# Comments and Addenda

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## Ward identities in gauge theories of gravitation with higher derivatives*

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The quantization of gauge theories of gravitation with $k^{-4}$ propagator is investigated with the functional integration method. It is found that gauge-fixing terms in these theories are quite different from those in ordinary gauge theories. Ward identities are then derived and used to show that there is no renormalization of the longitudinal parts of the two-point Green's functions, just as expected. It is also shown that in order to satisfy these identities the bare graviton propagator has to come from the term of the square of the Riemann curvature tensor in the Lagrangian.

Recent efforts by a number of authors[1] have shown that Einstein's theory of gravitation is not renormalizable at the one-loop level. Free gravitation leads to a renormalizable $S$ matrix at the one-loop calculation. However, the interacting version in which scalar fields, photon or fermion fields, or even Yang-Mills fields are included leads to counterterms in the one-loop calculation of the form $C^2{}_{\mu\nu\rho\sigma}$, the Weyl or conformal tensor, indicating nonrenormalizability. It is rather unlikely that inclusion of all possible interactions in the calculation of the single loop would result in a value of zero for the coefficient of the offending term.

Forced with the almost certainty that Einstein's gravitational equations are not renormalizable, it is of paramount interest to consider alternative theories that might reward us with the gift of renormalizability. Such theories include among others the recent suggestion by Yang[2] or theories with Lagrangians of the form $aR_{\mu\nu}{}^2+bR^2+cR$.[3]

Theories of this form, however, almost invariably are higher than quadratic in derivatives and their quantization is not at all clear. Questions such as the identification of the propagator, irreducible vertex functions, gauge-fixing terms, and the nature of the ghost propagator come immediately to sharp focus as soon as one attempts any quantization scheme that would lead to a consistent perturbation expansion.

An obvious way of dealing with such a problem is the determination through functional integration and methods established by Faddeev and Popov of the Ward identities as a tool towards the problem of internal consistency. Therefore, we begin by defining the generating functional in the usual way:

$$W[J]=\int [dh]\exp\left\{i\int [L(\bar{g})+h_{\mu\nu}J^{\mu\nu}-\tfrac{1}{2}F^2]d^4x\right\}\Delta[h]\,, \tag{1}$$

where $h_{\mu\nu}$ is defined by

$$\bar{g}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}$$

with $g_{\mu\nu}$ the flat-space metric, and the terms $\frac{1}{2}F^2$ and $\Delta[h]$ denoting the gauge-fixing term and ghost Lagrangian, respectively.

In this particular case the form of $F^2$ can be determined by imposing the requirement that $F^2$ be a scalar of dimension four, since terms with dimension higher than four would definitely invalidate possible renormalizability. This implies that, since $h_{\mu\nu}$ is dimensionless in these theories, $F^2$ must be quartic in derivatives.

By choosing the gauge condition

$$C_\mu \equiv \partial^\rho h_{\rho\mu}=0$$

in order to quantize the theory, there are two possible invariant forms, namely $(\partial^\mu C_\mu)^2$ and $(\partial_\mu C_\nu)^2$, that fulfill the above requirement. Therefore the most general form that $F^2$ can take is

$$F^2 \equiv \frac{1}{\alpha}(\partial^\mu C_\mu)^2 - \frac{2}{\beta}(\partial_\mu C_\nu)^2, \tag{2}$$

where $\alpha$ and $\beta$ are the two gauge-fixing parameters.

Here, we immediately recognize that the structure of the gauge-fixing term is quite different from that of the ordinary gauge theories, where the gauge-fixing term is just the square of the gauge function. It is peculiar only to this particular kind of gauge theory in which the Lagrangian is a quadratic function of the Riemann tensor or of the curvature scalar.

