Bound states of the quasipotential equation for two spin-$\frac{1}{2}$ particles in strongly coupled quantum electrodynamics

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The bound states of two spin-$\frac{1}{2}$ particles in strongly coupled quantum electrodynamics are examined using a local quasipotential equation recently proposed by Todorov. Because of repulsive recoil effects displayed by the spatial part of the vector potential, complex energies do not appear for large α as with the static-limit equations, and very deep binding is obtained for α sufficiently large. No cutoffs or charge distributions need to be introduced to obtain this result. The generalized $^1S_0$ positronium system examined here appears to be confined to a thin spherical shell in this deep-binding limit.

I. INTRODUCTION: QUASIPOTENTIAL EQUATION FOR TWO SPIN-$\frac{1}{2}$ PARTICLES

Using a recently proposed formulation of the two-body bound-state problem in quantum electrodynamics,[1] we examine the consequences of allowing the coupling constant α to become arbitrarily large. In the standard static-limit equations, such as the Klein-Gordon equation and the Dirac equation, complex energies set in for α of the order of 1. This does not occur if a realistic charge distribution is empirically given to the heavy nucleus. In this paper, we demonstrate that such complex energies do not appear when the recoil effects are taken into account for the case of two equal-mass point particles. The mechanism is the spatial component of the four-vector potential which, in the second-order form of the equation studied here, provides a very singular but repulsive $1/r^4$ behavior at the origin.

The particular system we examine is basically a generalized $^1S_0$ positronium atom in the ground state. By "generalized" we mean that we are not considering real positronium, for which α is the fixed number $\frac{1}{137}$. The coupling constant is allowed to take on arbitrary values between 10^{-1} and 10^4. The basic equation we shall work with is a local relativistic Schrödinger equation for two spin-$\frac{1}{2}$ particles. Todorov's original formulation of this equation involved a 16-component wave function. A 4×4 matrix formulation of this equation was given recently by the present author.[2] In this paper, we shall use a covariant four-component form recently proposed by Aneva, Karchev, and Rizov.[3] The rest of this section contains a review of some aspects of their work and indicates how we shall use the resultant quasipotential for large α.

In the next section (II) the main result of the paper is presented. After a discussion of the spectrum for the case of singlet "positronium" without the $\vec{V}^2$ term and a review of the physical significance of the appearance of complex energies for large α, the spectrum and wave function with full gauge invariance are given. The recoil effects displayed by the vector potential $\vec{V}$, together with the $1/r^4$ radial dependence for $\vec{V}^2$, lead to a hard core that grows outward for increasing α. As α becomes very large, the limiting configuration of this generalized positronium atom is a thin spherical shell.

Our motivation for considering strongly coupled QED is based in part on Schwinger's dyon hypothesis for hadrons.[4] Consequently the charge we speak of in this paper may be regarded as a general electric or magnetic charge and the bound-state configuration found may be regarded, in the sense to be defined below, as a nonperturbative approximation to the distribution of such constituents in hadrons. The results of our paper are also relevant for the vector-gluon model.

A. Helicity form of the homogeneous equation for bound states

We consider the elastic scattering of two spin-$\frac{1}{2}$ particles of mass m_1 and m_2 with opposite charge. Their momenta and spin projections are displayed in the following relation between the S matrix and the T matrix:

$$\begin{aligned}\langle q_1,\kappa_1;\, q_2,\kappa_2|S|p_1,\lambda_1;p_2,\lambda_2\rangle &= 4p_1^0p_2^0(2\pi)^6\delta(\vec{p}_1-\vec{q}_1)\delta(\vec{p}_2-\vec{q}_2)\delta_{\lambda_1\kappa_1}\delta_{\lambda_2\kappa_2}\\ &\quad+(2\pi)^4 i\,\delta(p_1+p_2-q_1-q_2)\\ &\quad\times T(q_1\kappa_1,q_2\kappa_2;\,p_1\lambda_1,p_2\lambda_2)\,. \qquad (1.1)\end{aligned}$$

The label 1 refers to the negatively charged particle and the label 2 to the positively charged particle. For the initial spin projections we use the variable λ and for the final ones we use the variable κ. The quasipotential equation to be written down below is an off-mass-shell, on-energy-shell equation. Going off-shell is done in such a way

that

$$p_1^2 - p_2^2 = q_1^2 - q_2^2 = m_2^2 - m_1^2 \ . \tag{1.2}$$

The reader is referred to an article by Crater and Naft for a detailed examination of the consequences of this assumption in a classical context.[5] The total four-momentum is

$$P = p_1 + p_2 = q_1 + q_2 \ , \tag{1.3}$$

and in the center-of-mass (c.m.) frame it has only a time component:

$$P = (E_1 + E_2, \vec{0}) = (w, \vec{0}), \quad w = (-P^2)^{1/2} \ . \tag{1.4}$$

The variables E_1 and E_2 are defined in an invariant way by

$$\begin{aligned} E_1 &= \frac{-p_1 \cdot P}{w} = \frac{-q_1 \cdot P}{w} = \frac{w^2 + m_1^2 - m_2^2}{2w} \ , \\ E_2 &= \frac{-p_2 \cdot P}{w} = \frac{-q_2 \cdot P}{w} = \frac{w^2 + m_2^2 - m_1^2}{2w} \end{aligned} \tag{1.5}$$

and the relative momenta p and q are given by

$$\begin{aligned} p &= \frac{E_2 p_1 - E_1 p_2}{w} \ , \\ q &= \frac{E_2 q_1 - E_1 q_2}{w} \ . \end{aligned} \tag{1.6}$$

