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Dynamical symmetry breaking as a bootstrap

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This paper is concerned with the bootstrap nature of dynamical symmetry breaking and with the infrared origins of the mass of the electron. We present a general calculational procedure for handling the situation in which a composite operator $\bar{\Psi}\Psi$ acquires a dynamical vacuum expectation value. We apply our procedure to finite quantum electrodynamics to show how the infrared divergences of the theory self-consistently cause $\langle\bar{\Psi}\Psi\rangle$ to become nonzero nonperturbatively so that it can provide a scale for a purely dynamical electron mass. Since no Goldstone boson need accompany this spontaneous breakdown, the electron mass bootstraps itself about the γ_5 degenerate vacuum. The mechanism also yields a new eigenvalue condition for the fine-structure constant. We discuss the deep interplay between the ultraviolet and the infrared, a characteristic feature of dynamical symmetry breaking. We use this interplay to show how to extend the Wilson operator-product expansion to the situation in which there is a degenerate vacuum. We discuss the possibility that the infrared structure of the weak interaction provides a dynamical origin for the Gell-Mann, Oakes, and Renner Hamiltonian. We indicate briefly the possibility that anomalous dimensions can soften a 4-Fermi interaction sufficiently to make it renormalizable.

I. INTRODUCTION

It is now apparent that the degenerate vacuum is an important feature of the theory of elementary particles, and it is very welcome in a sense since it takes into account in an essential way the presence of an infinite number of degrees of freedom. As such, it then permits particle theory to be understood as a many-body problem, and if something like the bootstrap philosophy is to make contact with conventional field theory it should ultimately be through the degeneracy of the vacuum. One of the major drawbacks to achieving this is that the standard way of destabilizing the vacuum has been through the introduction of highly artificial input scalar tachyons. Though very useful for discussing soft-pion physics and the Goldstone phenomenon in models such as the σ model, this approach gives no information at all about the dynamics underlying the degeneracy of the vacuum. Thus we must look for dynamical mechanisms which can cause the vacuum to become degenerate. The idea for instance that the pion would be a collective excitation associated with a dynamically induced fermion mass was first proposed by Nambu and Jona-Lasinio,[1] who pursued an extremely useful parallel with the theory of phase transitions, where spontaneous breakdown is a long-range order or infrared effect. Despite being one of the most attractive and compelling ideas in particle physics, the concept of purely dynamical masses essentially lay dormant until very recently, except for the important program developed by Johnson, Baker, and Willey[2-7] for obtaining a completely finite formulation of quantum electrodynamics (finite QED). In their program a purely dynamical electron mass is required in order to solve the ultraviolet problem of mass renormalization. Following the development of spontaneously broken gauge theories interest has been renewed in dynamical symmetry breaking,[8-12] but as of yet there has only been a limited discussion[9,12] of the infrared nature of the problem. This paper is the second of a series[13] which attempts to isolate the specific infrared dynamics which would cause the vacuum to become degenerate, and presents a discussion of how to treat the situation in which a composite operator $\bar{\psi}\psi$, acquires a vacuum expectation value.

Though current interest centers on spontaneously broken gauge theories, we shall concentrate in this work only on the question of breakdown of a global symmetry, chiral invariance, since this is the necessary first step toward understanding the more complicated local extension. Throughout we shall take dynamical symmetry breaking to mean that the composite $\bar{\psi}\psi$ acquires a vacuum expectation value (independent of whether or not there is an associated bound-state particle), rather than that radiative corrections cause an input fundamental scalar to acquire an expectation value, as is discussed in Refs. 9 and 14. Moreover, we shall consider only finite QED since this theory is rich enough to bring out the main features required for dynamical symmetry breaking, and reserve for future analysis the extension to the non-Abelian case.

Discussion of dynamical mass generation usually begins with a study of the self-consistent equation for the fermion self-energy[15]

$$\{\gamma_5, \Sigma(p)\}_+ = \int d^4k\, K(p,k,0) S(k) \{\gamma_5, \Sigma(k)\}_+ S(k), \tag{1}$$

the analog of the Bardeen-Cooper-Schrieffer (BCS) gap equation. Here everything is expressed in terms of the unrenormalized kernel and propagators. Provided that the theory exists without subtraction, it then follows that the chiral symmetry is spontaneously broken in the nontrivial solution to Eq. (1), with there being an accompanying Goldstone particle. However, in theories which require renormalization Eq. (1) is essentially destroyed so that the Goldstone mode is lost. (In fact our interest in discussing dynamical symmetry breaking in finite QED rather than in, say, the Goldstone realization of the gluon model is because the very fact of renormalization appears to exclude the latter possibility. We shall defer discussion of this point, however, until Sec. V.) In renormalizable theories it is more convenient to consider $\tilde{\Gamma}_S$, the insertion of the composite mass operator $\bar{\psi}\psi$ into the renormalized inverse electron propagator. This Green's function satisfies the Bethe-Salpeter equation (the kinematics is presented in Fig. 1)

$$m\,\tilde{\Gamma}_S(p, p+q, q) = m_0 Z_2 + m\int d^4k\, \tilde{K}(p, k, q)\,\tilde{S}(k) \times \tilde{\Gamma}_S(k, k+q, q)\,\tilde{S}(k+q)\;. \tag{2}$$

Moreover, $\tilde{\Gamma}_S$ is recognized as the mass insertion in the Callan-Symanzik equation[16]

$$\left[m\,\frac{\partial}{\partial m} + \beta(\alpha)\,\frac{\partial}{\partial \boldsymbol{\alpha}} - 2\gamma_F(\alpha)\right]\tilde{S}^{-1}(p) = -m\left[1 - \gamma_\theta(\alpha)\right]\,\tilde{\Gamma}_S(p, p, 0)\;, \tag{3}$$

where $\gamma_\theta(\alpha)$ is the anomalous part of the dimension of $\bar{\psi}\psi$ [the full anomalous dimension being given by $d_\theta(\boldsymbol{\alpha}) = 3 + \gamma_\theta(\boldsymbol{\alpha})$]. The asymptotic solution to Eq. (3) is readily given in finite QED where $\beta(\boldsymbol{\alpha}) = \gamma_F(\boldsymbol{\alpha}) = 0$ as[16]

$$\tilde{S}^{-1}(p) = Z_2\left[\not{p} - m\left(\frac{-p^2}{m^2}\right)^{(1/2)\,\gamma_\theta(\alpha)}\right],$$
$$m\,\tilde{\Gamma}_S(p, p, 0) = m\,Z_2\left(\frac{-p^2}{m^2}\right)^{(1/2)\gamma_\theta(\alpha)}\;. \tag{4}$$

In our notation Z_2^{-1} introduced here is the c-number anomaly of the electron anticommutator in the finite gauge, i.e., the gauge-independent part of the electron wave-function renormalization constant. Should $\gamma_\theta(\boldsymbol{\alpha})$ be negative it then follows that the bare mass m_0 vanishes in the limit of infinite cutoff[5] $[m_0 \sim m(\Lambda^2/m^2)^{(1/2)\gamma_\theta}]$ so that Eq. (2) now becomes homogeneous. (This is in fact the point where finite QED departs from the conventional Goldstone mode, which itself may be thought of as occurring in theories where m_0 is zero identically, i.e., even before the cutoff is removed.) The import of the asymptotic vanishing of m_0 is that Eq. (3) allows us to recognize $m\tilde{\Gamma}_S(p, p, 0)$ as the nonleading part of the fermion propagator which then bootstraps itself self-consistently in a now homogeneous Bethe-Salpeter equation. Because of the presence of a nonvanishing bare mass prior to the removal of the cutoff, it is then found that the situation corresponds to an (anomalously) nonconserved axial-vector current in the solution, so that the Goldstone theorem does not apply.[17,18,5] The crucial point in understanding dynamical symmetry breaking is to now appreciate that the existence of any nontrivial solution to Eq. (2) at all nonetheless corresponds to a γ_5 degenerate vacuum, and we shall discuss this point explicitly in Sec. II. Thus despite the nonconserved current we still have a degenerate vacuum and also a self-consistent gap-type equation, features characteristic of dynamical symmetry breaking, with renormalization's only role being to exclude the Goldstone boson. It must be stressed that the phenomenon of a degenerate vacuum and a nonconserved current is far more general than the original context of finite QED, as it is the only way a fermion can acquire a dynamical mass once there are homogeneous Bethe-Salpeter equations in the fermion sector, without any commitment being needed as to the specific structure of the kernel.

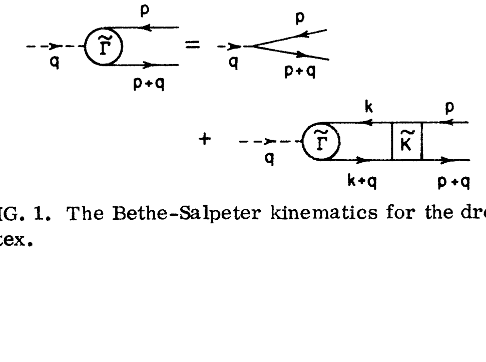

FIG. 1. The Bethe-Salpeter kinematics for the dressed vertex.

Now though we have the solution given in Eq. (4), we have not yet isolated the dynamics which will actually force the system to choose the nontrivial case of $m \neq 0$. As we discussed in Ref. 13, this is the infrared problem. The self-consistent solution to this problem will be presented in Sec. III, where we discuss how to handle the composite operator $\bar{\psi}\psi$. We will show there that the infrared divergences generated by the power solution of Eq. (4) indeed nonperturbatively make the massive theory energetically favorable provided

$$\gamma_\theta(\boldsymbol{\alpha}) + 1 = 0\;, \tag{5}$$

a new eigenvalue condition for α. The scale for masses is then set by $\langle\bar{\psi}\psi\rangle$, which though composite is not associated with a scalar bound state.

One of the major advantages of dynamical symmetry breaking is that there are no explicit soft operators in the theory (such as a wrong-sign σ mass term), but that the soft operators, e.g. the tadpole mass $m\bar{\psi}\psi$, are generated dynamically. Consequently, the input theory is purely dimensionless and the scale is then generated by $\langle\bar{\psi}\psi\rangle$, with the input massless theory being both ultraviolet and infrared divergent at once. Thus the ultraviolet problem and the infrared problem are solved simultaneously, the ultraviolet by $m_0=0$ and the infrared by $m\neq 0$, or both together by Eq. (5). This is perhaps the main value of Adler's essential singularity in $\beta(\alpha)$ (see Ref. 19) which permits the same coupling constant to control both the physical and asymptotic regions at once. Because of these remarks we can then expect infrared effects to show up in the ultraviolet, and we will use this fact in Sec. IV as a basis for extending the Wilson operator-product expansion[20] to the case where there is a degenerate vacuum. This will enable us to set up an alternative mass bootstrap which will realize Eq. (5).

For completeness we shall review the question of the absence of the Goldstone boson in Sec. V, where we also discuss the distinction between renormalized and physical masses. The renormalized mass describes the particle content of quantum fluctuations about a given vacuum, whereas the physical mass is the position of that vacuum. Their equivalence gives the bootstrap condition of Eq. (5).

The underlying thrust of the program that we have set up in this paper is to prove a theorem that a massless particle cannot have a charge, or that a charged particle cannot stay massless. We shall discuss the status of such a theorem in Sec. VI, where we shall also argue that finite QED is the relativistic generalization of the BCS theory of superconductivity. We conclude with some general comments in Sec. VII, where we briefly indicate a possible extension of our ideas to the non-Abelian case, and briefly discuss the relation of our ideas to the construction of a renormalizable 4-Fermi interaction.

II. THE γ_5 DEGENERACY OF A MASSIVE FERMION THEORY

In this section we shall review the discussion presented by Nambu and Jona-Lasinio[1] on the γ_5 degeneracy of massive fermion theories. Our presentation here is pedagogical in part, but the reader will find this section to be a useful orientation for the subsequent analysis of this paper. In their classic paper Nambu and Jona-Lasinio demonstrate by construction that a free massive fermion theory possesses an infinity of physically equivalent descriptions. This is done by constructing the Bogoliubov-Valatin transformation between the massive theory and a family of phase equivalent underlying massless theories. Consider massless and massive quantized fermion fields satisfying

$$i\gamma_\mu\partial^\mu\psi^{(0)}(x)=0\ , \tag{6a}$$

$$(i\gamma_\mu\partial^\mu-m)\psi^{(m)}(x)=0\ , \tag{6b}$$

$$\psi^{(0)}(x)=\psi^{(m)}(x)\ \text{at}\ x_0=0\ . \tag{6c}$$

A standard Fourier decomposition yields

$$\psi^{(i)}(x)=\frac{1}{V^{1/2}}\sum_{\vec{p},s}[u^{(i)}(\vec{p},s)a^{(i)}(\vec{p},s)e^{-ip\cdot x}+v^{(i)}(\vec{p},s)b^{(i)\dagger}(\vec{p},s)e^{ip\cdot x}]\ , \tag{7}$$

where $i=0,m$, and where each spinor is restricted to its own mass shell and is normalized so $u^\dagger u=1$. Since both sets of creation and annihilation operators satisfy the same anticommutation relations they must be related by a unitary canonical transformation. Introducing

$$\lambda_p^\pm=\left[\frac{1}{2}\left(1\pm\frac{|\vec{p}|}{(\vec{p}^2+m^2)^{1/2}}\right)\right]^{1/2}\ , \tag{8}$$

we obtain from the normalization condition of Eq. (6c)

$$\begin{aligned} a^{(m)}(\vec{p},s)&=\lambda_p^+a^{(0)}(\vec{p},s)+\lambda_p^-b^{(0)\dagger}(-\vec{p},s)\ ,\\ b^{(m)}(\vec{p},s)&=\lambda_p^+b^{(0)}(\vec{p},s)-\lambda_p^-a^{(0)\dagger}(-\vec{p},s)\ . \end{aligned} \tag{9}$$

We define the vacua of the two theories as $\Omega^{(0)}$ and $\Omega^{(m)}$ so that

$$a^{(i)}(\vec{p},s)\Omega^{(i)}=b^{(i)}(\vec{p},s)\Omega^{(i)}=0\ . \tag{10}$$

From Eqs. (9) and (10) we then find that

$$\Omega^{(m)}=\prod_{\vec{p},s}[\lambda_p^+-\lambda_p^-a^{(0)\dagger}(\vec{p},s)b^{(0)\dagger}(-\vec{p},s)]\Omega^{(0)}\ , \tag{11}$$

so that the massive vacuum is an infinite superposition of pairs of zero-mass particles. The overlap between the two vacua is given by

$$\langle\Omega^{(0)}|\Omega^{(m)}\rangle=\exp\left(\sum_{\vec{p},s}\ln\lambda_p^+\right)\ , \tag{12}$$

which vanishes in the limit of an infinite number of modes. Thus the unitarity of the canonical transformation is maintained by a limit in which there is an infinite number of vanishing matrix elements for the infinite complete set of states such that $0\times\infty=1$. (A similar situation is met in discussing Haag's theorem.) Thus in this limit there is no physical measurement which can connect the two systems based on $\Omega^{(0)}$ and $\Omega^{(m)}$, and hence the decoupling is a specific feature of an infinite number of degrees of freedom.

