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Separable solutions for directly interacting particle systems*

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The problem of constructing a representation of the Poincaré group corresponding to a directly interacting system of a finite number of particles and satisfying the condition that the interaction be separable is considered by expansion of the group generators in powers of $1/c^2$. It is established that the problem has a solution to order $1/c^2$, but, except in special cases, the solution requires that the interaction contain three-body terms to order $1/c^2$ if it is the sum of two-body terms only to nonrelativistic order. Furthermore, there is considerable arbitariness in the $1/c^2$-order interaction term, and we discuss the possiblity and significance of removing this arbitrariness by a unitary transformation. Finally, we discuss higher-order terms in $1/c^2$, where we present arguments to show that an N-particle system will eventually have some N-body interaction terms at some order in $1/c^2$ even though it contains only two-body terms nonrelativistically, and we then present some applications.

I. INTRODUCTION

In a paper[1] published in 1961 one of the present authors considered the problem of describing relativistically a system of particles using as dynamical variables only the familiar canonical dynamical variables of position $\vec{r}_\mu$, momentum $\vec{p}_\mu$, and spin $\vec{s}_\mu$ for a system of a finite number N of particles ($\mu = 1, 2, \ldots, N$). This was an attempt to carry out a program proposed by Dirac[2] in 1949 and led, in a somewhat roundabout manner, to results found earlier by Bakamjian and Thomas.[3] In that 1961 paper, however, a requirement which went beyond those proposed by Dirac and Bakamjian and Thomas was suggested for such theories to be physically sensible, namely that they possess a property called separability of the interaction, but which has also been referred to as cluster separability or cluster decomposition.[4] This separability problem was not solved in that paper but was taken up by Coester,[5] who succeeded in showing that one could construct separable interactions for systems of three particles. The method there employed could not be extended to a larger number of particles. Since 1961 a considerable number of papers on the subject of direct relativistic interactions in particle systems have appeared.[6] Some of these have dealt with the problem itself, others have been concerned with the handling of internal and external interactions of such systems, partially in connection with electromagnetic interactions of such systems and the low-energy theorems, and still others because of possible relevance to quark models. We do not propose to review these papers here since they are not directly relevant to the central theme of this paper.[7]

Our present interest in the subject was in fact stimulated by a paper of Shirokov[8] as well as one by Zhivopistsev, Perolomov, and Shirokov,[9] both published in 1959, of which we only recently became aware, together with a private communication from Coester that the results achieved by these authors were consistent with separability of the interaction. As in the 1961 paper referred to earlier, the approach to a relativistic interaction between particles is made through an expansion of the generators of the Poincaré group in powers of $1/c^2$, but no attempt is made to go beyond this order of accuracy. We also shall limit our primary attention to this same order but shall comment on higher-order results. What we show below is that the result of Shirokov[8] is a particular relativistic description to order $1/c^2$ for a system of two particles which is trivially separable, but the extension to the N-body system by Zhivopistsev, Perolomov, and Shirokov (ZPS),[9] in which the interaction is written as the sum over all pairs of particles of the interaction obtained by Shirokov in Ref. 8, is only correct in special instances, such as when the interparticle potentials depend nonrelativistically only on their spatial separations. We shall show further that a covariant and separable extension can in fact be made to an N-body system with direct interactions including all spin and momentum (and isospin) dependent nonrelativistic potentials, but that this in general requires that the interaction terms in the potential energy to order $1/c^2$ involve three-body terms as well, and more-body terms to higher order.

Furthermore, we shall show that one can easily construct a more general separable solution than that of Shirokov and consequently a more general separable solution for the N-body extension. We shall then note the possibility of removing this arbitrariness by a unitary transformation, and discuss the significance of such a transformation. Finally, we shall discuss higher-order terms in $1/c^2$, and, briefly, applications of our results.