Under the general covariance transformation

$$h_{\mu\nu} \to h_{\mu\nu} + D_\mu\lambda_\nu + D_\nu\lambda_\mu, \tag{3}$$

where $D_\mu$ denotes the covariant derivative, $\Delta[h]$ is determined from the gauge function $C_\mu$ to be

$$\Delta[h] = \det\left(\frac{\delta C_\sigma}{\delta\lambda_\tau}\right) \equiv \det M_{\tau\sigma}, \tag{4}$$

where

$$M_{\tau\sigma} = \partial^\rho D_\rho g_{\tau\sigma} + \partial_\tau D_\sigma .$$

From the point of view of Feynman diagrams the determinant $\Delta[h]$ is a sum of closed loops with respect to which a vector fermion of vanishing mass propagates. In those terms the functional $W[J]$ can be expressed as an integral of a local action by introducing the ghost vector field $A_\rho$:

$$W[J] = \int [dh][d\bar{A}][dA] \times \exp\left\{ i\int [L(g) + h_{\mu\nu}J^{\mu\nu} - \tfrac{1}{2}F^2 + \bar{A}_\rho M^{\rho\sigma}A_\sigma]d^4x\right\}. \tag{5}$$

Now, with respect to Eq. (4) the measure $\Delta[h]dh$ and the Lagrangian are invariant[4] under the gauge tranformation of Eq. (3). Performing this transformation to the generating functional, with the parameters $\lambda_\rho$ defined by

$$M_{\rho\sigma}\lambda^\sigma = \Lambda_\rho ,$$

where $\Lambda_\rho$ are arbitrary functions, one obtains[5]

$$\left[-F_\rho\left(\frac{1}{i}\frac{\partial}{\partial J}\right) + J^{\mu\nu}\frac{\delta h_{\mu\nu}}{\delta\lambda_\sigma}\left(\frac{1}{i}\frac{\partial}{\partial J}\right)M^{-1}{}_{\sigma\rho}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]W[J] = 0 \tag{6}$$

as a result of the variation of the functional with respect to $\Lambda_\rho$. Here

$$F_\rho \equiv \frac{\delta}{\delta\Lambda_\rho}(\tfrac{1}{2}F^2) = -\frac{1}{\alpha}\partial_\rho\partial^\mu C_\mu + \frac{2}{\beta}\Box C_\rho$$

and

$$\frac{\delta h_{\mu\nu}}{\delta\lambda_\sigma} = D_\mu\delta^\sigma_\nu + D_\nu\delta^\sigma_\mu ,$$

where use is made of Eqs. (2) and (3). Equation (6) is in a sense the master Ward identity from which relations among Green's functions can be obtained.

Differentiating Eq. (6) with respect to $J_{\mu\nu}(y)$ and setting $J_{\mu\nu} = 0$, then going over to momentum space, one obtains

$$-\left(\frac{1}{\alpha}k_\rho k_\sigma k_\eta - \frac{2}{\beta}k^2 k_\rho g_{\eta\sigma}\right)P^{\rho\sigma}{}_{\mu\nu}(k^2) = k_\mu D^0_{\eta\nu}(k^2) + k_\nu D^0_{\eta\mu}(k^2), \tag{7}$$

where

$$D^0_{\eta\nu} = \frac{g_{\eta\nu} - k_\eta k_\nu/2k^2}{k^2} .$$

The right-hand side of Eq. (7) is essentially the term $(\delta h_{\mu\nu}/\delta\lambda_\sigma)M^{-1}{}_{\sigma\rho}$ appearing in Eq. (6), in momentum space, and can be obtained by noticing that $M_{\mu\beta} = \partial^\nu(\delta h_{\mu\nu}/\delta\lambda_\beta)$ and the identity

$$MM^{-1} = 1 .$$

We recognize immediately that $D^0_{\eta\nu}(k^2)$ is just the bare vector ghost propagator obtained from inverting the quadratic form $k_\mu k_\nu + k^2 g_{\mu\nu}$ in the ghost Lagrangian. Also note that the function $P_{\rho\sigma,\mu\nu}(k^2)$ is the graviton propagator.