Now the massless theory is chiral-invariant under the transformations

$$\psi^{(0)} \to e^{i\alpha\gamma_5}\psi^{(0)} ,$$
$$a^{(0)}(\vec{p}, \pm) \to e^{\mp i\alpha} a^{(0)}(\vec{p}, \pm) , \qquad (13)$$
$$b^{(0)\dagger}(\vec{p}, \pm) \to e^{\pm i\alpha} b^{(0)\dagger}(\vec{p}, \pm) .$$

We can thus construct a new canonical transformation

$$a_\alpha^{(m)}(\vec{p}, \pm) = \lambda_p^+ e^{\mp i\alpha} a^{(0)}(\vec{p}, \pm) + \lambda_p^- e^{\pm i\alpha} b^{(0)\dagger}(-\vec{p}, \pm) , \qquad (14)$$
$$b_\alpha^{(m)}(\vec{p}, \pm) = \lambda_p^+ e^{\mp i\alpha} b^{(0)}(\vec{p}, \pm) - \lambda_p^- e^{\pm i\alpha} a^{(0)\dagger}(-\vec{p}, \pm)$$

to give us new creation and annihilation operators and a new vacuum

$$\Omega_\alpha^{(m)} = \prod_{\vec{p}, \pm} [\lambda_p^+ - \lambda_p^- e^{\pm 2i\alpha} a^{(0)\dagger}(\vec{p}, \pm) b^{(0)\dagger}(-\vec{p}, \pm)]\, \Omega^{(0)} . \qquad (15)$$

From Eq. (14) we can then construct a new $\psi_\alpha^{(m)}$, by Fourier-expanding as in Eq. (7) only in terms of $a_\alpha^{(m)}$ this time, which will have exactly the same particle content as $\psi^{(m)}$ since $\{a_\alpha^{(m)}, a_\alpha^{(m)\dagger}\}_+ = 1$, $a_\alpha^{(m)}\Omega_\alpha^{(m)} = 0$, etc. Now the crucial point is to realize that $\psi_\alpha^{(m)}$ is not the γ_5 rotated $\psi^{(m)}$ in the massive theory; $\psi_\alpha^{(m)}$ is constructed entirely via the canonical transformation by making γ_5 rotations in the underlying massless theory. No discussion need be given of whether the massive theory is γ_5-invariant, and this is the origin of the fact that certain dynamically induced massive fermion theories have a nonconserved massive axial-vector current, even though all $\psi_\alpha^{(m)}$ describe equivalent physical theories. Under the transformation of Eq. (14) the Hamiltonian of the theory

$$H^{(m)} = \sum_{\vec{p}, s} (\vec{p}^2 + m^2)^{1/2} [a^{(m)\dagger}(\vec{p}, s) a^{(m)}(\vec{p}, s) - b^{(m)}(\vec{p}, s) b^{(m)\dagger}(\vec{p}, s)] \qquad (16)$$

becomes

$$H_\alpha^{(m)} = \sum_{\vec{p}, s} (\vec{p}^2 + m^2)^{1/2} [a_\alpha^{(m)\dagger}(\vec{p}, s) a_\alpha^{(m)}(\vec{p}, s) - b_\alpha^{(m)}(\vec{p}, s) b_\alpha^{(m)\dagger}(\vec{p}, s)] . \qquad (17)$$

Thus we find that

$$\langle \Omega^{(m)} | H^{(m)} | \Omega^{(m)} \rangle = \langle \Omega_\alpha^{(m)} | H_\alpha^{(m)} | \Omega_\alpha^{(m)} \rangle = -2 \sum_{\vec{p}} (\vec{p}^2 + m^2)^{1/2} \qquad (18)$$

and

$$\langle \Omega^{(m)} | \bar{\psi}^{(m)} \psi^{(m)} | \Omega^{(m)} \rangle = \langle \Omega_\alpha^{(m)} | \bar{\psi}_\alpha^{(m)} \psi_\alpha^{(m)} | \Omega_\alpha^{(m)} \rangle = -\frac{2}{V} \sum_{\vec{p}} \frac{m}{(\vec{p}^2 + m^2)^{1/2}} \qquad (19)$$

as required since each $\Omega_\alpha^{(m)}$ provides an equivalent description of the same massive fermion. [It is important to note that the quantity that appears in Eq. (19) is not

$$\cos 2\alpha \langle \Omega_\alpha^{(m)} | \bar{\psi}_\alpha^{(m)} \psi_\alpha^{(m)} | \Omega_\alpha^{(m)} \rangle + i \sin 2\alpha \langle \Omega_\alpha^{(m)} | \bar{\psi}_\alpha^{(m)} \gamma_5 \psi_\alpha^{(m)} | \Omega_\alpha^{(m)} \rangle$$

since, as we have already stated, we are not making a γ_5 rotation in the massive theory. In this paper whenever we use the term γ_5 degeneracy we have in mind the invariance of Eq. (19) rather than an explicit transformation on the massive vacuum, i.e., we define the degeneracy through matrix elements only.] Further,

$$\langle \Omega_\alpha^{(m)} | \Omega^{(m)} \rangle = \exp\left\{ \sum_{\vec{p}, \pm} \ln[1 + (e^{\mp 2i\alpha} - 1)(\lambda_p^-)^2] \right\} , \qquad (20)$$

which also vanishes in the limit of an infinite number of modes, so again no physical measurement can link these vacua, and all of them define nonoverlapping Fock spaces. The one we choose to define the physical fermion is just a matter of taste.

Thus the point of our analysis is that once there is a massive fermion at all there is then an infinite degeneracy which can be exhibited by constructing an underlying massless theory, so that to relate $\Omega^{(m)}$ to $\Omega_\alpha^{(m)}$ we have to go through Eqs. (11) and (15), which is not the same as trying to rotate $\Omega^{(m)}$ directly into $\Omega_\alpha^{(m)}$ in the massive theory itself. In fact, when stated in this way it would appear quite difficult to obtain a conserved massive axial-vector current. However, we have not yet found a general rule for determining whether particular massive theories have a conserved current, and for the moment this must be discussed on a case by case basis. Indeed the example we have presented of a free massive fermion is itself one in which the current is not conserved, whereas in the σ model the current of course is conserved. However, what happens there is that the axial-vector current contains not just the fermion bilinear but also a meson term, and it is their interplay which maintains current conservation (and hence a massless pion) in the translated mode. In this case the fermion mass arises through the tadpole $g\langle\sigma\rangle$ contribution of Fig. 2(a), whose distinguishing feature is that it contributes a constant p-independent shift to $\Sigma(p)$. Exactly the same situation is met in the ladder approximation (one loop) to the 4-Fermi interaction discussed in Refs. 1 and 12 [Fig. 2(b)]. In this model there is a dynamical bound-state scalar tachyon whose shifting also produces a constant contribution to $\Sigma(p)$,

FIG. 2. (a) The scalar tadpole contribution to the fermion self-energy in the σ model. (b) The one-loop contribution to the fermion self-energy in an interacting 4-Fermi theory in the self-consistent vacuum.

with the massive current remaining conserved. It would be of interest to see whether the bound state in this model is preserved in higher orders when $\Sigma(p)$ becomes dependent on p. If the bound state is lost, then the momentum dependence of $\Sigma(p)$ may be the general indicator of whether the massive current is conserved or not.

As an example of a situation in which the massive theory has no conservation we return to our analysis of finite QED [where $\Sigma(p)$ has a momentum dependence] begun in the Introduction. Since the theory is renormalizable there are now Bethe-Salpeter equations. As well as Eq. (2) we also have the equation for the pseudoscalar insertion, $i\bar\psi\gamma_5\psi$,

$$m\tilde\Gamma_P(p,p+q,q) = m_0 Z_2\gamma_5 + m\int d^4k\,\tilde K(p,k,q)\tilde S(k) \times\tilde\Gamma_P(k,k+q,q)\tilde S(k+q)\,. \quad (21)$$

Because of Eq. (4) the axial-vector Ward identity

$$\frac{Z_2}{Z_A}q^\mu\tilde\Gamma_{\mu5}(p,p+q,q)=\tilde S^{-1}(p+q)\gamma_5+\gamma_5\tilde S^{-1}(p) +2m\frac{Z_S}{Z_P}\tilde\Gamma_P(p,p+q,q) \quad (22)$$

reduces at $q_\mu=0$ to (no Goldstone pole)

$$2m\frac{Z_S}{Z_P}\tilde\Gamma_P(p,p,0)-2m\,\tilde\Gamma_S(p,p,0)\gamma_5=0\ , \quad (23)$$

which is a completely chiral-symmetric relation for asymptotic p, which holds since $\tilde\Gamma_S$, $\tilde\Gamma_P$ asymptotically satisfy the same bootstrap equations. Moreover, bootstrap equations exist for all linear combinations $\cos2\alpha\,\tilde\Gamma_S+\sin2\alpha\,\tilde\Gamma_P$, and it is just a matter of convenience which one we identify with the mass term $\tilde\Sigma(p)$ and which with the divergence of the current. Thus in the massive theory the γ_5 invariance is not a property of the equations of motions themselves, but rather it is a feature of the set of homogeneous bootstrap equations for the insertions of the various $\cos2\alpha\,\bar\psi\psi+i\sin2\alpha\,\bar\psi\gamma_5\psi$ into the fermion propagator. This complements our discussion of Eq. (19). There is an intuitive way to understand this lack of current conservation suggested by our previous discussion of the σ model. In the present case we are missing the additional scalar and pseudoscalar bound states which could interplay with the fermion mass to maintain $\partial^\mu j_{\mu5}=0$. Thus, to repeat, once there are homogeneous self-consistent bootstrap equations for the fermion composites the solution has no conserved current.

To determine whether or not the massive solution is energetically favored we require the vacuum energy density difference

$$\epsilon(m)=\frac{1}{V}\left[\langle\Omega^{(m)}|H^{(m)}|\Omega^{(m)}\rangle-\langle\Omega^{(0)}|H^{(0)}|\Omega^{(0)}\rangle\right] \quad (24)$$

to be negative.[1] For our example of a free theory this is readily given as

$$\begin{aligned}\epsilon(m)&=-\frac{2}{V}\sum_{\vec p}\left[(\vec p^{\,2}+m^2)^{1/2}-|\vec p|\right]\\&=-\frac{2}{V}\sum_{\vec p}\left(\frac{1}{2}\frac{m^2}{|\vec p|}-\frac{1}{8}\frac{m^4}{|\vec p|^3}+\cdots\right)\\&=-\text{quadratic}+\text{logarithmic}\ .\end{aligned} \quad (25)$$

In Ref. 1 the authors worked with a cutoff so that $\epsilon(m)$ was indeed negative in their work. We shall see next in Sec. III where we show now to construct $\epsilon(m)$ for an interacting theory that after renormalizing the theory the logarithmic divergence takes over and completely alters the stability properties of the theory.

III. BOOTSTRAP CALCULATION OF THE VACUUM ENERGY DIFFERENCE

Before discussing the specific question of dynamical symmetry breaking by composite operators we review briefly the situation which obtains when there is a fundamental field. The most convenient framework is that described by Coleman and Weinberg.[9,21] We start from Schwinger's generating functional, $W(J)$, in which the field of interest, ϕ, is coupled linearly to an external source, J, with

$$e^{iW(J)}=\langle 0^+|0^-\rangle_J =\left\langle 0\left|T\exp\left(i\int d^4x[\mathcal{L}_0(x)+J(x)\phi(x)]\right)\right|0\right\rangle . \quad (26)$$

Here 0^-, 0^+ are respectively the vacua long before and long after $J(x)$ has acted (i.e., we consider J to act smoothly within a large but finite four-dimensional box of volume L^3T). A c-number classical field is then introduced through the definition

$$\phi_c(x)=\frac{\delta W}{\delta J(x)}=\left.\frac{\langle 0^+|\phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle}\right|_J\ , \quad (27)$$

which enables us to construct an effective action functional

$$\Gamma(\phi_c)=W(J)-\int d^4x\, J(x)\phi_c(x)\ . \tag{28}$$

This $\Gamma(\phi_c)$ can be expanded in two ways, either in a functional Taylor series about $\phi_c=0$ or in powers of momentum about the point where all external momenta vanish, i.e.,

$$\Gamma(\phi_c)=\sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n\, \Gamma^{(n)}(x_1,\ldots,x_n)\times\phi_c(x_1)\cdots\phi_c(x_n)$$
$$=\int d^4x[-V(\phi_c)+\tfrac{1}{2}(\partial_\mu\phi_c)^2Z(\phi_c)+\cdots]\ , \tag{29}$$

where we have introduced the effective potential $V(\phi_c)$ which satisfies

$$V(\phi_c)=\sum_n \frac{1}{n!}\Gamma^{(n)}(q_i=0)\phi_c{}^n\ . \tag{30}$$

The significance of $V(\phi_c)$ is that[21]

$$V(\phi_c)=\langle S|H_0|S\rangle-\langle N|H_0|N\rangle\ , \tag{31}$$

i.e., it is the energy difference between the normal and spontaneously broken vacua of a given Hamiltonian density, H_0. Note that this is not the same energy difference as that of Eq. (24). Thus we start off in a normal vacuum in which $\langle N|\phi|N\rangle=0$, and find that the dynamics is such that there is another vacuum with a lower energy in which $\langle S|\phi|S\rangle\neq 0$. In order to keep the particle interpretation we usually shift the field at this point so that $\langle S|\phi'|S\rangle=0$, but this is only a matter of convenience. The physics of spontaneous breakdown is in the fact that $|S\rangle$ and $|N\rangle$ are different vacua, and the shifting is only performed so that the no-particle second-quantized Fock space vacuum will coincide with the (first-quantized) state of lowest energy.

