}}e^{-er}\right), \quad r \to \infty.$$

The Dirac equation for a fermion in the vortex line connects the spinor components in pairs:

$$ie^{-i\theta}\left[\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta} + eg\right]\begin{pmatrix}\psi_4\\ -\psi_2\end{pmatrix} = m\begin{pmatrix}\psi_1\\ \psi_3\end{pmatrix} - \omega\begin{pmatrix}\psi_1\\ -\psi_3\end{pmatrix},$$

$$ie^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta} - eg\right]\begin{pmatrix}\psi_3\\ -\psi_1\end{pmatrix} = m\begin{pmatrix}\psi_2\\ \psi_4\end{pmatrix} + \omega\begin{pmatrix}-\psi_2\\ \psi_4\end{pmatrix}.$$

Again we can show analytically that the only bound states are at threshold, $|\omega| = m$. Take $\psi_2 = \psi_3 = \psi_4 = 0$ and $m = \omega$. Then

$$e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta} - eg\right]\psi_1(r, \theta) = 0.$$

This separates trivially, and we find

$$\psi_1(r, \theta) = \text{const} \times e^{il\theta}r^l \exp\left(e\int_0^r g(r')dr'\right),$$

where $l = 0, \pm\frac{1}{2}, \pm 1, \ldots$ in order that ψ_1 be invariant under rotations of 4π. Further restrictions on l come from the boundary conditions at the origin and the requirement of normalizability. For $r^2|\psi_1|^2$ to be finite at the origin it is necessary that $l \geq -1$. As before, ψ_1 must go to zero faster than $r^{-3/2}$ as $r \to \infty$; $g(r) \sim -p/er$ as $r \to \infty$, so we require $l < p - \frac{3}{2}$.

It is interesting that for large p (large number of flux quanta in the vortex line), the bound states for various angular momenta $-1 \leq l \leq p - 2$ are degenerate in energy. There are no states more strongly bound, since for $l = 0$ $\psi_1(r, 0)$ has no nodes.

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[4] The Henyey method finds the solution to the difference equation generated by (1) on a finite set of discrete points. The equations are solved on a finite interval $[0, R]$, and boundary conditions are imposed by fixing the values of f and g at the end points: $f(0) = g(0) = 0$, $f(R) = 1$, $g(R) = -2/R$. R is chosen so that it falls well out in the asymptotic region.

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