On the mass shell, their common c.m. magnitude is

$$p^2 = q^2 = b^2 = \frac{w^2}{4} - \frac{m_1^2 + m_2^2}{2} + \frac{(m_1^2 - m_2^2)^2}{4w^2} \ . \tag{1.7}$$

By definition, then, the vectors p and q are orthogonal to P. In the c.m. frame,

$$\begin{aligned} p_1 &= (E_1, \vec{p}), \quad p_2 = (E_2, -\vec{p}) \ , \\ q_1 &= (E_1, \vec{q}), \quad q_2 = (E_2, -\vec{q}) \ , \end{aligned} \tag{1.8}$$

and the relative momenta have only spatial components:

$$\begin{aligned} p &= (0, \vec{p}), \\ q &= (0, \vec{q}) \ . \end{aligned} \tag{1.9}$$

Given these preliminaries, we postulate the following relativistic Lippmann-Schwinger equation:

$$T(q_1\kappa_1, q_2\kappa_2; p_1\lambda_1, p_2\lambda_2) + V(q_1\kappa_1, q_2\kappa_2; p_1\lambda_1, p_2\lambda_2) + \int\int d^4k_1 d^4k_2 V(q_1\kappa_1, q_2\kappa_2; k_1\mu_1, k_2\mu_2) G(k_1k_2, \mu_1\mu_2, \mu_1'\mu_2') T(k_1\mu_1', k_2\mu_2'; p_1\lambda_1, p_2\lambda_2) = 0 \ . \tag{1.10}$$

The Green's function G is taken to be

$$G(k_1k_2, \mu_1\mu_2, \mu_1'\mu_2') = \frac{\delta(k_1 + k_2 - P)\delta(k \cdot P)\delta\mu_1\mu_1'\delta\mu_2\mu_2'}{4\mu\left(\frac{k_1^2 + m_1^2}{2m_1} + \frac{k_2^2 + m_2^2}{2m_2} - \iota\epsilon\right)} \ . \tag{1.11}$$

The relative variable k is defined as

$$k = \frac{E_2 k_1 - E_1 k_2}{w} \tag{1.12}$$

and μ is the reduced mass $m_1 m_2/M$. The total mass is $M = m_1 + m_2$. There are two δ functions in the Green's function. The first imposes momentum conservation for the intermediate state and, together with the second, is sufficient to enforce the on-energy-shell assumption in the c.m. frame for the intermediate variables.

The above quasipotential equation is completely Lorentz invariant. In the c.m. frame we write T and V in the form

$$T = T_w(\vec{q}, \vec{p}; \kappa_1, \kappa_2; \lambda_1, \lambda_2), \quad V = V_w(\vec{q}, \vec{p}; \kappa_1, \kappa_2; \lambda_1, \lambda_2) \ , \tag{1.13}$$

and the sum that appears in the denominator of the Green's function is proportional to

$$\frac{w}{M}\left(\frac{k_1^2 + m_1^2}{2m_1} + \frac{k_2^2 + m_2^2}{2m_2}\right) = \frac{\vec{k}^2 - b^2}{2m_w} \tag{1.14}$$

where m_w is the relativistic reduced mass variable

$$m_w = \frac{m_1 m_2}{w} \ . \tag{1.15}$$

Hence the quasipotential equation takes on the form

$$T_w(\vec{q},\vec{p};\kappa_1,\kappa_2;\lambda_1,\lambda_2)+V_w(\vec{q},\vec{p};\kappa_1,\kappa_2;\lambda_1,\lambda_2)+\int\frac{d^3k}{(2\pi)^3}\frac{V_w(\vec{q},\vec{k};\kappa_1,\kappa_2;\mu_1,\mu_2)T_w(\vec{k},\vec{p};\mu_1,\mu_2;\lambda_1,\lambda_2)}{2w(\vec{k}^2-b^2-i\epsilon)}=0 \qquad (1.16)$$

in the c.m. frame.

This equation defines in a perturbative way the quasipotential V in terms of the scattering amplitude T:

$$\begin{aligned} V_1&=-T_1,\\ V_2&=-T_2+T_1GT_1\ . \end{aligned} \qquad (1.17)$$

For Hermitian V, this T satisfies the on-mass-shell elastic unitarity condition

$$T_w(\vec{q},\vec{p};\kappa_1,\kappa_2;\lambda_1,\lambda_2)-T_w^*(\vec{q},\vec{p};\kappa_1,\kappa_2;\lambda_1,\lambda_2)$$
$$=\pi i\int\frac{d^3k}{w}\,T_w^*(\vec{q},\vec{k};\kappa_1,\kappa_2;\mu_1,\mu_2)T_w(\vec{k},\vec{p};\mu_1,\mu_2;\lambda_1,\lambda_2)\delta(\vec{k}^2-b^2)\ . \qquad (1.18)$$