The utility of Eq. (30) is that we usually do not know $|S\rangle$ and so we look for the true vacuum by summing an infinite series about $|N\rangle$, the vacuum for the Green's functions $\Gamma^{(n)}(q_i=0)$, i.e., we recognize the $\Gamma^{(n)}$ as quantum fluctuations about $|N\rangle$ since we made a Taylor series expansion of Eq. (29) about the point $\langle N|\phi|N\rangle=0$. It should not be thought that the action of the source has taken us from $|N\rangle$ to $|S\rangle$. The source here has no physical significance and was only introduced to obtain Eq. (30), which can then be discussed in the absence of sources. Though the source has no physical significance, it can be used as a mathematical fiction for interpreting the ferromagnetic self-consistent Weiss mean field as a bootstrap phenomenon.[22] In calculating the position of the minimum of $V(\phi_c)$, i.e., $\langle S|\phi|S\rangle$, it is convenient to imagine a situation in which we switch on an external field, B, make an infinite summation of perturbation theory diagrams about $|N\rangle$ in its presence, and then switch off B at the end and obtain a nontrivial result. Thus the bootstrap nature of the solution is due to the noninterchangeability of summing diagrams and switching off B. In the terminology of Ref. 23 this is the use of the unshifted or ϕ lines. Alternatively we could write down a nonlinear equation for $\langle S|\phi|S\rangle$ directly in the shifted or ϕ'-line description of Ref. 23 [the equation $V'(\phi_c)=0$], which again is a self-consistent bootstrap equation. Indeed (see e.g., Ref. 24) the characteristic feature of condensed matter below the critical point is a spontaneously broken phase with an associated order parameter whose bootstrap nature is that it satisfies a self-consistent equation of a form analogous to Eq. (1).

Though we have indicated that the source is a mathematical device, there is a possible source of confusion since an external magnetic field does play a physical role for a ferromagnet below the critical point. What happens there is that the ferromagnet usually starts off in an impure state prior to preparation, with the ground state being described by an incoherent superposition of all the different degenerate vacua, $|S_i\rangle$, of the system. In that impure state the ensemble average of the magnetization is zero (as is the case if we are in the $|N\rangle$ vacuum above the critical point, though we are not of course). If we now apply a weak external magnetic field and then remove it again the system will now be forced into and then remain in a pure state built only on one of the $|S_i\rangle$. In this pure state the magnetization is nonzero and remains pointing in the direction of the original external magnetic field.

Up to this point we have discussed phase-transition theory so as to indicate why symmetry breaking would be a bootstrap phenomenon, and have presented the formalism suitable for studying the problem when there is a scalar field in the theory. We turn now to the study of composite operators. Now in principle the problem has (presumably) already been solved by Eq. (30), which could equally well have been constructed by coupling a composite operator to an external source. However, in that case the $\Gamma^{(n)}$ have no simple diagrammatic significance and for the moment Eq. (30) seems intractable. Now of course we would only be interested in $V(\langle\bar\psi\psi\rangle)$ if we were actually in the Goldstone mode, with H_0 being a chiral-invariant Hamiltonian density. As we have already remarked the finite QED mode has a nonconserved current, so that $V(\langle\bar\psi\psi\rangle)$ is not in fact the object of interest, since rather we wish to study a theory which has an explicit mass term present. We shall now obtain a suitable alternative to $V(\langle\bar{\psi}\psi\rangle)$ which will be of use in finite QED and which will have the same utility in discussing broken symmetry. [Apart from our own interest in $\epsilon(m)$ in what follows, a knowledge of it actually permits an evaluation of $V(\langle\bar{\psi}\psi\rangle)$, a point we will return to below].

In constructing the effective potential we used an auxiliary device, $W(J)$. $W(J)$ is also an energy difference and it is of interest to ask which.[21] Consider a constant $J(x)$ inside the four-dimensional box, so that

$$W(J) = -\int d^4x\ \epsilon(J)\ . \tag{32}$$

Then

$$\langle 0^+|0^-\rangle_J = e^{-iL^3T\epsilon(J)}\ . \tag{33}$$

Thus at $t=-\infty$ we have a Hamiltonian density H_0 with a vacuum Ω_0 which passes smoothly into a density $H_0-J\phi$ with a vacuum Ω_J during the time interval T, and then passes smoothly back into H_0 at $t=+\infty$. The vacua at $t=\pm\infty$ are vacua of the same H_0 and can thus only differ by a phase, the one given in Eq. (33). However, the phase is determined by the theory in which J acts, i.e., by the state of the system at $t=0$, so that it gives the energy of the ground state of the perturbed system as discussed in the method of adiabatic switching developed by Gell-Mann and Low. Thus[21]

$$\epsilon(J) = \langle \Omega_J|(H_0 - J\phi)|\Omega_J\rangle - \langle\Omega_0|H_0|\Omega_0\rangle\ , \tag{34}$$

which we recognize as the energy difference of Eq. (24) (when $J\phi$ is replaced by $-m\bar{\psi}\psi$), so that we are now ready to extend Eq. (25) to an interacting theory. [In passing we remark that Eq. (34) is the relativistic analog of the statement in statistical mechanics that the thermodynamic potential is the logarithm of the partition function in the grand canonical ensemble.]

Though we have now generalized Eq. (25) to an interacting theory, $\epsilon(m)$ is not quite what we want yet since it compares energy differences of different Hamiltonian densities which is not itself of immediate physical significance. Using the Bogoliubov-Valatin transform, however, it is easy to show that $\langle\Omega^{(0)}|H^{(0)}|\Omega^{(0)}\rangle$ and $\langle\Omega^{(0)}|H^{(m)}|\Omega^{(0)}\rangle$ are equal, as is to be expected since $\bar{\psi}\psi$ possesses a vanishing expectation value in the chiral-invariant vacuum $\Omega^{(0)}$. Consequently, we can rewrite $\epsilon(m)$ in the form

$$\epsilon(m) = \langle\Omega^{(m)}|H^{(m)}|\Omega^{(m)}\rangle - \langle\Omega^{(0)}|H^{(m)}|\Omega^{(0)}\rangle\ , \tag{35}$$

so that it now compares different eigenstates of the same $H^{(m)}$. However, by construction $\Omega^{(m)}$ is the true vacuum of $H^{(m)}$ [up to the infinite degeneracy of Eq. (15)], whereas $\Omega^{(0)}$ is just some other possible eigenstate of $H^{(m)}$. Thus for the theory to be able to support a mass term at all in a consistent manner we require $\epsilon(m)$ to be negative. Should $\epsilon(m)$ prove to be positive the theory is simply not consistent. We thus see the difference (and similarity) between Eq. (31) and Eq. (35). In Eq. (31) we compare different vacua of an invariant Hamiltonian density, whereas in Eq. (35) we compare different vacua of a noninvariant Hamiltonian density. For stability both are required to be negative.

In passing we should just remark that at the time of writing Ref. 13 we were not sure of the physical significance of $\epsilon(m)$, and in that paper we unfortunately referred to it as an effective potential. To avoid confusion we shall use the term effective potential exclusively for $V(\langle\bar{\psi}\psi\rangle)$ and refer to $\epsilon(m)$ as the vacuum energy difference.

We can now proceed to evaluate $\epsilon(m)$ in the case of interest. Thus suppose we have a massless theory in which a mass arises dynamically through the appearance of a mass term $m\bar{\psi}\psi$ in the massive theory. The mass term is a true mass term, but formally we can treat it as a source term (with the source now acquiring a physical significance),[13] so that

$$\epsilon(m) = \sum_n \frac{1}{n!} G_{(0)}^{(n)}(q_i=0)m^n\ , \tag{36}$$

since $\epsilon(m)$ generates the Green's functions of the insertions of $\bar{\psi}\psi$ into the vacuum functional. Note that the $G_{(0)}^{(n)}$ are the $\bar{\psi}\psi$ connected Green's functions of the massless theory ($m=0$). Unlike Eq. (30) the coefficients in Eq. (36) have a simple graphical interpretation precisely because $\bar{\psi}\psi$ is a composite insertion and are easy to calculate. As we stressed in Ref. 13 our construction of Eq. (36) is achieved precisely because the fermion mass term is linear, and thus our construction can only work because the composite $\bar{\psi}\psi$ gives the dynamically induced mass term we are looking for. Our approach would not be useful for studying situations in which, say, a quartic composite acquires an expectation value. There is thus a deep connection between the fact that $\bar{\psi}\psi$ acquires a vacuum expectation value and the fact that $m\bar{\psi}\psi$ is the mass term suggested by the Dirac equation, and this connection itself is a bootstrap.

To see that everything is consistent let us now calculate Eq. (36) for a free theory. The whole theory is given by the set of graphs of Fig. 3,

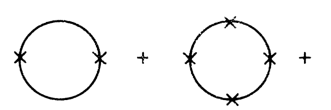

FIG. 3. The exact set of graphs used to calculate the vacuum energy difference, $\epsilon(m)$, in a free fermion theory. The propagators are massless.

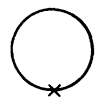

FIG. 4. The single tadpole graph of a free massive fermion theory. The propagator is massive.

where each vertex is just the bare vertex and where each propagator is massless, so that the vacuum energy difference is given by[13]

$$\epsilon(m) = -\frac{m^4}{32\pi^2}\left[4\frac{\Lambda^2}{m^2} - 1 - 2\ln\left(\frac{\Lambda^2}{m^2}\right)\right]$$
$$= -\text{quadratic} + \text{logarithmic}, \qquad (37)$$

i.e., the same divergence structure as Eq. (25), though now we are using a covariant cutoff. Moreover, by definition

$$\epsilon'(m) = \langle \Omega^{(m)} | \bar\psi\psi | \Omega^{(m)} \rangle \;, \qquad (38)$$

whose graphical structure is given by the one-loop graph of Fig. 4 in which the propagator is massive. Thus we confirm Eq. (38) by noting that Eq. (19) is the derivative of Eq. (25).

Now how do we remove the divergences of Eq (37)? We discuss first the normal-ordering prescription. When we normal-order we have to specify which vacuum we are normal-ordering with respect to. Consider first the massless theory. In the massless theory $\langle \Omega^{(0)} | \bar\psi\psi | \Omega^{(0)} \rangle$ is already zero without normal-ordering since Fig. 4 vanishes for a massless propagator. Moreover, none of the graphs of Fig. 3 will be affected at all by normal-ordering, so we can consider them as already normal-ordered with respect to $\Omega^{(0)}$, i.e., the first graph for instance is given by

$$G^{(2)}_{(0)} = \langle \Omega^{(0)} | T(:\bar\psi(x)\psi(x)::\bar\psi(y)\psi(y):) | \Omega^{(0)} \rangle \,. \quad (39)$$

We now sum the series of Fig. 3 and calculate a nonvanishing form for Eq. (38). This is because $\bar\psi\psi$, while normal-ordered with respect to $\Omega^{(0)}$, is not obliged to be normal-ordered with respect to $\Omega^{(m)}$. Thus even if we normal-order the underlying massless theory there will still be tadpole graphs appearing in the massive theory. Since normal-ordering the massive theory will not affect Eq. (37) either we must look for another prescription. Moreover, normal-ordering is not really a good prescription since as well as the infinities

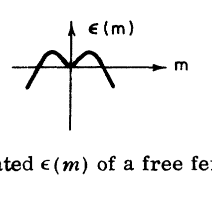

FIG. 5. The calculated $\epsilon(m)$ of a free fermion theory.

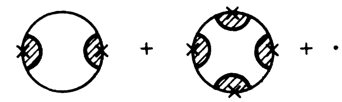

FIG. 6. The one-loop approximation to $\epsilon(m)$ for an interacting theory. The shaded blob vertex represents the complete dressed scalar vertex. The propagators are massless.

of the free theory it also takes out the finite parts. Thus we shall normal-order only the massless theory to keep the electrostatic charge of the vacuum finite, but shall use a different prescription, namely counterterms, in order to handle the additional infinities met when we introduce a mass. Thus we add m^2 and m^4 counterterms to Eq. (37). Because of the infrared divergence in $G^{(4)}_{(0)}(q_i = 0)$ we follow Coleman and Weinberg by introducing an arbitrary mass M, so that, as described in Ref. 13,

$$\epsilon(m) = -\frac{m^4}{32\pi^2}\left[2\ln\left(\frac{m^2}{M^2}\right) - 1\right], \qquad (40)$$

with M chosen so that $\epsilon'(M) = 0$, to take out the tadpole of Fig. 4, so that the renormalized fermion mass is given by $m = M$. Equation (40) is plotted in Fig. 5, and we see now that the massive theory ($m = M$) is no longer energetically favored. We believe this to be a possibly serious defect of a free Fermi theory, which we shall remedy below by generating a dynamical mass.