Equation (7) tells us that there is no renormalization of the part of the two-point Green's function, which is longitudinal in all four indices, and therefore difficulties associated with unphysical polarizations are taken care of by the vector ghost particle in the usual way as in ordinary gauge theories. As far as the unitarity of the $S$ matrix is concerned there might be inherent ghost problems associated with the $k^{-4}$ behavior of the graviton propagator which must be investigated separately.[6]

Since Eq. (7) is satisfied order by order in perturbation theory, it would be of interest to examine it for specific examples. However, the extremely complicated nature of these theories makes the calculation of even the lowest correction loop to the graviton propagator utterly unattenable. In this respect even the calculation of the bare graviton propagator is time consuming but otherwise straightforward. For instance, if one uses as a Lagrangian the following expression

$$L = -\tfrac{1}{2}R^\mu{}_{\nu\alpha\beta}R^\nu{}_{\mu\rho\sigma}\bar{g}^{\alpha\rho}\bar{g}^{\beta\sigma}(-\bar{g})^{1/2}, \tag{8}$$

where $R^\mu{}_{\nu\alpha\beta}$ is the Riemann curvature tensor, one obtains for the quadratic form

$$W_{\mu\nu,\rho\sigma} = k^4 I_{\mu\nu,\rho\sigma} - k^2(g_{\mu\rho}k_\sigma k_\nu + g_{\nu\sigma}k_\mu k_\rho)\left(1-\frac{1}{\beta}\right) + \left(1-\frac{1}{\alpha}\right)k_\mu k_\nu k_\rho k_\sigma\,, \qquad (9)$$

where $I_{\mu\nu,\rho}$ the identity is defined by

$$I_{\mu\nu,\rho\sigma} = \tfrac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\nu\rho}g_{\mu\sigma})\,.$$

Note that the gauge-fixing terms have been included in the quadratic term. Inversion of Eq. (9) by requiring

$$W_{\mu\nu,\rho\sigma}P^{0\,\rho\sigma}{}_{\eta\xi} = -I_{\mu\nu,\eta\xi}$$

leads to the bare graviton propagator

$$P^0_{\rho\sigma,\eta\xi} = \frac{1}{k^4}I_{\rho\sigma,\eta\xi} + \frac{B}{2k^6}(g_{\rho\eta}k_\sigma k_\xi + g_{\sigma\xi}k_\rho k_\eta) + \frac{B}{2k^6}(g_{\rho\xi}k_\sigma k_\eta + g_{\eta\sigma}k_\rho k_\xi) + \frac{D}{k^8}k_\rho k_\sigma k_\eta k_\xi$$

with

$$B = (\beta - 1)$$

and

$$D = \frac{(1-1/\alpha) + (1/\beta - 1/\alpha)2(\beta-1)}{(1/\alpha - 2/\beta)}\,.$$

The Feynman gauge would correspond to the choices of $\alpha = \beta = 1$ and the Landau gauge is the one for $\beta \to 0$, and any $\alpha$. A unitary gauge can also be chosen by taking $C_\mu = \partial^i h_{i\mu}$, where the index $i$ runs over 1, 2, 3, and taking the limit $\alpha \to 0$, $\beta \to 0$. In this case the propagator has a pole at $k^4 = 0$ only for the physical polarizations, while the rest of the terms of the propagator have no pole and therefore do not contribute to the absorptive part of the $S$ matrix.

As a result of the application of the operator $(1/\alpha)k_\rho k_\sigma k_\eta - (2/\beta)k^2 k_\rho g_{\eta\sigma}$ of Eq. (7) to $P^0_{\mu\nu,\eta\xi}$, we obtain

$$\left(\frac{1}{\alpha}k_\rho k_\sigma k_\eta - \frac{2}{\beta}k^2 k_\rho g_{\sigma\eta}\right)P^{0\,\rho\sigma}{}_{\mu\nu}(k^2) = \frac{k_\eta k_\mu k_\nu}{k^4}\left[\frac{1+2(\beta-1)+D}{\alpha} - \frac{2(D+\beta-1)}{\beta}\right] - \frac{1}{k^2}(k_\mu g_{\eta\nu} + k_\nu g_{\eta\mu}) = -[k_\mu D^0_{\eta\nu}(k^2) + k_\nu D^0_{\eta\mu}(k^2)]\,.$$

In this respect, therefore, the propagator obtained from the Langrangian of Eq. (8) satisfies the derived Ward identity at least to zeroth order.

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