This can be seen by formally solving $T+V+VGT=0$. One finds[1]

$$T-T^*=T^*(G-G^*)T\ , \qquad (1.19)$$

which leads to (1.18) and

$$G_w(\vec{k})=\frac{1}{2w}\frac{1}{\vec{k}^2-b^2-i\epsilon}\ . \qquad (1.20)$$

In Appendix A we show that (1.16) leads to the following homogeneous quasipotential equation (or relativistic Schrödinger equation) for two spin-$\frac{1}{2}$ particles:

$$G_w^{-1}(\vec{q})\Phi_w(\vec{q};\kappa_1,\kappa_2)+\int\frac{d^3k}{(2\pi)^3}V_w(\vec{q},\vec{k};\kappa_1,\kappa_2;\mu_1,\mu_2)\Phi_w(\vec{k};\mu_1,\mu_2)=0\ . \qquad (1.21)$$

The momentum-space wave function is $\Phi_w(\vec{q};\kappa_1,\kappa_2)$, and its relation to T and G is given in Appendix A also.

B. Four-component forms of the homogeneous equation for two spin-½ particles

The matrix form of the quasipotential is defined by

$$V_w(\vec{q},\vec{k};\kappa_1,\kappa_2;\mu_1,\mu_2)=\bar{u}_{\kappa_1}(\vec{q})\bar{u}_{\kappa_2}(-\vec{q})V_w(\vec{q},\vec{k})u_{\lambda_2}(-\vec{p})u_{\lambda_1}(\vec{p})$$
$$=\bar{u}_{\kappa_1}(\vec{q})_{\alpha_1}\bar{u}_{\kappa_2}(-\vec{q})_{\alpha_2}V_w(\vec{q},\vec{k})^{\alpha_1\alpha_2}_{\beta_1\beta_2}u_{\lambda_2}(-\vec{p})_{\beta_2}u_{\lambda_1}(\vec{p})_{\beta_1}\ . \qquad (1.22)$$

The indices α and β go from 1 to 4 and the quasipotential $V_w(\vec{q},\vec{k})$ is a 16×16 matrix. The spinors are given in the Pauli representation by

$$\begin{aligned} u_{\lambda_1}(\vec{p})&=\begin{pmatrix}(E_1+m_1)^{1/2}e_{\lambda_1}\\ (E_1-m_1)^{1/2}(\vec{p}\cdot\vec{\sigma}/|\vec{p}|\,)e_{\lambda_1}\end{pmatrix},\\ u_{\lambda_2}(-\vec{p})&=\begin{pmatrix}(E_2+m_2)^{1/2}e_{\lambda_2}\\ -(E_2-m_2)^{1/2}(\vec{p}\cdot\vec{\sigma}/|\vec{p}|\,)e_{\lambda_2}\end{pmatrix}, \end{aligned} \qquad (1.23)$$

where e_λ ($\lambda=\pm\frac{1}{2}$) are two-component spinors, eigenvectors of σ_3

$$\sigma_3e_\lambda=2\lambda e_\lambda,\quad \lambda=\pm\tfrac{1}{2} \qquad (1.24)$$

normalized by

$$\bar{e}_\kappa e_\lambda=\delta_{\kappa\lambda}\ . \qquad (1.25)$$

In this representation

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \vec{\gamma}=\begin{pmatrix}0&\vec{\sigma}\\-\vec{\sigma}&0\end{pmatrix}. \qquad (1.26)$$

Given these preliminaries, a four-component quasipotential U_w is defined by[3]

$$V_w(\vec{q},\vec{k};\kappa_1,\kappa_2;\lambda_1,\lambda_2)=\bar{e}^{a_1}_{\kappa_1}\bar{e}^{a_2}_{\kappa_2}U_w(\vec{q},\vec{k})^{b_1\,b_2}_{a_1\,a_2}e^{b_1}_{\lambda_1}e^{b_2}_{\lambda_2}\ . \tag{1.27}$$

The homogeneous equation (1.21) now takes the form

$$2w(\vec{q}^2-b^2)\phi_w(\vec{q})+\int\frac{d^3k}{(2\pi)^3}U_w(\vec{q},\vec{k})\phi_w(\vec{k})=0\ , \tag{1.28}$$

where ϕ is a four-component wave function.

The quasipotential to be used arises from the exchange diagram:

$$V_w^{(1)}(\vec{q},\vec{k})=\frac{-e_1e_2\gamma_\mu^{(1)}\gamma^{(2)\mu}}{(\vec{q}-\vec{k})^2}$$
$$\equiv\frac{4\pi\alpha\gamma_\mu^{(1)}\gamma^{(2)\mu}}{(\vec{q}-\vec{k})^2}\ . \tag{1.29}$$