In an interacting theory we no longer know the $G^{(n)}_{(0)}$ Green's functions exactly, so we shall make the loop approximation of Fig. 6, essentially by inserting the complete dressed vertices of Eq. (4) into Fig. 3. The bootstrap nature of this procedure is apparent. However, we recall that Eq. (4) was the solution for the dressed vertex in the massive theory, whereas we need the vertices of the massless theory for Eq. (36). For the massless theory we shall have to renormalize off-shell by introducing a subtraction point, μ. [Equation (3) is renormalized on-shell.] If we consider the first graph of Fig. 7 (lowest-order perturbation theory), we note that it is a vertex renormalization, not a mass renormalization, and possesses a logarithmic divergence in the Landau gauge even with internal massless propagators. Thus unlike the vertex Γ_μ of the insertion of $:\bar\psi\gamma_\mu\psi:$ into the fermion propagator, we see that Γ_S is not finite in massless fermion QED in the finite gauge. However, the divergences in Γ_S are removed non-

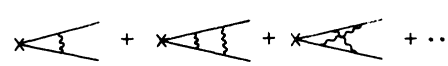

FIG. 7. The graphs which contribute to the renormalization of $\Gamma_S(p, p, 0)$ in massless fermion QED.

perturbatively at the eigenvalue by summing the whole series of Fig. 7, which then compensates the bare vertex. Thus after a nontrivial multiplicative renormalization by Z_S $[=(\Lambda^2/\mu^2)^{(1/2)\gamma_\theta}\to 0]$ we obtain a scaling equation which holds for all momenta since we are still in the massless theory. Hence

$$\tilde{\Gamma}^{(0)}_{S,\mu}(p,p,0)=Z_2C(\alpha)\left(\frac{-p^2}{\mu^2}\right)^{(1/2)\gamma_\theta(\alpha)} \tag{41}$$

for all momenta, with $C(\alpha)$ being an arbitrary function of the coupling constant. [Note that since we are at the eigenvalue changes in μ do not correspond to changes in the effective coupling constant. We fix the eigenvalue once and for all from the photon sector alone so as to then control the asymptotic behavior of the massless theory. Because of the essential singularity in $\beta(\alpha)$ we can then renormalize the fermion sector of the massive theory on-shell, with the physical coupling constant then being given by the same eigenvalue. Thus the parameter μ has only limited significance and does not serve as a scale for the eventual dynamical electron mass.[13]]

The situation described here is analogous to Wilson's discussion of the massless Thirring model.[25] In that model the composite $:\bar{\psi}\psi:$ acquires a negative anomalous dimension nonperturbatively (the model is easy to solve since there is no coupling constant renormalization, so it behaves in the fermion sector as does finite QED at the eigenvalue), so that there is a nontrivial gauge-independent renormalization in the theory with Eq. (41) holding for all momenta since the fermion is massless. Returning now to finite QED we find that we have complete conformal invariance with anomalous dimensions in the massless $\Omega^{(0)}$ vacuum, and that information alone will soon be seen to be sufficient to destabilize $\Omega^{(0)}$ and take us to $\Omega^{(m)}$ where Eq. (3) holds instead.

The calculation now proceeds as in Ref. 13. From Fig. 6 we obtain

$$\begin{aligned}\epsilon(m)&=i\int\frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(-1)}{2n}\,\mathrm{Tr}\left\{\left[-iZ_2C(\alpha)\right]^2\left(\frac{-p^2}{\mu^2}\right)^{\gamma_\theta(\alpha)}\left(\frac{i\,Z_2^{-1}}{\not p}\right)^2\right]m^2\right\}^n\\&=\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\ln\left[1-C^2(\alpha)\frac{m^2}{p^2}\left(\frac{-p^2}{\mu^2}\right)^{\gamma_\theta(\alpha)}\right].\end{aligned} \tag{42}$$

This form can be conveniently rewritten by defining a massive propagator for *all* momenta

$$\tilde{S}_\mu^{-1}(p)=Z_2\left[\not p-mC(\alpha)\left(\frac{-p^2}{\mu^2}\right)^{(1/2)\gamma_\theta(\alpha)}\right], \tag{43}$$

which satisfies

$$m\frac{\partial\tilde{S}_\mu^{-1}(p)}{\partial m}=-m\tilde{\Gamma}^{(0)}_{S,\mu}(p,p,0)\,. \tag{44}$$

Thus $\epsilon(m)$ can be rewritten compactly as

$$\epsilon(m)=i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\ln\left(\frac{Z_2^{-1}}{\not p}\tilde{S}_\mu^{-1}(p)\right), \tag{45}$$

with $Z_2^{-1}/\not p$ being the fully dressed propagator of massless QED in the finite gauge. Further,

$$\epsilon'(m)=-i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,\tilde{\Gamma}^{(0)}_{S,\mu}(p,p,0)\tilde{S}_\mu(p)\,, \tag{46}$$

so that $\langle\Omega^{(m)}|\bar{\psi}\psi|\Omega^{(m)}\rangle$ is given by the tadpole graph of Fig. 8, which is calculated with the dressed vertex and dressed propagator. In Ref. 13 we have calculated $\epsilon(m)$ for different values of $\gamma_\theta(\alpha)$ and the results are presented in Figs. 9, 10, and 11 for the respective cases $0>\gamma_\theta(\alpha)>-1$, $\gamma_\theta(\alpha)=-1$, $\gamma_\theta(\alpha)<-1$. [The ambiguity found in Ref. 13 in the case $0>\gamma_\theta(\alpha)>-1$ will be resolved in Sec. IV.] In the case of most interest, $\gamma_\theta(\alpha)=-1$, we have

$$\epsilon(m)=-\frac{C^2(\alpha)m^2\mu^2}{16\pi^2}\left[\ln\left(\frac{\Lambda^4}{C^2(\alpha)m^2\mu^2}\right)+1\right]. \tag{47}$$

We add a counterterm

$$-\frac{C^2(\alpha)m^2\mu^2}{16\pi^2}\ln\left(\frac{C^2(\alpha)M^2\mu^2}{\Lambda^4}\right),$$

chosen so $\epsilon'(M)=0$; so that after renormalizing we obtain

$$\epsilon(m)=\frac{C^2(\alpha)m^2\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2}{M^2}\right)-1\right], \tag{48}$$

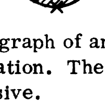

FIG. 8. The tadpole graph of an interacting theory in the one-loop approximation. The vertex is dressed and the propagator is massive.

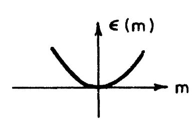

FIG. 9. The stable $\epsilon(m)$ obtained in $0>\gamma_\theta(\alpha)>-1$.

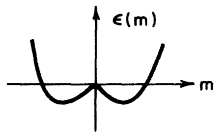

FIG. 10. The double-well $\epsilon(m)$ obtained in $\gamma_\theta(\alpha) = -1$.

where the arbitrary M now sets the scale for the fermion mass. The tadpole graph is now removed by choosing $m = M$, with the massive theory now being energetically favored. We thus see that after renormalizing there is a nontrivial choice for the mass that we can put into the propagator in Fig. 8 which causes the tadpole graph to vanish identically (without any need to normal-order the massive theory). This is a bootstrap phenomenon in which the condition $\epsilon'(m) = 0$ has a nontrivial solution,[26] and is the analog of the situation met when discussing the scalar field case where $V'(\phi_c) = 0$ nontrivially. An alternative and essentially equivalent physical description is to think of m as a Lagrange multiplier for $\bar\psi\psi$,[21] so that, as is typical in statistical mechanics, the Lagrange multiplier is nonzero in the physical solution and acquires a physical significance, in this case that of a mass. Thus what has happened is that the infrared divergences generated by the solution of Eq. (41) (which itself came originally from the ultraviolet) have accumulated nonperturbatively in the series of Fig. 6 to destabilize the origin and force us to a nonzero mass. Moreover, the counterterm we used removed both the ultraviolet and infrared divergences from Eq. (47) at once. We thus demonstrate the deep interplay between the infrared and the ultraviolet which occurs when there is dynamical symmetry breaking.

Having demonstrated the utility of the loop approximation we shall conclude this section by discussing its range of validity. We note first that our loop summation is essentially nonperturbative since it uses dressed vertices and dressed propagators, so that it corresponds to infinite summations of dressings to bare loops. This of course immediately poses a double-counting problem. We shall now argue that such double-counting problems are very mild, and that our loop summation may even be exact. The summation of Eq. (42) is a summation of graphs which are either ultraviolet divergent or infrared divergent when all external

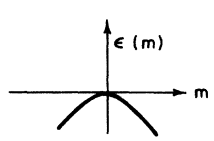

FIG. 11. The unstable $\epsilon(m)$ obtained in $\gamma_\theta(\alpha) < -1$.

momenta vanish or both. Thus our model provides a specific parametrization of those divergences by utilizing the exact conformal invariance of the underlying massless theory. For instance, the 2-point function of Eq. (39) is fixed uniquely to be

$$G^{(2)}_{(0)}(x,y) = \frac{E(\alpha)}{4\pi^4}\frac{\mathrm{Tr}(\hat{x}-\hat{y})(\hat{y}-\hat{x})}{\mu^{2\gamma_\theta}[(x-y)^2]^{(1/2)[d_\theta+1]}[(y-x)^2]^{(1/2)[d_\theta+1]}}\,, \tag{49}$$

where $E(\alpha)$ is an arbitrary function of the coupling constant, normalized so that $E(\alpha) = 1$ for a free Fermi theory. We Fourier transform $G^{(2)}_{(0)}$ treating it as a product of two propagators, so as not to miss the tip of the light-cone singularity in Eq. (49). Thus

$$G^{(2)}_{(0)}(q^2) = -iE(\alpha)\mu^{-2\gamma_\theta}2^{6-2d_\theta}\frac{\Gamma^2(\frac{5}{2}-\frac{1}{2}d_\theta)}{\Gamma^2(\frac{1}{2}d_\theta+\frac{1}{2})} \times\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[\not{p}(\not{p}+\not{q})]}{[p^2(q+p)^2]^{5/2-(1/2)d_\theta}}\,, \tag{50}$$

to be compared with the model value [which differs from the first term of $\epsilon(m)$ by a combinatoric factor] of

$$G^{(2)}_{(0)}(q_\mu = 0) = -i\,4C^2(\alpha)\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\left(\frac{-p^2}{\mu^2}\right)^{\gamma_\theta} = -\frac{C^2(\alpha)\mu^{-2\gamma_\theta}}{4\pi^2}\frac{(p^2)^{1+\gamma_\theta}}{(1+\gamma_\theta)}\bigg|_0^\infty\,. \tag{51}$$

Thus we see that the model correctly reproduces both the ultraviolet and infrared divergences of the exact $G^{(2)}_{(0)}$ at $q_\mu = 0$, being exact up to an over-all numerical factor.

The reason why there is no double-counting problem for $G^{(2)}_{(0)}$ is because there is no overlapping-divergence problem. A Schwinger-Dyson equation for $G^{(2)}_{(0)}$ would ordinarily equate it to a loop integral in which one vertex is dressed and the other bare, and thus give a different parametrization of the divergences than that correctly found in our model where all vertices are dressed. However, such a Schwinger-Dyson equation for $G^{(2)}_{(0)}$ only holds in theories where $G^{(2)}_{(0)}$ is in fact the proper self-energy part of some fundamental scalar field with the vertex being dressed by the scalar field itself. This of course is not the case in QED where only the vacuum polarization satisfies a Schwinger-Dyson equation. Thus the absence of a scalar field in QED neatly avoids the problem of overlapping divergences in $G^{(2)}_{(0)}$ and permits our model to determine it exactly.

For the 4-point function $G^{(4)}_{(0)}$ we cannot make the same analysis as for $G^{(2)}_{(0)}$ since conformal invariance does not fix $G^{(4)}_{(0)}$ uniquely. (Even if it did it is doubtful that the conformal structure could be Fourier-transformed analytically, this being the situation met in the soluble Thirring model apparently.) Consider, however, the special case in which all external momenta are scaled to zero together. Then since $G_{(0)}^{(4)}$ is dimensionless

$$G_{(0)}^{(4)}(q_\mu \to 0) \sim \left(\frac{-q^2}{\mu^2}\right)^{2\gamma_\theta} \tag{52}$$

since q^2 and μ^2 are the only scales in the theory. However, the model gives

$$G_{(0)}^{(4)}(q_\mu = 0) = \frac{C^4(\alpha)\mu^{-4\gamma_\theta}}{4\pi^2}\left.\frac{(p^2)^{2\gamma_\theta}}{2\gamma_\theta}\right|_0^\infty , \tag{53}$$

and hence successfully parametrizes the infrared divergence of Eq. (52) when $\gamma_\theta < 0$. The argument can be carried out for the $2n$-point function in the same fashion with conformal invariance correctly parametrizing the infrared divergence of each term in the summation of Eq. (42). Thus the only thing still to be fixed is the relative numerical weight of each of the terms that appear in the summation.

To fix the relative weights we return for simplicity to the formula of Eq. (47) which we obtained in the special case where $\gamma_\theta = -1$. We note that in this case only $G_{(0)}^{(2)}$ possesses an ultraviolet-divergent logarithm, providing $\epsilon(m)$ with a term of the form $C^2(\alpha)m^2\mu^2\ln\Lambda^4$. Thus on purely dimensional grounds the infinite summation of the infrared divergences of all the other Green's functions must produce a compensating term of the form $C^2(\alpha)m^2\mu^2\ln m^2\mu^2$ and thus remove the dependence on the infrared divergence coming from the lower limit in Eq. (51), and our model is a specific parametrization of the relative weights which produces such a compensation. For the moment, however, we are unable to show that this is the only way that it can be done. Apart from this possible question of uniqueness it then appears that our loop summation with dressed vertices may well be exact, and as such it provides an exact determination of the ground-state energy of massive quantum electrodynamics by appealing only to the conformal properties of the underlying massless theory.