Using (1.23), (1.26), (1.27), and (1.29) leads to[3]

$$U_w^{(1)}(\vec{q},\vec{k})=\frac{-4\pi\alpha}{(\vec{q}-\vec{k})^2}[4Ew-(\vec{q}-\vec{k})^2]+4\pi\alpha\left[\frac{E_1}{E_2+m_2}+\frac{E_2}{E_1+m_1}-\frac{(\vec{q}-\vec{k})^2}{4(E_1+m_1)(E_2+m_2)}\right]$$
$$-\frac{8i\pi\alpha}{(\vec{q}-\vec{k})^2}\left[1+\frac{E_2}{E_1+m_1}-\frac{(\vec{q}-\vec{k})^2}{4(E_1+m_1)(E_2+m_2)}\right](\vec{q}\times\vec{k})\cdot\vec{\sigma}_1$$
$$-\frac{8i\pi\alpha}{(\vec{q}-\vec{k})^2}\left[1+\frac{E_1}{E_2+m_2}-\frac{(\vec{q}-\vec{k})^2}{4(E_1+m_1)(E_2+m_2)}\right](\vec{q}\times\vec{k})\cdot\vec{\sigma}_2$$
$$+4\pi\alpha\left[\delta_{ij}-\frac{(q-k)_i(q-k)_j}{(\vec{q}-\vec{k})^2}+\frac{1}{(E_1+m_1)(E_2+m_2)}\frac{(\vec{q}\times\vec{k})_i(\vec{q}\times\vec{k})_j}{(\vec{q}-\vec{k})^2}\right]\sigma_{1i}\sigma_{2j}\ , \tag{1.30}$$

where

$$\vec{\sigma}_1=\vec{\sigma}\times1,\quad\vec{\sigma}_2=1\times\vec{\sigma},\quad[\vec{\sigma}_{1i},\vec{\sigma}_{2j}]=0\ . \tag{1.31}$$

The variable E is the energy of the effective particle and is defined by

$$Ew=E_1E_2+b^2=\frac{w^2-m_1^2-m_2^2}{2}\ . \tag{1.32}$$

Another form is

$$E^2=m_w^2+b^2\ . \tag{1.33}$$

This form emphasizes its role as the energy of the effective particle. Following Todorov[1] and the work in Ref. 3 we modify (1.28) to take into account gauge invariance. The modified equation has the operator form

$$[(E-V^0)^2-(\vec{p}-\vec{V})^2-m_w^2+D+\vec{F}_1\cdot\vec{\sigma}_1+\vec{F}_2\cdot\vec{\sigma}_2+\sigma_{1i}H_{ij}\sigma_{2j}]\psi_w=0\ . \tag{1.34}$$

The operators V^μ, D, F_i, and H_{ij} can be identified by equating the $(\vec{q},\vec{k})$ matrix elements of terms linear in α to U_w (see Ref. 3):

$$2EV^0(\vec{q},\vec{k})-(\vec{q}+\vec{k})\cdot\vec{V}(\vec{q},\vec{k})-D(\vec{q},\vec{k})-\vec{F}_1(\vec{q},\vec{k})\cdot\vec{\sigma}_1-F_2(\vec{q},\vec{k})\cdot\vec{\sigma}_2-\sigma_{1i}H_{ij}(\vec{q},\vec{k})\sigma_{2j}=\frac{1}{2w}U_w^{(1)}(\vec{q},\vec{k})\ . \tag{1.35}$$

The operators F and H are readily identifiable, but there is no unique way of identifying V^μ and D. Following Todorov,[1] we make the choice of gauge where V^0 is the Coulomb potential:

$$V^0=\frac{-4\pi\alpha}{(\vec{q}-\vec{k})^2}\ ,$$
$$\vec{V}=\frac{-2\pi\alpha(\vec{q}+\vec{k})}{w(\vec{q}+\vec{k})^2}\ . \tag{1.36}$$

In coordinate space

$$V^0 = \frac{-\alpha}{r}\ ,$$
$$\vec{V} = \frac{-i\alpha}{2w}\frac{\vec{r}}{r^3} I_S \tag{1.37}$$

where I_S is the space-reflection operator

$$I_S f(\vec{r}) = f(-\vec{r})\ . \tag{1.38}$$

The remaining operator D can then be evaluated from (1.30), (1.35), and (1.36).[3]

In obtaining the coordinate form of (1.34) it is important to note that

$$\nabla_i \nabla_j \frac{1}{r} = \frac{3r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3} - \frac{4\pi}{3}\delta(\vec{r})\delta_{ij}\ . \tag{1.39}$$

We find

$$\begin{aligned}
&\Bigg\{ -\vec{\nabla}^2 - \frac{2E\alpha}{r} - \frac{\alpha^2}{r^2} + \frac{2\pi\alpha\delta(\vec{r})}{w} + \frac{\alpha^2}{4w^2 r^4} + \frac{2\pi\alpha\delta(\vec{r})}{w}\left(\frac{E_1}{E_2+m_2} + \frac{E_2}{E_1+m_1}\right) + \frac{\pi\alpha\delta(\vec{r})}{w}\frac{1}{(E+m_1)(E_2+m_2)}\vec{\nabla}^2 \\
&+ \frac{\alpha}{wr^3}\vec{L}\cdot(\vec{\sigma}_1+\vec{\sigma}_2) + \frac{\alpha}{wr^3}\vec{L}\cdot\left(\frac{E_2}{E_1+m_1}\vec{\sigma}_1 + \frac{E_1}{E_2+m_2}\vec{\sigma}_2\right) + \frac{4\pi\alpha}{3w}\delta(\vec{r})\vec{\sigma}_1\cdot\vec{\sigma}_2 + \frac{\alpha}{2w}\frac{1}{r^3}\left(\frac{3\vec{r}\cdot\vec{\sigma}_1\,\vec{r}\cdot\vec{\sigma}_2}{r^2} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right) \\
&+ \frac{\alpha}{2w}\frac{1}{(E_1+m_1)(E_2+m_2)}\Bigg[\frac{1}{2r^5}(\vec{\sigma}_1\cdot\vec{\sigma}_2 - \{\vec{L}\cdot\vec{\sigma}_1, \vec{L}\cdot\vec{\sigma}_2\}) - \frac{1}{4}\Bigg(\vec{\sigma}_1\cdot\vec{\sigma}_2\Bigg[\Big\{\frac{1}{r^3} + \frac{4\pi}{3}\delta(\vec{r}), \vec{\nabla}^2\Big\} + 2\nabla_i\left(\frac{1}{r^3} + \frac{4\pi}{3}\delta(\vec{r})\right)\nabla_i\Bigg] \\
&\qquad - \Big\{\vec{\sigma}_1\cdot\vec{\nabla}\vec{\sigma}_2\cdot\vec{\nabla}, \frac{1}{r^3} + \frac{4\pi}{3}\delta(\vec{r})\Big\} \\
&\qquad - \sigma_1\cdot\vec{\nabla}\left(\frac{1}{r^3} + \frac{4\pi}{3}\delta(\vec{r})\right)\vec{\sigma}_2\cdot\vec{\nabla} \\
&\qquad - \vec{\sigma}_2\cdot\vec{\nabla}\left(\frac{1}{r^3} + \frac{4\pi}{3}\delta(\vec{r})\right)\vec{\sigma}_1\cdot\vec{\nabla}\Bigg)\Bigg]\Bigg\}\psi_w(\vec{r}) = b^2\psi_w(\vec{r})\ .
\end{aligned} \tag{1.40}$$

Our aim in this paper is to examine the high-α nonperturbative form of the spectrum for a generalized singlet-positronium atom. Although dealing with a particle bound state, we shall not be concerned with the $^3S_1$ solution which needs the annihilation contribution.[6] From a perturbative study of this equation in Ref. 3, it can be easily shown that the only part of the quasipotential equation (1.40) that contributes to order α^4 to the $^1S_0$ state is

$$-\nabla^2 - \frac{2E\alpha}{r} - \frac{\alpha^2}{r^2}\psi_w(\vec{r}) = b^2\psi_w(\vec{r})\ . \tag{1.41}$$

This simplification occurs to this order in part because the $\vec{L}\cdot\vec{\sigma}_i$ terms and the tensor terms do not contribute to the singlet S state and for equal masses the spin-independent contact terms in line 1 cancel cancel with the spin-spin contact term in line 2. The remaining terms do not contribute to order α^4.[3]

On the other hand, the spectrum from this eigenvalue equation can be determined exactly and leads to[1,7]

$$w_{nl} = \Bigg\{2m^2\Bigg[1 + \Bigg(1 + \frac{\alpha^2}{(n+\frac{1}{2}\{[(2l+1)^2 - 4\alpha^2]^{1/2} - 2l - 1\})^2}\Bigg)^{-1/2}\Bigg]\Bigg\}^{1/2} \tag{1.42}$$

What significance can we ascribe to this spectrum for large α? We regard the appearance of complex energies as a deficiency in the quasipotential given in (1.41) for large α. Beyond this, however, there is the larger question of the significance of a nonperturbative calculation of a spectrum based on a first-order diagram.

Solving for the bound-state eigenvalues of the homogeneous quasipotential equation "exactly" (by numerical methods if necessary) is equivalent to finding the poles of an infinite sum of iterations of a diagram of the type represented by the quasipotential.[8] In this sense of the word, such a solution represents a nonperturbative approximation to the true spectrum of the given field theory. It is conceivable that the qualitative aspects of such

an approximation to the spectrum and the resultant bound-state configuration might not be drastically altered by radiative corrections.

In a recent paper, Atkinson and Crater examined nonperturbative corrections to the spectrum of a pure scalar version of QED (scalar photons and particles).[9] Radiative corrections were incorporated nonperturbatively by a Padé approximant evaluation of the quasipotential. For the range of coupling considered it was found that this modified quasipotential produced no significant departure from the nature of the spectrum. By and large, it remained of the Coulomb type in the range up to $\alpha \sim 1$. Intrinsic coupling-constant limitations inherent in that model prevented considerations of larger α. Such limitations on the coupling-constant size are also present here, as is evident in the spectrum (1.42).

No large-α calculation can be done unless the problem of complex energies evident in (1.42) is eliminated. It is desirable to do this in a way that does not involve an *ad hoc* cutoff or charge distribution. Ideally such a crucial and drastic qualitative modification of the spectrum and potential would arise naturally from the theory. We claim that the mechanism for such a modification comes from the imposition of gauge invariance.

In Eq. (1.40) the highly repulsive energy-dependent term

$$\vec{V}^2 = \frac{\alpha^2}{4w^2r^4} \tag{1.43}$$

arises as a consequence of imposing gauge invariance. This term is also common to the quasipotential in interactions between spin-0 and spin-$\frac{1}{2}$, and spin-0 and spin-0 particles.[1] For small α, it does not contribute to order α^4 in the spectrum. In fact, for S waves, its perturbative contribution is not even defined. However, as we shall see below, its nonperturbative influences are well defined and play a crucial role in altering the qualitative nature of the spectrum. In particular, it eliminates the appearance of complex energies and leads to very deep binding for large α.