IV. DEGENERATE-VACUUM WILSON EXPANSIONS

The main theme of our work is that the ultraviolet and infrared regions are related, and we now use this fact to discuss the question of what happens to the Wilson expansion when there is a degenerate vacuum. In massless fermion QED at the eigenvalue or also in the massless Thirring model we have the standard expression (ignoring additional terms which will be irrelevant to the subsequent discussion)

$$T(\psi(x)\bar\psi(0)) = \langle \Omega^{(0)}|T(\psi(x)\bar\psi(0))|\Omega^{(0)}\rangle + Z_2^{-1}D(\alpha)\frac{1}{(\mu^2x^2)^{-(1/2)\gamma_\theta}} :\psi(0)\bar\psi(0): , \tag{54}$$

where $\langle\Omega^{(0)}|T(\psi(x)\bar\psi(0))|\Omega^{(0)}\rangle$ is just a free massless propagator in the finite gauge. Equation (54) follows simply by summing the series of Fig. 7. (Note that there will be no singularity at all unless $\gamma_\theta < 0$.) Here the dots mean that we have normal-ordered with respect to $\Omega^{(0)}$ so that Eq. (54) is an identity when taken in the $\Omega^{(0)}$ vacuum. Now in our case there is a new vacuum $\Omega^{(m)}$ in which $:\bar\psi\psi:$ can take an expectation value, as we discussed in the previous section. If this happens we then generate a nonleading mass term in $\langle\Omega^{(m)}|T(\psi(x)\bar\psi(0))|\Omega^{(m)}\rangle$, the true propagator of the theory. Thus $\bar\psi\psi$ acquires an expectation value from the infrared which then shows up in matrix elements of Eq. (54) in the ultraviolet. To make our discussion as general as possible let us work in a theory with space-time dimension D so that the dimension of $:\bar\psi\psi:$ is given by $d_\theta = \gamma_\theta + D - 1$. Recalling that

$$\int d^Dx \frac{e^{ip\cdot x}}{(-x^2)^\lambda} = i\pi^{D/2}2^{D-2\lambda}\frac{\Gamma(\frac{1}{2}D-\lambda)}{\Gamma(\lambda)}(-p^2)^{\lambda-(1/2)D}, \tag{55}$$

we thus obtain for the true propagator of the theory

$$\tilde S(p) = \frac{Z_2^{-1}}{\not p} + Z_2^{-1}D(\alpha)\frac{\sigma\mu^{\gamma_\theta}\pi^{D/2}2^{D+\gamma_\theta}}{\mathrm{Tr}1} \times\frac{\Gamma(\frac{1}{2}D+\frac{1}{2}\gamma_\theta)}{\Gamma(-\frac{1}{2}\gamma_\theta)}(-p^2)^{-(1/2)(D+\gamma_\theta)}, \tag{56}$$

where

$$(2\pi)^D\delta^D(p_\mu)\sigma = -i\int d^Dx\, e^{ip\cdot x}\langle\Omega^{(m)}|\mathrm{Tr}:\psi(x)\bar\psi(x):|\Omega^{(m)}\rangle , \tag{57}$$

with σ also being equal to $\epsilon'(m)$. For large momenta the inverse propagator is given as

$$\tilde S^{-1}(p) = Z_2\not p - Z_2D(\alpha)\frac{\sigma\mu^{\gamma_\theta}\pi^{D/2}2^{D+\gamma_\theta}}{\mathrm{Tr}1} \times\frac{\Gamma(\frac{1}{2}D+\frac{1}{2}\gamma_\theta)}{\Gamma(-\frac{1}{2}\gamma_\theta)}(-p^2)^{(1/2)(1-d_\theta)} . \tag{58}$$

Thus we construct the true propagator by changing the vacuum. Now in the Introduction we presented an alternative way of constructing the propagator in which the quantum fluctuations associated with the fields contained particles of mass m and yielded Eq. (4). [The difference between this propagator and $\tilde S_\mu^{-1}(p)$ of Eq. (43) is not significant, since, as we shall show in Sec. V, there is a renormalization invariance equation for the mass so that changes in μ can be absorbed in $C(\alpha)$.] Equations (43) and (58) are two equivalent descriptions of the same physical situation. Thus we see that the mass arises because $\bar{\psi}\psi$ acquires a vacuum expectation value, and we obtain a bootstrap condition from the momentum dependence

$$1-d_\theta=d_\theta+1-D \tag{59}$$

or

$$d_\theta=\tfrac{1}{2}D,\quad \gamma_\theta=1-\tfrac{1}{2}D\ , \tag{60}$$

so that we recover our condition $\gamma_\theta(\alpha)=-1$ when $D=4$ as required.

Moreover, we can also discuss the case $D=2$. (We need not consider $D>4$ because of renormalizability.) In $D=2$ we would require $\gamma_\theta=0$, which would mean no singularity in Eq. (54); thus the mass bootstrap is not achievable in two dimensions. Further in the Thirring model we have $\gamma_\theta<0$, so that any bare mass would vanish asymptotically as in finite QED. Specifically Wilson[25] found that $d_\theta=(1-\lambda)/(1+\lambda)$, where λ is positive for a repulsive current-current interaction, so that $d_\theta<1$. Thus the fermion does not acquire a mass, as is already known from Johnson's solution to the model. In the Thirring model therefore both the bare and renormalized masses are zero, so the fact of γ_θ being negative is only a necessary condition that the mass be dynamical, with Eq. (60) providing the sufficient one. To confirm that the massless Thirring model is energetically stable we also have to show that $\langle\Omega^{(m)}|H^{(0)}|\Omega^{(m)}\rangle-\langle\Omega^{(0)}|H^{(0)}|\Omega^{(0)}\rangle$ is positive, i.e., that $V(\langle\bar{\psi}\psi\rangle)$ has a minimum in the $\Omega^{(0)}$ vacuum, so that $\Omega^{(0)}$ is the true vacuum of the massless theory. Calculating $\epsilon(m)$ in the massive Thirring model can give us information about the consistency of the massive theory but does not tell us anything directly about the stability of the massless theory. However, a knowledge of $\epsilon(m)$ allows us to construct $V(\langle\bar{\psi}\psi\rangle)$ indirectly from Eq. (28) in those cases when $\epsilon(m)$ has a simple enough structure. Indeed our loop summation for $\gamma_\theta<0$ gives

$$\epsilon(m)=-\tfrac{1}{4}(m^2C^2\mu^{-2\gamma_\theta})^{1/(1-\gamma_\theta)}\ \csc\left(\frac{\pi}{1-\gamma_\theta}\right), \tag{61}$$

which is completely finite and hence free of any renormalization ambiguity, and has the structure of Fig. 11. Thus from Eq. (27) we can construct $\langle\bar{\psi}\psi\rangle$ as a function of m, so that from Eq. (28) we obtain

$$\begin{aligned}V(\langle\bar{\psi}\psi\rangle)&=\epsilon(m)-m\langle\bar{\psi}\psi\rangle\\&=(1+\gamma_\theta)\left\{\left(\frac{\mu^2}{4}\right)^{\gamma_\theta}\left[(1-\gamma_\theta)\sin\left(\frac{\pi}{1-\gamma_\theta}\right)\right]^{(1-\gamma_\theta)}\frac{\langle\bar{\psi}\psi\rangle^2}{C^2}\right\}^{1/[1+\gamma_\theta]}\end{aligned} \tag{62}$$

In the range $0>\gamma_\theta>-1$ we thus see that $V(\langle\bar{\psi}\psi\rangle)$ is stable about $\Omega^{(0)}$ having the structure of Fig. 9, so that the massless vacuum is self-consistent. Moreover, we also note that for $\gamma_\theta<-1$ the Hamiltonian is unbounded since $V(\langle\bar{\psi}\psi\rangle)$ is negatively divergent at the origin; we thus recover the well-known result that the massless Thirring model is unbounded below in $d_\theta<0$.

Returning now to the massive Thirring model we note that $\epsilon(m)$ of Eq. (61) is negative so that the massive vacuum $\Omega^{(m)}$ in the theory where there is a mass term in the Lagrangian survives our consistency check with the massive model apparently also being acceptable. (Since m is not a parameter to be varied—each value of m defines a different theory—we cannot conclude that the massive theory is unbounded below in Fig. 11.) However, we can only satisfy the condition $\epsilon'(m)=0$ at $m=0$ and can thus never remove the tadpole in the massive case. Unfortunately, it is not clear whether this additional requirement is necessary,[26] and consequently we are unable to exclude the massive Thirring model for the moment. Further study of the nature of the requirement that $\epsilon'(m)$ vanish is necessary in order to determine whether the massive Thirring model is an acceptable theory. While discussing two dimensional field theories we would like to make one additional remark, namely that for a free Fermi theory in $D=2$ $\epsilon(m)$ has the structure of Fig. 10 after removing a single logarithm, so a free massive fermion is stable in $D=2$ to contrast with the situation obtained in $D=4$.

So far we have studied the momentum dependence of Eqs. (43) and (58) and obtained the bootstrap condition of Eq. (60). However, the equivalence of the two propagators contains further information as it also equates the numerical coefficients of the nonleading parts of those propagators. We shall now show that this further constraint is also satisfied identically when $\gamma_\theta=-1$. To do this we shall need an additional relation between $E(\alpha)$ and $C(\alpha)$ other than the one given by equating Eqs. (50) and (51) (restricting ourselves now to $D=4$), viz.,

$$E(\alpha)2^{6-2d_\theta}\frac{\Gamma^2(\frac{5}{2}-\frac{1}{2}d_\theta)}{\Gamma^2(\frac{1}{2}d_\theta+\frac{1}{2})}=C^2(\alpha)\ . \tag{63}$$

The extra relation will be supplied by studying the consistency of various operator-product expansions. In massless fermion QED conformal invariance yields

$$\langle\Omega^{(0)}\,|\,T(\psi(x):\bar{\psi}(z)\psi(z):\bar{\psi}(y))\,|\,\Omega^{(0)}\rangle = Z_2^{-1}\frac{A(\alpha)}{4\pi^4}\,\frac{(\hat{y}-\hat{z})(\hat{z}-\hat{x})}{\mu^{\gamma_\theta}\,[(y-z)^2(z-x)^2]^{(1/2)[d_\theta+1]}\,[(x-y)^2]^{(1/2)[3-d_\theta]}}\ . \tag{64}$$

Thus from the consistency of Eqs. (49), (54), and (64) obtained by letting x approach y in Eq. (64) we find that

$$A(\alpha)=D(\alpha)E(\alpha)\ . \tag{65}$$

Moreover, we can Fourier transform Eq. (64) analytically at zero momentum transfer using the method described in Ref. 27. After amputating the fermion legs with $Z_2\not{p}$ inverse propagators we obtain

$$\begin{aligned}\bar{\Gamma}^{(0)}_{S,\mu}&(p,p,0)\\ &=Z_2A(\alpha)2^{1-d_\theta}(d_\theta-1)\\ &\times\frac{\Gamma^2(\frac{5}{2}-\frac{1}{2}d_\theta)\Gamma(\frac{3}{2}-\frac{1}{2}d_\theta)\Gamma(d_\theta-2)}{\Gamma^2(\frac{1}{2}+\frac{1}{2}d_\theta)\Gamma(\frac{1}{2}+\frac{1}{2}d_\theta)\Gamma(3-d_\theta)}\left(\frac{-p^2}{\mu^2}\right)^{(1/2)\gamma_\theta}.\end{aligned} \tag{66}$$

The coefficient in Eq. (66) was previously called $Z_2C(\alpha)$ in Eq. (41). Hence we can now express Eq. (58) entirely in terms of $C(\alpha)$ and proceed to compare it with Eq. (43).

If we now approach the special value $d_\theta=2$ from below we find first that

$$-\frac{A(\alpha)}{|2-d_\theta|}=C(\alpha)\ . \tag{67}$$

Since we require $mC(\alpha)(-p^2/\mu^2)^{(1/2)\gamma_\theta}$ to be the dynamical mass term in the first place $C(\alpha)$ must be finite. Thus $A(\alpha)$ must vanish when $d_\theta=2$. In passing we then remark that since Eq. (64) is presumably nontrivial the vanishing of $A(\alpha)$ in turn implies the vanishing of Z_2 when $d_\theta=2$, where we recall again that Z_2 is the gauge-independent part of the electron wave-function renormalization constant. We mention this point here only because it may prove to be interesting, but will not affect the work of this paper since objects like $\epsilon(m)$ are independent of Z_2. We now eliminate $A(\alpha)$ and $E(\alpha)$ to obtain

$$C(\alpha)D(\alpha)=-4|2-d_\theta|\ . \tag{68}$$

Hence the consistency of Eqs. (58) and (43) requires

$$\sigma=-\frac{m\mu^2C^2(\alpha)}{4\pi^2|2-d_\theta|}\ , \tag{69}$$

where σ is also to be obtained from Eq. (46). However, before we make the comparison we must point out that the short-distance analysis that we have just made has not taken into account the counterterms used to renormalize $\epsilon(m)$ and hence we must compare Eq. (69) with the unrenormalized form of $\epsilon'(m)$. In a sense the counterterms take us outside of QED as a closed theory, a point which we will discuss in more detail below. So long as we stay within pure QED, however, the unrenormalized $\epsilon(m)$ of Eq. (47) is infinite when $d_\theta=2$ (but still negative-definite so that the massive theory is energetically favored). For our purposes constructing $\epsilon'(m)$ from Eq. (47) is not the most convenient way of parametrizing the infinity in Eq. (69). Instead we first construct $\epsilon(m)$ in $d_\theta<2$ where it is completely convergent, viz.,[13]

$$\epsilon(m)=\frac{[C^2(\alpha)m^2\mu^{-2\gamma_\theta}]^{2/(1-\gamma_\theta)}}{16\pi}\ \csc\left(\frac{\pi(1+\gamma_\theta)}{1-\gamma_\theta}\right). \tag{70}$$

If we now analytically continue onto the pole at $\gamma_\theta=-1$ we obtain

$$\epsilon'(m)=-\frac{m\mu^2C^2(\alpha)}{4\pi^2|1+\gamma_\theta|}\ , \tag{71}$$

which agrees indentically with Eq. (69), to demonstrate the complete consistency between the two different ways of constructing the nonleading part of the fermion propagator.

We turn now to the difficult question of the significance of the counterterms we introduced in Sec. III. As we have just seen in deriving Eq. (71) we can have a consistent formulation of the theory in which both $\epsilon(m)$ and $\epsilon'(m)$ are negatively divergent with the ultimate final form for the propagator of Eq. (43) still being completely finite. [A possible zero in Z_2 would not affect the position of the pole in Eq. (43).] In that respect the unrenormalized structure found for $\epsilon(m)$ still leads to a perfectly acceptable physical system, unless there is some measurement which is sensitive to the logarithmic divergences in $\epsilon(m)$ or $\epsilon'(m)$. Since we will suggest in Sec. VII that $\epsilon'(m)$ is indeed measurable as a tadpole contribution to the Gell-Mann, Oakes, and Renner Hamiltonian we shall therefore favor some renormalization scheme. In renormalizing $\epsilon(m)$ we then have to go outside of QED as a closed theory. Moreover, in order to obtain the counterterm used in deriving Eq. (48), i.e., a term proportional to m^2, we must add into the Lagrangian a term of the form $(\bar{\psi}\psi)^2$, so that a 4-Fermi interaction is induced into QED as a renormalization counterterm, this being a hitherto unsuspected role for the 4-Fermi interaction. This intriguing possibility obviously demands further investigation, and so far we only have some speculations which we will present in Secs. VI and VII.