In this paper, radiative corrections will not be considered. One might speculate on the basis of the results of Atkinson and Crater[9] that these corrections would not have significant additional qualitative effects on the spectrum and the bound-state configuration. There are other terms in (1.40) that could also be considered. However, since the term (1.43) is spin independent and related to maintaining gauge invariance, we shall in this work restrict ourselves to it and to numerical solutions of the ground state of the quasipotential in Eq. (1.41) with this term added in.

II. GAUGE INVARIANCE AND DEEP BINDING

A. Spectrum without terms due to gauge invariance

On the basis of our discussions in Sec. I, the equation for singlet "positronium" we shall examine is

$$\left(-\vec{\nabla}^2 - \frac{2E\alpha}{r} - \frac{\alpha^2}{r} + \frac{\alpha^2}{4w^2r^4}\right)\psi_w(\vec{r}) = b^2\psi_w(\vec{r}) . \tag{2.1}$$

The attractive $1/r^2$ part of the potential arises from the V^{0^2} term that comes from imposing gauge invariance. This term, of course, is the only one of such origin in the static-limit equation. The next term, the repulsive $1/r^4$ term, arises from the $\vec{V}^2$ contribution and comes from including the vector part of the potential. This term is zero in the static limit. Without these two terms one has the hydrogen-type equation

$$\left(-\vec{\nabla}^2 - \frac{2E\alpha}{r}\right)\psi_w(\vec{r}) = b^2\psi_w(\vec{r}) . \tag{2.2}$$

Writing this equation as

$$\left(\frac{-\nabla^2}{2E} - \frac{\alpha}{r}\right)\psi_w(\vec{r}) = \frac{b^2}{2E}\psi_w(\vec{r}) \tag{2.3}$$

allows us to use the analogy with hydrogen to write the solution to the eigenvalue problem as

$$\frac{b^2}{2E} = \frac{-E\alpha^2}{2n^2} . \tag{2.4}$$

Now $E^2 = m_w^2 + b^2$ and hence

$$E^2 = \frac{m_w^2}{1+\alpha^2/n^2} = \left(\frac{w^2-2m^2}{2w}\right)^2 \tag{2.5}$$

or

$$w^2 = 2m^2\left(1 + \frac{1}{(1+\alpha^2/n^2)^{1/2}}\right), \tag{2.6}$$

which displays an O(4) symmetry typical of hydrogen. For the ground state,

$$w = m\left(2 + \frac{2}{(1+\alpha^2)^{1/2}}\right)^{1/2} \tag{2.7}$$

and the ground-state wave function is

$$\psi_w^0 = \frac{e^{-r/a}}{(\pi a^3)^{1/2}} , \tag{2.8}$$

where

$$a = \frac{1}{E\alpha} . \tag{2.9}$$

Notice that, without the V^{0^2} terms, w is real for all α. In fact, as $\alpha \to \infty$, w approaches $\sqrt{2}\,m$. Evidently very deep binding does not occur in this case. Also, as $\alpha \to \infty$, the radius a approaches $2/m$.

We point out these results here for later comparison with the case when full gauge invariance is imposed.

B. Spectrum with $V^{0\,2}$ term included

The spectrum (1.42) for this case is obtained by recognizing that the effect of the sharply attractive $-\alpha^2/2m_w r^2$ term is to redefine by analytic continuation from l to λ the angular momentum variable. This λ variable is defined by

$$\lambda(\lambda+1) = l(l+1) - \alpha^2 \ . \tag{2.10}$$

Imposing gauge invariance in this particular way, without the vector potential contribution $\vec{V}^2$, leads to complex energies if $\alpha > \frac{1}{2}$. This is ordinarily associated with the Dirac and Klein-Gordon equations for a particle in an external Coulomb field for a point nucleus with $Z\alpha > 1$. The ground-state total c.m. energy is equal to $(2+\sqrt{2})^{1/2}m$ for $\alpha = \frac{1}{2}$.

Case has demonstrated that the appearance of complex energies in the static limit of relativistic bound-state equations indicates that the Coulomb potential is singular if α is large enough.[10] This manifests itself as an essential singularity in the radial wave function at the origin for α beyond a critical value. By a more careful handling of this singularity in α, Case shows that one may in fact obtain a real spectrum for larger α but that the resulting spectrum depends on an undetermined constant. He interprets this constant as a cutoff parameter to be determined by using a more realistic potential that does not display such singular behavior. As we shall see in our example in the next section, this effective cutoff is provided by QED itself (without radiative corrections) in the case of Todorov's quasipotential approach to the two-body problem.

C. The spectrum and wave function with the $\vec{V}^2$ term

Equation (2.1) includes the $V^{0\,2}$ and $\vec{V}^2$ terms. The term that is proportional to V^0 is $1/r$, and the one proportional to $\{\vec{V}, \cdot \vec{p}\}$ cancels in the approximation we are considering with the spin-spin terms.[11] The sharply attractive $1/r^2$ potential is counterbalanced very near the origin by the even more singular $1/r^4$ potential. As noted earlier, this repulsive term does not contribute significantly when α is small. However, as we shall see, its effects are significant for larger α. It has two effects. First of all, it permits real w for all α. Second, in the generalized positronium atom, the constituent particles are shielded from one another by the hard-core influence of the $1/r^4$ potential. This latter point can be demonstrated analytically as very near the origin one has for the S state the equation

$$\left(\frac{-d^2}{dr^2} + \frac{\alpha^2}{8w^2 r^4}\right) u = 0 \ , \tag{2.11}$$

where u is the radial part of the wave function. Solving this yields

$$u \underset{r\to 0}{\sim} r e^{-r_0/r} \ , \tag{2.12}$$

where

$$r_0 = \frac{\alpha}{2\sqrt{2}\,w} \ . \tag{2.13}$$

One cannot solve (2.1) analytically. A numerical evaluation of the eigenvalues and the eigenfunctions must be used.[12] We are interested in the ground state only. The technique we used is outlined in Appendix B. In Table I we present the numerical results for the ground-state eigenvalue computed at various values of $\alpha = 0.1$ to $\alpha = 10^4$. Also given in this table are the values of r (r_1 and r_2) between which most of the wave function resides. From this information on the confinement region of the constituents we extract a rough measure of the Fermi momentum of the effective particle.

III. DISCUSSION OF RESULTS

The tabulated results demonstrate several points. First of all, real energies appear for large α. The vector potential $\vec{V}$ accounts for this fact. Since $\vec{V}$ would be zero in the static limit, this implies that recoil effects are responsible for the elimination of complex energies for large α in the two-body case. The second point is that very deep binding occurs where α is large. For large α, the attractive $1/r^2$ and repulsive $1/r^4$ potentials dominate. As a result the effective particle is in a well with a minimum at about

$$r_{\min} = \frac{1}{\sqrt{2}\,w} \tag{3.1}$$

and a depth of

$$V_{\min} = -\frac{1}{2}\frac{\alpha^2 w^3}{m^2} \tag{3.2}$$

The third point is that for large coupling the particles appear to be confined in a rather thin shell,[13] the inner radius ranging up to 95% of the outer radius in the range of α considered. Even so, as indicated in column 6 of the table, the Fermi momentum is not relativistic. Notice that the outer radius contracts for increasing α up until $\alpha \sim 10$. This is a typical Coulomb behavior. Beyond that value the Coulomb effects are dominated by the $V^{0\,2}$ and $\vec{V}^2$ terms.

In a paper now in preparation, we shall examine

TABLE I. Total c.m. energy and wave function for generalized singlet positronium in the ground state. The variables are in units (or inverse units) of the common mass m of the constituents. The variables r_1 and r_2 are the inner and outer confinement radii for this system. The c.m. energy is w and the sixth column provides a measure of the Fermi momentum.

α	w	r_1	r_2	r_1/r_2	$(r_2-r_1)^{-1}$
0.1	1.9975	0.008	200	0.000 04	0.005
0.2	1.9898	0.01	100	0.000 1	0.009
0.3	1.976	0.02	70	0.000 4	0.01
0.4	1.956	0.03	50	0.000 6	0.02
0.6	1.894	0.04	30	0.001	0.03
0.8	1.806	0.05	25	0.002 5	0.045
1	1.704	0.06	20	0.003	0.05
2	1.259	0.1	10	0.01	0.1
3	1.010	0.2	7	0.02	0.15
4	0.8616	0.25	6.5	0.035	0.15
6	0.6902	0.35	6.5	0.055	0.15
8	0.5913	0.5	5.5	0.085	0.2
10	0.5252	0.6	6	0.1	0.2
20	0.3660	1	6	0.2	0.25
30	0.2974	1.5	6.5	0.25	0.2
40	0.2569	2.0	7.0	0.3	0.2
60	0.2092	2.5	7.5	0.35	0.2
80	0.1809	3.0	8.0	0.35	0.2
100	0.1617	4.0	8.0	0.5	0.25
200	0.1142	6.0	9.0	0.65	0.3
300	0.093 16	7.5	11	0.7	0.3
400	0.080 66	9	13	0.7	0.3
600	0.065 84	11	14	0.8	0.3
800	0.057 01	13	17	0.8	0.3
1000	0.050 99	16	20	0.8	0.3
2000	0.036 05	22	25	0.9	0.3
3000	0.029 43	27	30	0.9	0.3
4000	0.025 49	32	36	0.9	0.3
6000	0.020 81	40	43	0.9	0.3
8000	0.018 02	46	50	0.9	0.3
10 000	0.016 14	52	55	0.95	0.3

the trajectory $w(n, l)$ of such generalized positronium particles. There are other items to be considered also. We have not considered in this present work the effects of the residual δ-function terms and even more singular spin-dependent factors as well as the triplet state. Radiative corrections have also been ignored. The δ function and other such singular terms must be replaced by finite-range potentials. Otherwise they could not, in a nonperturbative sense, contribute even for S states because of the repulsive $1/r^4$ term. The length scale will be set by including radiative corrections nonperturbatively. This will also modify V^0, V^{0^2}, and $\vec{\nabla}^2$ and the spin-dependent terms.