The renormalization prescription we introduced in Sec. III not merely removed the infinity from $\epsilon'(m)$, but the finite part as well, so we will now discuss an alternative prescription in which $\epsilon'(m)$ remains nonvanishing. This alternative prescription is due to Cornwall, Jackiw, and Tomboulis (Ref. 14), whose work bears some similarity to ours. They also constructed an effective action for composite operators whose form is

$$\epsilon^{\mathrm{CJT}}(m) = i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\ln\left(\frac{Z_2^{-1}}{\not p}\,\tilde S_\mu^{-1}(p)\right) + Z_2\not p\tilde S_\mu(p) - 1\right] \qquad (72)$$

in one loop, to be compared with our Eq. (45). Thus the two additional terms of Eq. (72) may be thought of as representing some counterterms added to Eq. (45). Indeed calculating $\epsilon^{\mathrm{CJT}}(m)$ for any negative γ_θ leads to a finite negative result and in particular at $\gamma_\theta = -1$ [care must be taken in performing the actual calculation to introduce a combinatoric factor as in Eq. (42)] we find

$$\epsilon^{\mathrm{CJT}}(m) = -\frac{C^2(\alpha) m^2 \mu^2}{16\pi^2}\;, \qquad (73)$$

so that it corresponds to setting $M = m$ in Eq. (48). In this case the tadpole is now both finite and nonzero, i.e.,

$$\sigma = -\frac{C^2(\alpha) m \mu^2}{8\pi^2}\;. \qquad (74)$$

Moreover, we also remark that for a free theory $\epsilon^{\mathrm{CJT}}(m)$ is found to be logarithmically divergent and negative, so that it would still require an additional renormalization; alternatively we can say that $\epsilon^{\mathrm{CJT}}(m)$ is completely finite when the mass is dynamical. The nice feature of Eq. (72) is that the two additional terms also depend on $\tilde S_\mu^{-1}(p)$, and hence the counterterms do not alter the structure of the propagator, so that, after renormalizing $\epsilon(m)$, the propagator still possesses the same scaling behavior. More importantly the $\tilde S_\mu^{-1}(p)$ required for Eq. (72) is the exact propagator for all momenta and not merely the asymptotic propagator. We then have to conclude that when $\gamma_\theta = -1$ Eq. (43) gives the exact electron propagator of quantum electrodynamics even on-shell, so that we can identify $m = C(\alpha)\mu$ by normalizing to Eq. (4) with a pole at $\not p = m$; with this choice $\epsilon^{\mathrm{CJT}}(m) = -m^4/16\pi^2$, so that the physically insignificant parameter μ has finally disappeared from the theory. This viewpoint that Eq. (4) is exact is reinforced by our derivation of Eq. (58), since we obtained the nonleading term from the infrared, so that there is nothing else left in the theory to generate further nonleading terms in Eq. (58), with the obtained term being both ultraviolet (asymptotic) and infrared (nonasymptotic) at once.[28]

Finally we then conclude this section by remarking again that the bootstrap condition $\gamma_\theta = -1$ is the nontrivial consequence of the fact that infrared effects show up in the short-distance limit, or conversely that conformal invariance contains sufficient information to force $\bar\psi\psi$ to acquire a nonvanishing vacuum expectation value.

V. ON THE NATURE OF THE BOUND-STATE PROBLEM

As we stressed in the Introduction the question of bound states is not related to the question of the degeneracy of the vacuum, or more precisely to the question of the existence of the infinite family of equivalent physical descriptions constructed in Sec. II. The Goldstone theorem only requires a pole if there is a conserved current. As we have seen, the fact that the vacuum is degenerate is not by itself sufficient to force current conservation, so there need not be an accompanying Goldstone boson. Though this point is already understood, we would like to make some clarifying remarks about the nature of the bound-state problem to bring out the distinction between the finite QED mode and the Goldstone mode.

We discuss first the Goldstone mode. In theories in which the Lagrangian is γ_5-invariant but the vacuum is not, the scalar part of the fermion self-energy satisfies the unrenormalized gap equation, Eq. (1).[15] In this case m_0 vanishes identically, and the physical mass is self-consistently determined. The unrenormalized pseudoscalar vertex Γ_P of the insertion of $i\bar\psi\gamma_5\psi$ into the inverse fermion propagator satisfies

$$\Gamma_P(p, p+q, q) = \gamma_5 + \int d^4k\, K(p,k,q)\, S(k) \times \Gamma_P(k, k+q, q)\, S(k+q)\;. \qquad (75)$$

Note that Γ_P is not the insertion of $\partial^\mu j_{\mu 5}$ into the inverse propagator, since $\partial^\mu j_{\mu 5} = 0$, so that Γ_P does not appear in the axial-vector Ward identity, but is nonetheless a well-defined Green's function of the theory. Now if Eq. (1) is of the Fredholm type its consistency with Eq. (75) would then require the presence of a pseudoscalar pole in Γ_P at $q^2 = 0$ whose residue satisfies

$$G(p, p+q) = \int d^4k\, K(p, k, q) S(k) G(k, k+q) \times S(k+q)\;. \qquad (76)$$

Alternatively we could make the same analysis for the axial-vector vertex $\Gamma_{\mu 5}$ to obtain Eq. (76) again where now $G(p, p+q)$ is the residue of the pole in $\Gamma_{\mu 5}$. We then obtain consistency with Eq. (1) (the Goldberger-Treiman relation) by going to the pole in the axial-vector Ward identity in the familiar manner. Thus to establish the Goldstone mode in a given theory requires to show that there is a pole in $\Gamma_{\mu 5}$ or in Γ_P, to show that Eqs. (1) and (76) have a nontrivial solution, and finally to show that such a nontrivial solution possesses lower energy than the trivial one. As far as we know this program has only been carried out completely in the BCS theory of superconductivity and in the Hartree-Fock and loop approximations to 4-Fermi theories discussed in Refs. 1 and 12. We are not aware that this program has ever been carried out in its entirety in the vector gluon model. What is usually done in the literature is to show that Eqs. (1) and (76) have an explicit nontrivial solution, say in some approximation such as the ladder approximation which provides a summation of an infinite class of graphs. This of course, like the Ward identity, only shows that the assumption of a pole is consistent. To establish the pole requires a return to the inhomogeneous Bethe-Salpeter equations satisfied by $\Gamma_{\mu 5}$ or Γ_P to see whether the same ladder approximation will actually generate a pole in those Green's functions in the first place. Despite the warning of Baker, Johnson, and Lee[15] that this may well not happen in a renormalizable theory since the kernel may not be compact, this point has never been fully resolved in the literature. Moreover, there has also never been any demonstration that the Goldstone mode is energetically favored in the gluon model either, which would require a calculation of $V(\langle\bar{\psi}\psi\rangle)$. Thus to repeat, constructing an explicit solution to Eq. (1) is a necessary but totally insufficient criterion for establishing the Goldstone mode.

We turn now to the case where there is a bare mass m_0 in the theory. In such a case Eq. (1) is modified as[18] $(S^{-1}(p) = \not{p} - m_0 - \Sigma(p))$

$$\{\gamma_5, \Sigma(p)\}_+ = \int d^4k\, K(p,k,0) S(k) \{\gamma_5, \Sigma(k)\}_+ S(k) + 2m_0 \int d^4k\, K(p,k,0) S(k)\gamma_5 S(k) \,. \tag{77}$$

Now of course there is no Goldstone mode and $\partial^\mu j_{5\mu} = 2m_0 i\bar{\psi}\gamma_5\psi$. The unrenormalized Ward identity gives

$$\{\gamma_5, \Sigma(p)\}_+ + 2m_0\gamma_5 - 2m_0\Gamma_P(p,p,0) = 0 \,, \tag{78}$$

so that Eq. (77) is recovered directly from Eqs. (75) and (78). In passing we note an interesting consequence of Eq. (78); it requires $\Gamma_P(p,p,0)$ to be pure γ_5 and have no term of the form $\not{p}\gamma_5$, both for asymptotic and nonasymptotic momenta. Now Eq. (77) is true for any m_0 finite or otherwise. Suppose, however, that m_0 vanishes when the cutoff is removed, as is the case in finite QED. In that case Eq. (77) contains an ambiguity of the form $0\times\infty$ since the γ_5 projection of the kernel is noncompact. We shall resolve this ambiguity by making a subtraction so that Eq. (77) becomes

$$\{\gamma_5, \Sigma(p)\}_+ - \{\gamma_5, \Sigma(p')\}_+ = \int d^4k [K(p,k,0) - K(p',k,0)] \times S(k)\{\gamma_5, \Sigma(k)\}_+ S(k) \,, \tag{79}$$

with the integration over the kernel now being finite. Now the correct behavior of the self-energy when p and p' are far off the mass shell is given by replacing the propagators in the kernel by their free forms. This is done either by disregarding vacuum polarization insertions in the photon propagator[2] or by summing the theory loopwise at an eigenvalue for the coupling constant.[19] Johnson[18] has given a general method for extracting the asymptotic solution to equations such as Eq. (79). He introduces (A denotes asymptotic)

$$f\left(\frac{p^2}{p'^2}\right) = 1 + \int d^4k [K_A(p,k,0) - K_A(p',k,0)] \frac{1}{k^2} f\left(\frac{k^2}{p'^2}\right), \tag{80}$$

so that

$$\frac{d}{dp^2} \ln\{\gamma_5, \Sigma(p)\}_+ \sim \frac{m\gamma_5}{p^2} f'(1) \,, \tag{81}$$

where m is arbitrarily chosen to set the scale. Thus for asymptotic momenta

$$\{\gamma_5, \Sigma(p)\}_+ \sim m\gamma_5 \left(\frac{-p^2}{m^2}\right)^{f'(1)} \,. \tag{82}$$

If $f'(1)$ is negative we note that this solution also reproduces itself in Eq. (1) (the negative power supplies the necessary damping for the integration) so that Eq. (1) exists in this case without renormalization. This is of course the reason why finite QED is so called. Since Eq. (1) exists without renormalization if $f'(1)$ is negative it would then appear that we have an opportunity to obtain the Goldstone mode after all if we have Eq. (82). However, such appearances are illusory, as is best seen by reconsidering Γ_P. If we make a subtraction in Eq. (75) to eliminate the bare vertex we obtain

$$\Gamma_P(p,p,0)-\Gamma_P(p',p',0)=\int d^4k[K(p,k,0)-K(p',k,0)] \times S(k)\Gamma_P(k,k,0)S(k)\ , \tag{83}$$

which is identical in form to Eq. (79), so that

$$\Gamma_P(p,p,0)\sim\gamma_5\left(\frac{-p^2}{m^2}\right)^{f'(1)}\ . \tag{84}$$

This form for Γ_P is inconsistent with Eq. (75) even if $f'(1)$ is negative since the bare vertex survives asymptotically. Consequently, Γ_P needs to be renormalized [it is not clear whether this renormalization is related to that required for $\epsilon(m)$] even in what is called finite QED. We shall perform this renormalization multiplicatively instead of by subtracting by defining $\tilde{\Gamma}_P=Z_P\Gamma_P$, where

$$Z_P\sim\left(\frac{\Lambda^2}{m^2}\right)^{f'(1)} \tag{85}$$

and vanishes in the limit of infinite cutoff. Consequently, after renormalization $\tilde{\Gamma}_P$ now satisfies a homogeneous equation, Eq. (21), in fact with $m_0Z_2=mZ_P=0$. Thus the inhomogeneous bare vertex term is removed when $Z_P=0$, and $\tilde{\Gamma}_P(p,p+q,q)$ can now no longer generate a pole associated with trying to treat Eq. (1) as a Fredholm eigenvalue problem. (Moreover, even if it transpired that the potential pole just happened to decouple from $\tilde{\Gamma}_P$ while still appearing in $\tilde{\Gamma}_{\mu5}$, the fact of the vanishing of Z_P has already shown us that the γ_5 projection of the kernel is noncompact, so we now know that the bound-state problem is not of the Fredholm type.) Since m_0Z_2 is the renormalizing factor for Γ_P we can now also proceed to derive the Callan-Symanzik equation, Eq. (3), and hence identify $2f'(1)$ with $\gamma_\theta(\alpha)$. Technically this is effected by introducing $Z_\theta^{-1/2}$ which renormalizes $\bar\psi\psi$ so that $Z_S=Z_P=Z_2Z_\theta^{-1/2}$ renormalizes Γ_S. If we define

$$\gamma_\theta=m\frac{\partial}{\partial m}[\ln Z_\theta^{1/2}(g_\Lambda,\Lambda/m)|_{g_\Lambda,\Lambda}]\ , \tag{86}$$

Eq. (4) then follows at the eigenvalue. In this way m_0 is introduced as a multiplicative renormalization constant rather than an additive one with $m_0Z_\theta^{1/2}=m$, so that

$$m_0(\bar\psi\psi)_0=m(\bar\psi\psi)_{\rm ren}\ . \tag{87}$$

[In fact our mass bootstrap may be thought of as reinterpreting a vertex renormalization already present in the massless theory in Eq. (41) as a mass renormalization in the massive Eq. (3).] Thus the fact of anomalous dimensions signals for us the fact that there was a nontrivial renormalization in the theory which itself can be traced back to the fact that the γ_5 projection of the kernel was noncompact. Hence theories with anomalous dimensions are not expected to possess bound states, a point we have already stressed.[27] Actually the argument is not quite complete since there may be theories in which the kernel is not of the Fredholm type with there still being possible bound states, a suggestion made recently by Jackiw and Johnson.[10] This possibility seems unlikely in cases which possess the conventional type of kernel met in renormalizable field theories, but may perhaps occur in a bootstrap type situation where the kernel already contains the bound state itself. Such possibilities, however, still remain to be explored.