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APPENDIX A: RELATION BETWEEN THE QUASIPOTENTIAL EQUATION AND THE RELATIVISTIC SCHRÖDINGER EQUATION FOR TWO SPIN-$\frac{1}{2}$ PARTICLES

We define a Green's function

$$g_w(\vec{q}, \vec{p}; k_1, k_2; \lambda_1, \lambda_2)$$

by

$$T_w(\vec{q}, \vec{p}; k_1, k_2, \lambda_1, \lambda_2) = -(2\pi)^3 G_w^{-1}(\vec{p})(\vec{p}-\vec{q})\delta_{\lambda_1 k_1}\delta_{\lambda_2 k_2} + G_w^{-1}(\vec{q}) g_w(\vec{q}, \vec{p}; k_1, k_2; \lambda_1, \lambda_2) G_w^{-1}(\vec{p}). \quad \text{(A1)}$$

Substitution of this into (1.16) leads to

$$G_w^{-1}(\vec{q}) g_w(\vec{q}, \vec{p}; k_1, k_2; \lambda_1, \lambda_2) + \int \frac{d^3k}{(2\pi)^3} V_w(\vec{q}, \vec{k}; k_1, k_2; \mu_1, \mu_2) \times g_w(\vec{k}, \vec{p}; \mu_1, \mu_2; \lambda_1, \lambda_2) = 0 . \quad \text{(A2)}$$

We impose the standard assumption that the Green's function has a pole in the c.m. frame at $w = m_B$ (c.m. mass of bound state) with the factorizable residues being associated with the wave function in the following way:

$$g_w(\vec{q}, \vec{p}; k_1, k_2; \lambda_1, \lambda_2) = \frac{\Phi_w(\vec{p}; \lambda_1, \lambda_2)\Phi_w(\vec{q}; k_1, k_2)}{m_B - w} + \text{regular terms} . \quad \text{(A3)}$$

Substituting this into (A2) leads to the relativistic Schrödinger equation (1.21).

APPENDIX B: TECHNIQUE FOR SOLVING NONLINEAR EIGENVALUE PROBLEM

The eigenvalue equation to solve is

$$\frac{-d^2}{dr^2} - \frac{2E\alpha}{r} - \frac{\alpha^2}{r^2} + \frac{\alpha^2}{w^2 r^4} u(r) = b^2(w) u(r). \quad \text{(B1)}$$

This equation has the general form

$$H(\alpha, \lambda) u = \lambda u . \quad \text{(B2)}$$

First one must make a fairly accurate guess for λ. Call this value λ_0. One must also use a reasonable guess u_0 for the wave function. This can be obtained by a simple variational method. The method we are about to describe is called the power method.[12] The eigenvalue λ_0 is an estimate of the ground-state level. Now

$$(\hat{u}_0, H(\alpha, \lambda_0)\hat{u}_0) \equiv \lambda_0 \ , \tag{B3}$$

with u_0 the normalized form of u_0. Let $\{\hat{u}^i; i = 1, 2, \ldots\}$ be a complete set of orthonormalized states corresponding to the Hamiltonian $H(\alpha, \lambda)$ so that

$$H(\alpha, \lambda_0)\hat{u}^i = \lambda_0^i \hat{u}^i \tag{B4}$$

and

$$\hat{u}_0 = \sum_i a_1^0 u^i \ . \tag{B5}$$

We define

$$u_1 = [H(\alpha, \lambda_0) - \lambda_0]^{-1}\hat{u}_0 \ . \tag{B6}$$

Using (C4) and (C5) to compute the inner product

$$(u_0, u_1) = \sum_i a_i^0 a_i^0 (\lambda_0^i - \lambda_0)^{-1} \tag{B7}$$

we see that if the original wave function was an accurate guess so that $a^0 \sim 1$ then this product may be close to $1/(\lambda_0^1 - \lambda_0)$. Next u_1 is normalized to

$$\hat{u}_1 = \sum_i \hat{u}^i a_i^1 \ , \tag{B8}$$

where

$$a_i^1 = \frac{a_i^0(\lambda_0^i - \lambda_0)^{-1}}{|\sum_j a_j^0 a_j^0 (\lambda_j - \lambda_0)^{-2}|^{1/2}} \ . \tag{B9}$$

If the original eigenvalue guess λ_0 was close to λ_0^1, the correct ground-state level, then a_1^1 will be closer to 1 in magnitude than a_1^0.

Now define

$$u_2 = [H(\alpha, \lambda_0) - \lambda_0^1]^{-1} u_1 \ . \tag{B10}$$

Then the inner product

$$(u_1, u_2) = \sum_i a_i^1 a_i^1 (\lambda_0^1 - \lambda_0)^{-1} \tag{B11}$$

will be closer to $1/(\lambda_0^1 - \lambda_0)$ than the previous inner product. This cycle is repeated until the desired accuracy is obtained, the end result being as close to λ_0^1 as one may specify. The next step involves iterating the whole procedure [(B3) through (B11)] with λ^0 in the Hamiltonian. This is continued until the eigenvalue guessed and used in the Hamiltonian is equal within specified limits to the eigenvalue computed. An outline of this next step using Padé approximants is outlined in Appendix B of Ref. 9.

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[11]This refers to the cancellation of δ functions in this equal-mass case alluded to earlier. This cancellation is exact in a perturbative sense. Strictly speaking, the high-order residual δ function will not contribute because of the highly repulsive $1/r^4$ potential.

[12]Eugene Isaacson, *Analysis of Numerical Methods* (Wiley, New York, 1966). An alternative method of computing the eigenvalues is the WKBJ method. Essentially the same eigenvalues are obtained as given in the table by the power method as outlined in Appendix B.

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