Though the γ_5 projection of the kernel is noncompact, the $\gamma_\mu\gamma_5$ projection of the kernel turns out to be compact at the finite QED eigenvalue.[27] This then raises the question of why there actually is no pole in $\tilde{\Gamma}_{\mu5}$ which still satisfies an inhomogeneous equation,

$$\tilde{\Gamma}_{\mu5}(p,p+q,q)=Z_A\gamma_\mu\gamma_5+\int d^4k\ \tilde{K}(p,k,q)\tilde{S}(k) \times\tilde{\Gamma}_{\mu5}(k,k+q,q)\tilde{S}(k+q)\ , \tag{88}$$

since $Z_A(=Z_1)$ is finite. We have recently noted that[27]

$$\tilde{\Gamma}_{\mu5}(p,p,0)=Z_A\gamma_\mu\gamma_5\ , \tag{89}$$

so that there is no zero mass pole in Eq. (88). Thus we satisfy the axial-vector Ward identity of Eq. (22) by associating the leading part of $\tilde{S}^{-1}$ with $\tilde{\Gamma}_{\mu5}$ and the nonleading part with $\tilde{\Gamma}_P$.

In passing we would like to make an additional remark about the Ward identity. Because of the essential singularity in $\beta(\alpha)$ (see Ref. 19) we can sum the theory loopwise. There is thus no need to consider closed fermion loops in the Bethe-Salpeter kernel, so that there is no intermediate two-photon state in the axial-vector Ward identity for $\tilde{\Gamma}_{\mu5}$ but just a continuous fermion line dressed to all orders with photon lines. Hence the perturbative triangle anomaly (which exists even for massless physical fermions) plays no role at all in the analysis of this paper. Loopwise summed QED is an example of a theory which possesses a nonperturbatively anomalously nonconserved axial-vector current (provided the physical fermion mass is nonzero) while possessing no triangle modification to $\tilde{\Gamma}_{\mu5}$ at all. [Though there may still be a modification to the vector, vector, axial-vector vertex it simply does not show up in Eq. (22).] Thus these two types of anomaly are on a different dynamical footing.

Though we have been referring to m_0 continually as the bare mass, we would like at this point to examine in more detail what the actual significance of the parameter m_0 is. We note first that it is important to distinguish between renormalized mass and translated physical mass and between asymptotic mass and input bare mass. In Eqs. (2) and (87) the parameters m_0 and m must both refer to the same vacuum be it $\Omega^{(0)}$ or $\Omega^{(m)}$. Like the invariant charge

$$e_0^2 D'_{\mu\nu}(q^2) = e^2 \tilde{D}_{\mu\nu}(q^2) , \qquad (90)$$

we note that according to Eq. (87) $m\bar{\psi}\psi$ is also a renormalization invariant. Thus using the usual renormalization-group approach we can distinguish between m_0 and m by moving the momentum space subtraction point into the asymptotic region. Thus m_0 is the effective asymptotic mass, which is only the same as the input bare mass of the Lagrangian if we are in the normal or bare vacuum. However, in our case we identify m with the position of the minimum in $\epsilon(m)$ by translating to the new vacuum. If we now do perturbation theory by exciting particles out of this degenerate vacuum then m_0 and m respectively parametrize the asymptotic and nonasymptotic scattering of these excitations. Moreover, it was our identification of m with $\langle\bar{\psi}\psi\rangle$, i.e., of the particle content of quantum fluctuations about a vacuum with the position of that vacuum, that led us to the bootstrap condition $\gamma_\theta(\alpha) = -1$. Thus m_0 is no longer the bare mass taken in the normal vacuum but rather it parametrizes the asymptotic behavior of $\tilde{\Gamma}_S(p, p, 0)$, a quantum fluctuation calculated in the massive vacuum, $\Omega^{(m)}$. Thus for us the vanishing of the effective asymptotic mass m_0 is the statement that

$$\lim_{p^2 \to -\infty} \tilde{\Gamma}_S(p, p, 0) \sim \left(\frac{-p^2}{m^2}\right)^{(1/2)\gamma_\theta(\alpha)} \to 0 , \qquad (91)$$

so that $\gamma_\theta(\alpha)$ has to be negative. Self-consistently we then find that the integration over the kernel in Eq. (2) exists since the negative-power behavior provides the necessary damping for the integral. The important content of Eq. (87) is not that $m_0\langle\bar{\psi}\psi\rangle_0$ and $m\langle\bar{\psi}\psi\rangle_{\text{ren}}$ are equal to each other, but rather that both are equal to a nonzero quantity. This quantity will be nonzero if the over-all multiplying factor m in Eq. (23) is nonzero, and as we have seen this is an infrared effect. The current is broken in the solution because we have to choose one of the set $\cos 2\alpha \tilde{\Gamma}_S + \sin 2\alpha \tilde{\Gamma}_P$ to be the self-energy, so that Eq. (23) forces the orthogonal combination to correspond to a nonconserved current. It is only in the underlying massless theory, the untranslated theory where $\langle\bar{\psi}\psi\rangle = 0$, that the current is conserved. However, because of Eq. (91) we see that chiral invariance tends to be restored in the deep Euclidean region, so that the short-distance behavior of the theory built on the degenerate vacuum will look rather like that of a chiral-invariant theory with a normal vacuum, with vacuum-degeneracy effects only showing up as discussed in Sec. IV [this would be a natural mechanism for asymptotic $SU(3)\times SU(3)$]. Further, since $m_0 = 0$, we see that $\delta m = m$, so that we obtain the interesting connection that $\delta m = M$, i.e., that asymptotic to non-asymptotic mass shift in the degenerate vacuum is equal to the shift between the vacua of Fig. 10. We thus have two equivalent ways of thinking of a mass shift.

There is one point which we have glossed over in regard to the nonconversation of the axial-vector current. It is very difficult to see how changing the vacuum from the underlying massless theory to the new massive physical theory could break current conservation, since this cannot be understood in a Lagrangian framework. In our work we have gone directly into the massive theory and checked self-consistency using Eq. (36). We have not begun in the massless theory and used Eq. (30) to find the massive one, and consequently have not needed to face this problem directly, since it is mainly one of interpretation. A possible resolution of this difficulty is that perhaps even in what we have been referring to as the underlying massless theory there actually is a nonconserved axial-vector current whose matrix elements all vanish in the conformal invariant $\Omega^{(0)}$ vacuum. (Indeed there certainly will be some extra terms in the Lagrangian of the underlying massless theory, namely the counterterms induced to renormalize $\Gamma_S^{(0)}$ and $\Gamma_P^{(0)}$ terms which in a massive theory are usually recognized as mass counterterms.) In such a case the axial-vector Ward identity would still lead us to infer that the bare (asymptotic) and renormalized (nonasymptotic) masses of the fermion taken in $\Omega^{(0)}$ are zero. Thus in the wrong vacuum we do not feel the fact of the nonconservation. It is only when we change to the true $\Omega^{(m)}$ vacuum that we see a nonzero physical mass which still looks as though its associated bare mass is zero, in the sense of Eq. (91).

There is an additional way of describing the situation we find ourselves in. We appear to have a symmetry which is broken both in the vacuum and in the Lagrangian. This would be like a σ model with a wrong sign for the mass term to break the symmetry in the vacuum to which is then added a linear term to break the symmetry in the Lagrangian but in such a way so as not to remove the degeneracy of the vacuum. (We gave this example merely to define the different types of breaking, though clearly we are in a situation which cannot be achieved with fundamental scalars in a Lagrangian framework.) What we have with dynamical symmetry breaking is that the massless vacuum becomes γ_5 degenerate to produce a mass term

$m\bar{\psi}\psi$. This mass term then provides the massless theory with a preferred direction (as though it were a linear σ term) but in such a way that the massive theory still has a completely degenerate vacuum. Thus the theory simultaneously bootstraps both a degenerate vacuum and a preferred direction. Thus the same mechanism both produces and "removes" the degeneracy of the vacuum. (In the language of soft-pion physics this would mean that f_π and m_π are switched on simultaneously. Further, the axial-vector current is partially conserved in the sense that its divergence has dimension $4+\gamma_\theta=2$, so that the divergence is a soft operator which can thus satisfy an unsubtracted dispersion relation.) Even though this mechanism is not yet completely understood, we nevertheless regard it as a most attractive feature of dynamical symmetry breaking, since it may then never be necessary in particle physics to actually find an independent mechanism at all for lifting the degeneracy of the vacuum.

VI. CAN A MASSLESS PARTICLE HAVE A CHARGE?

In this work we have essentially been trying to realize an idea which dates back to Lorentz, which is that the whole of the mass of the electron would be electromagnetic in origin. Lorentz introduced the classical electron radius for that purpose. However, at the time the idea could not be accepted because if an electron were really a ball of charge then it would not be stable under Coulomb repulsion. Also, of course, the discussion ignored completely the question of quantum fluctuations in the self-energy. We see now that the role of the quantum fluctuations is precisely to sum up the infrared divergences of the theory to produce long-range order (the physical mass), i.e., "superconducting" attraction, which then stabilizes the physical electron. Thus an initial massless electron in interaction with an infinite bath of massless pairs feels an attractive force if $\gamma_\theta(\alpha)=-1$ and undergoes a phase transition, which then produces a massive electron which will now scatter repulsively off other massive electrons. All of the attraction is used up in producing the mass, leaving only weak residual repulsive forces in the massive theory.

We would like to develop this analogy with phase-transition theory a little further. There are actually two formulations of the theory of superconductivity, one due to Bardeen, Cooper, and Schrieffer[29] and the other due to Bogoliubov.[30] Both start with an initial electron-phonon interaction which produces the attractive Cooper pair. The BCS approach is to then abstract from this fundamental electron-phonon interaction an induced electron-electron interaction of the 4-Fermi type, the reduced BCS Hamiltonian, which is then discussed on its own. This Hamiltonian is then minimized using a Hartree-Fock trial wave function analogous to that of Eq. (11) with the minimization condition being the self-consistent gap equation. Moreover, in the limit of an infinite number of degrees of freedom this trial wave function is found to be exact. The approach of Bogoliubov on the other hand is to stay with the electron-phonon interaction and make the Bogoliubov transform to the self-consistent vacuum. Bogoliubov then demands that the quantum flunctuations about this vacuum be finite, the method of compensation of dangerous diagrams, and obtains a constraint which turns out to be none other than the BCS gap equation. It is then immediately clear that these two approaches are synthesized together in the theory of finite QED as set up in this paper, with the sole exception being that we have lost the collective modes through renormalization, but still obtained the attraction. Our loop summation is thus seen to be the relativistic generalization of the BCS Hartree-Fock approximation. All of this then indicates that a nonperturbatively renormalizable 4-Fermi interaction can be constructed as the relativistic generalization of the theory of superconductivity.

Though we have shown that an electron can get its mass from its own charge, we have not gone the other way to show that it needs a mass in order to carry a charge. To do this we should show that the theory with $m=0$ identically (not to be confused with the theory obtained by letting $m\to 0$) is in fact a free theory. In fact we can give a partial answer to this question by calculating the effective potential of massless QED. From Eq. (70) we obtain directly

$$V(\langle\bar{\psi}\psi\rangle)=(\gamma_\theta+3)\left\{\left[\pi(1-\gamma_\theta)\sin\left(\frac{-\pi(1+\gamma_\theta)}{1-\gamma_\theta}\right)\right]^{1-\gamma_\theta}\times\frac{\langle\bar{\psi}\psi\rangle^4}{[4^{\gamma_\theta+1}C^2(\alpha)\mu^{-2\gamma_\theta}]^2}\right\}^{1/(\gamma_\theta+3)}, \tag{92}$$

which has the stable structure of Fig. 9. Thus the massless theory is self-consistent when $\gamma_\theta<-1$. Moreover, we can also continue onto $\gamma_\theta=-1$ where we find that $V(\langle\bar{\psi}\psi\rangle)$ vanishes identically. (This also confirms that the Goldstone mode is not realizable in the massless gluon model at our eigenvalue.) Now while it is true that $V(\langle\bar{\psi}\psi\rangle)$ vanishes if the theory is free [as may be seen from Eq. (62) when continued to $\gamma_\theta=0$], it is not clear that the vanishing of $V(\langle\bar{\psi}\psi\rangle)$ forces the theory to be free, though it does seem likely. Should it be the case that massless QED is free when $\gamma_\theta=-1$, we would then have a situation in which mass and charge bootstrap together. Though theorists favor a theorem which would forbid a massless particle from having a charge (in agreement with the present experimental situation vis-à-vis the neutrino and the photon) discussion of the question has so far centered on the infrared structure of perturbation theory.[31-33] The program we have presented in this paper is an attempt to set up a nonperturbative framework for studying the question.

The one open question raised by this work is whether the conditions $\beta(\alpha)=0$, $\gamma_\theta(\alpha)=-1$ are in fact compatible. *A priori* it seems to be somewhat difficult to tell, perhaps being as difficult as actually trying to solve for α. We shall conclude this section by indicating that a possible connection between $\beta(\alpha)$ and $\gamma_\theta(\alpha)$ may be realized by studying the one remaining unexplored sector of finite QED, namely the infrared structure of the vacuum polarization as the electron mass goes to zero. Adler[19] has discussed $\tilde{D}_{\mu\nu}$ in massive finite QED in the limit $m\to 0$. Since Z_3^{-1} is finite at the eigenvalue the absorbtive part of $\tilde{D}_{\mu\nu}$ vanishes for all q^2 except $q^2=0$ where there could be possible infrared divergences. These divergences can then give a contribution to Z_3^{-1} by putting a δ function into the spectral function sum rule and prevent the theory from being free ($Z_3=1$) when finite in the limit of vanishing electron mass. Then the photon propagator behaves asymptotically like Z_3^{-1}/q^2, where Z_3^{-1} contains infrared information, to demonstrate again the subtle interplay between the ultraviolet and the infrared. Thus the natural question to consider is whether the same infrared divergences which give the electron its mass are also the ones which contribute to the vacuum polarization spectral function. We hope these infrared effects will prove to be two aspects of the same phenomenon and thus relate the zeros of $\beta(\alpha)$ and $\gamma_\theta(\alpha)+1$. Such a study may also shed some light on the other question, whether a massless particle could have a charge.

VII. GENERAL COMMENTS

The approach we have used so far is to work in the self-consistent vacuum. It is of interest to ask what sort of an untranslated theory could produce the massive theory after translating $\bar{\psi}\psi$. We would expect the untranslated but dressed Lagrangian not to be just that of pure QED, but to also contain an induced effective chiral-invariant 4-Fermi interaction of the form $(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2$ with dimension $6+2\gamma_\theta(\alpha)=4$, so that the mass term would appear after translating. By making a Fierz transformation this can be written in the form $(\bar{\psi}\gamma_\mu\psi)^2-(\bar{\psi}\gamma_\mu\gamma_5\psi)^2$ in which this difference has dimension 4 even though each term has dimension 6. [This is reminiscent of Wilson's discussion[20] of the convergence of the Weinberg spectral function sum rules which are controlled by the u_i tadpoles, the non-Abelian equivalents of $\langle\bar{\psi}\psi\rangle$, with $\langle\Omega^{(m)}|T(V_\mu V_\nu)|\Omega^{(m)}\rangle-\langle\Omega^{(m)}|T(A_\mu A_\nu)|\Omega^{(m)}\rangle$ being dominated by operators of dimension less than 6.] The viewpoint suggested here then is that the tadpoles first acquire a nonvanishing vacuum expectation value through the infrared and then appear after translating to the degenerate vacuum as terms in the current-current operator-product expansion in the ultraviolet, so that the tadpoles are simultaneously infrared and ultraviolet objects in much the manner described in Sec. IV. This can only happen in theories in which dynamical symmetry breaking occurs without explicit soft operators. Moreover, since $\gamma_\theta(\alpha)=-1$ we appear to have a possible bootstrapped nonperturbatively renormalizable 4-Fermi interaction of dimension 4 in which the dynamical scalar tadpole replaces the usual scalar field which takes a vacuum expectation value, with the one distinguishing feature that we are spontaneously breaking a global rather than a local symmetry. A deeper appreciation of these points could be of relevance to non-Abelian gauge theories of the type developed by Weinberg.

The extreme viewpoint of our work is to say that the only role of the photon is to produce a fermion sector with anomalous dimensions, in a manner analogous to the derivation of the BCS reduced Hamiltonian. In our discussion of Sec. III we could have just postulated the conformal invariant Eq. (41) without needing to ask where it came from (i.e., without needing to make a graphical analysis) by writing

$$\tilde{\Gamma}^{(0)}_{S,\mu}(p,p,0)=\lambda\left(\frac{-p^2}{\mu^2}\right)^{(1/2)\gamma_\theta} \tag{93}$$

with an effective coupling constant λ, and then inserted the vertex into Fig. 6 to obtain the tadpole mass. We could think of Eq. (93) as being obtained from some possibly nonlocal 4-Fermi coupling carrying the additional power behavior of Eq. (93). Such a power would then solve the unitarity problem at high energies in 4-fermion scattering since the coupling is now no longer a pure s-wave point interaction, with anomalous dimensions softening the theory sufficiently to make it renormalizable if $d_\theta(\alpha)=2$. We have also seen that the summation of Fig. 6 produces long-range order or attraction. Thus after renormalizing $\epsilon(m)$ we can define a new M-dependent λ and an effective $\beta(\lambda)$ using the usual renormalization-group analysis of Ref. 9 which expresses the lack of physical significance of the actual value of M. (The physical significance of M is that it is nonzero.) This $\beta(\lambda)$ of course has nothing to do with our original $\beta(\alpha)$, since it describes renormalization of the 4-Fermi coupling and not of the electric charge. Since this $\beta(\lambda)$ produces attraction it should be negative since a positive $\beta(\lambda)$ is ordinarily repulsive, and this is indirectly confirmed since from Eq. (48) we see that

$$\epsilon''''(M) = -\frac{\lambda^2\mu^2}{4\pi^2 M^2} = -\kappa^2; \qquad (94)$$

i.e., the only dimensionless parameter associated with $\epsilon(m)$ is its fourth derivative, which we see is negative at the minimum energy $m=M$, and so we have an attractive theory. Moreover, we may also expect a renormalization-group equation for changes in M of the type suggested in Ref. 9 for $V(\phi_c)$ only with m replacing the classical field. This would be of the form

$$\left[M\frac{\partial}{\partial M} + \beta(\kappa^2)\frac{\partial}{\partial \kappa^2} - \gamma_m(\kappa^2) m\frac{\partial}{\partial m}\right]\epsilon(m) = 0. \qquad (95)$$

Thus from Eq. (48) we obtain

$$\beta(\kappa^2) = -4\kappa^2, \quad \gamma_m(\kappa^2) = -1 \qquad (96)$$

as required. [Though $\gamma_m(\kappa^2)$ bears no direct relation to $\gamma_\theta(\alpha)$—their identity would require $\alpha = 1/4\pi^2$—it is curious that it is also equal to -1 at the point $m=M$.] This argument is somewhat heuristic for the moment. Nonetheless it then invites the possibility that an Abelian gauge theory may bootstrap a 4-Fermi interaction which will then exhibit Bjorken scaling nonperturbatively. Clearly a lot more study will have to go into confirming this possibility.

We conclude this section by indicating a possible extension of our ideas to the case of a non-Abelian gauge theory. If our ideas are to apply directly in this case we first need conformal invariance with anomalous dimensions. To achieve this we need an additional infrared-stable fixed point away from the origin which will presumably need an infinite-order zero in order to take over the short-distance limit.[34] Thus we can unify the weak and electromagnetic interactions by giving them the same infinite-order eigenvalue. The treatment of the fermion sector will then proceed as in the Abelian case and we will be led to global dynamical symmetry breaking with some SU(3)-type tadpoles $u_i = \langle\bar\psi\lambda_i\psi\rangle$ acquiring dynamical vacuum expectation values. (The ψ's would presumably have to be field operators for the leptons since there does not appear to be any quark confinement in our approach as of yet, but only that the fermions get masses.) Thus the infrared structure of the weak interaction will provide a dynamical origin for the Gell-Mann, Oakes, and Renner Hamiltonian as we suggested in Ref. 35, and at the same time allow the tadpoles to show up in the ultraviolet current-current operator-product expansions as generalized mass terms as had been originally proposed by Wilson.[20] As discussed in Sec. V we will obtain relations like $m_n - m_p = u_3$, and the u_8 tadpole will give the Gell-Mann–Okubo mass formula as an exact relation. Thus the n-p mass difference and the SU(3) mass formula will be pure infrared effects. Moreover, because of the interplay between the ultraviolet and the infrared Wilson's u_3 tadpole will be able to provide an explanation of the $\eta\to 3\pi$ problem while fulfilling the infrared requirements of Ref. 35 at the same time.

Though the fermion sector appears capable of providing an appealing picture of the origin of the dynamical tadpoles there are unfortunately problems in the meson sector. We have seen that the scalar tadpoles are not associated with bound states, and hence they cannot generate a dynamical Higgs-Englert-Brout mechanism. Moreover, in the non-Abelian case there are additional infrared divergences in the three-massless-meson vertex which do not appear in the Abelian case. Though these divergences may perhaps eliminate the vector mesons from the spectrum altogether, none of the suggestions of this section will be able to materialize until this particularly severe infrared problem is understood.

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Added Note. In this added note we would like to make some additional clarifying remarks.

1. Though our discussion of Sec. II was made specifically in the case where there was a nonconserved current, we would like to explain here how the analysis can be adapted to apply in the Goldstone mode, this being the actual intention of Nambu and Jona-Lasinio. Our derivation of Eq. (18) was made in the case where a rotation was required both in the vacuum and in the Lagrangian. In the Goldstone mode we would instead require

$$\langle\Omega^{(m)}|H^{(0)}|\Omega^{(m)}\rangle = \langle\Omega_\alpha^{(m)}|H^{(0)}|\Omega_\alpha^{(m)}\rangle, \qquad (97)$$

where $H^{(0)}$ is chiral-invariant, and Eq. (97) does hold in the theory $\mathcal{L}^{(0)} = -\frac{1}{2}g[(\bar\psi\psi)^2 - (\bar\psi\gamma_5\psi)^2]$ analyzed in Ref. 1. Moreover, we must also show that each $|\Omega_\alpha^{(m)}\rangle$ is a true vacuum of $H^{(0)}$. To do this we rewrite $\mathcal{L}^{(0)}$ as

$$\mathcal{L}^{(0)} = -g\langle\bar\psi\psi\rangle\bar\psi\psi - \tfrac{1}{2}g[(\bar\psi\psi - \langle\bar\psi\psi\rangle)^2 - \langle\bar\psi\psi\rangle^2 - (\bar\psi\gamma_5\psi)^2]$$
$$= \mathcal{L}_1 + \mathcal{L}_2. \qquad (98)$$

In the Hartree-Fock approximation we solve $\mathcal{L}_1$ exactly first in order to find the self-consistent vacuum and then perturb about it with the residual $\mathcal{L}_2$. Thus to compare $|\Omega^{(m)}\rangle$ (where $\langle\bar{\psi}\psi\rangle \neq 0$) and $|\Omega^{(0)}\rangle$ (where $\langle\bar{\psi}\psi\rangle = 0$) we obtain from Eq. (37)

$$\epsilon(g\langle\bar{\psi}\psi\rangle) = -\frac{g^4\langle\bar{\psi}\psi\rangle^4}{32\pi^2}\left[\frac{4\Lambda^2}{g^2\langle\bar{\psi}\psi\rangle^2} - 1 - 2\ln\left(\frac{\Lambda^2}{g^2\langle\bar{\psi}\psi\rangle^2}\right)\right], \tag{99}$$

so that $|\Omega^{(m)}\rangle$ has lower energy in the Hartree-Fock approximation to the cutoff theory. Moreover

$$\epsilon'(g\langle\bar{\psi}\psi\rangle) = \langle\bar{\psi}\psi\rangle = -\frac{g^3}{4\pi^2}\left[\frac{\Lambda^2\langle\bar{\psi}\psi\rangle}{g^2} - \langle\bar{\psi}\psi\rangle^3\ln\left(\frac{\Lambda^2}{g^2\langle\bar{\psi}\psi\rangle^2}\right)\right]. \tag{100}$$

The nontrivial solution to Eq. (100) is then recognized as the self-consistent solution to the Schwinger-Dyson equation for the fermion self-energy in the ladder approximation [Fig. 2(b)] obtained in Ref. 1. Thus the analysis of Eq. (25) which was made in the non-chiral-invariant massive free theory carries over directly into the chiral-invariant interacting theory at the one-loop level because of the diagonalizing nature of the Hartree-Fock approximation. Thus even when we are in the Goldstone mode we must still calculate $\epsilon(m)$ when we study composite operators rather than $V(\langle\bar{\psi}\psi\rangle)$ of Eq. (30). Hence once there are composite operators $V(\langle\bar{\psi}\psi\rangle)$ has only limited utility independent of whether or not the current is conserved, with $\epsilon(m)$ containing all the relevant physical information.

2. We would like to reformulate a little the analysis of Sec V to explain exactly what is the underlying massless theory. Let us introduce $H^{(0)}$ as the Hamiltonian density of truly massless QED ($\mathcal{L}^{(0)} = \mathrm{KE} - e\bar{\psi}\gamma_\mu\psi A^\mu$ only). Let $|\Omega^{(0)}\rangle$ be the vacuum obtained by dressing the bare vacuum for the kinetic energy (KE) by the electromagnetic interaction so that $|\Omega^{(0)}\rangle$ is both chiral-invariant ($\langle\Omega^{(0)}|\bar{\psi}\psi|\Omega^{(0)}\rangle = 0$) and conformal-invariant. [The discussion of Eq. (92) shows that $|\Omega^{(0)}\rangle$ is the true vacuum of $H^{(0)}$.] At this stage the unrenormalized dressed vertex Γ_S of Fig. 7 behaves as

$$\Gamma_S(p,p,0) \sim \left(\frac{-p^2}{\Lambda^2}\right)^{(1/2)\gamma_\theta} \tag{101}$$

for all momenta up to the cutoff and describes the quantum fluctuations about the $|\Omega^{(0)}\rangle$ vacuum, with the electron propagator being given by $Z_2^{-1}/\not{p}$. In order to renormalize Γ_S we are obliged to induce a mass term $m_0(\bar{\psi}\psi)_0$ into the Lagrangian. Since massless QED is not closed under renormalization we thus have to change the theory and go to a new vacuum $|\Omega'\rangle$, which may not lie in the same Hilbert space as $|\Omega^{(0)}\rangle$. In fact, this is a specific realization of Wilson's idea that the u_i tadpoles are induced as renormalization counterterms to an underlying conformal-invariant theory $H^{(0)}$, and the main point of this paper is to ask what is the infrared dynamics which will enable the induced counterterm to drive itself consistently into a genuine mass term, i.e., to ask whether ψ can create massive particles out of $|\Omega'\rangle$ —so that $|\Omega'\rangle$ can be identified with $|\Omega^{(m)}\rangle$ with $\langle\Omega^{(m)}|\bar{\psi}\psi|\Omega^{(m)}\rangle \neq 0$, and with the electron propagator now being given by Eq. (4)—while creating massless particles out of $|\Omega^{(0)}\rangle$. Thus we see that m_0 is not the bare mass since we really do start without a mass term at all and are forced to induce a mass term just to obtain closure under renormalization. Then in order to ascertain whether the presence of a mass term and a now nonconserved axial-vector current actually means that the particles of the theory possess mass we have to find the true vacuum of the theory using the method of Sec. III.

3. Following these remarks we can now state the criterion for whether a chiral-invariant theory can possibly admit of the Goldstone mode—it has to be closed under renormalization; and hence we can now exclude the Goldstone mode in the Abelian vector-gluon model. Moreover, we see that at the two-loop level of corrections to Γ_S the models of Refs. 1 and 12 are not closed under renormalization. Hence those models will not be able to support the Goldstone mode in higher orders. On the other hand, the BCS theory does continue to support collective modes in higher orders since it has a built-in cutoff and thus never needs to be renormalized. Moreover, the $\mathcal{L}_2$-type corrections of Eq. (98) to the reduced BCS Hamiltonian are negligible in the thermodynamic limit so that the Hamiltonian remains diagonalized. It is in this sense that we can expect the parton model to be realized as a Hartree-Fock diagonalization in the asymptotically free example discussed in Sec. VII.

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