Our investigation will begin with the Lie algebra for the Poincaré group, which has been written down many times, but can be found in Ref. 1 in our present notation. We shall also keep our notation as close as possible to this reference and shall take certain of the results, such as the form of a "standard representation,"[1] directly from this paper; reference should therefore be made to it for any questions about matters which are not made adequately clear below.

II. FORMULATION OF PROBLEM

Since essentially all details of notation and our starting point are explained at length in Ref. 1, we shall pass quickly to the essential modifications in its development. With $\vec{r}_\mu, \vec{p}_\mu, \vec{s}_\mu$ the position, momentum, and spin operators for the μth particle of the system of particles we take the generators of space translations, $\vec{P}$, and of space rotation, $\vec{J}$, to be

$$\vec{P} = \sum_\mu \vec{p}_\mu, \quad \vec{J} = \sum_\mu (\vec{r}_\mu \times \vec{p}_\mu + \vec{s}_\mu). \tag{1}$$

For the generators of time translations, H, and Lorentz transformations (boosts), $\vec{K}$, we write

$$H = \sum_\mu H_\mu + U, \tag{2}$$

$$\vec{K} = \sum_\mu \left[\frac{1}{2c^2}(\vec{r}_\mu H_\mu + H_\mu \vec{r}_\mu) - \frac{\vec{s}_\mu \times \vec{p}_\mu}{H_\mu + m_\mu c^2} - t\vec{p}_\mu \right] + \vec{V}, \tag{3}$$

where

$$H_\mu = (m_\mu^2 c^4 + p_\mu^2 c^2)^{1/2}, \tag{4}$$

and U and $\vec{V}$ represent interaction terms. The Poincaré group Lie algebra imposes on U and $\vec{V}$ the requirements

$$[\vec{P}, U] = [\vec{J}, U] = 0, \quad [J_i, V_j] = i\epsilon_{ijk} V_k, \tag{5}$$

$$[V_i, P_j] = i\delta_{ij} U/c^2, \tag{6}$$

$$[U, \vec{K} - \vec{V}] + [H - U, \vec{V}] + [U, \vec{V}] = 0 \tag{7}$$

$$[K_i - V_i, V_j] + [V_i, K_j - V_j] + [V_i, V_j] = 0. \tag{8}$$

Equations (5) simply assert that U is translationally and rotationally invariant while $\vec{V}$ transforms as a vector under rotations.

We shall represent the expansion of any operator in powers of $1/c^2$ by series of the form

$$G = G^{(0)} + G^{(1)} + \cdots . \tag{9}$$

Expanding H, $\vec{K}$, U, and $\vec{V}$ in this way and using a "standard representation" in which $\vec{V}^{(0)} = 0$, one finds[1]

$$\vec{K}^{(0)} = M\vec{R} - t\vec{P}, \tag{10}$$

where

$$M = \sum_\mu m_\mu, \quad \vec{R} = \sum_\mu m_\mu \vec{r}_\mu / M. \tag{11}$$

We shall make repeated use below of the fact that the nonrelativistic center-of-mass $\vec{R}$ is canonically conjugate to the total momentum $\vec{P}$. One then has

$$[V_i^{(1)}, P_j] = i\delta_{ij} U^{(0)}/c^2, \tag{12}$$

$$[R_i, V_j^{(1)}] - [R_j, V_i^{(1)}] = 0, \tag{13}$$

$$M[\vec{R}, U^{(1)}] + [\vec{K}^{(1)} - \vec{V}^{(1)}, U^{(0)}] + [\vec{V}^{(1)}, H^{(0)} - U^{(0)}] + [\vec{V}^{(1)}, U^{(0)}] = 0, \tag{14}$$

where

$$\vec{K}^{(1)} - \vec{V}^{(1)} = \frac{1}{2c^2} \sum_\mu \left(\vec{r}_\mu \frac{p_\mu^2}{2m_\mu} + \frac{p_\mu^2}{2m_\mu} \vec{r}_\mu - \frac{\vec{s}_\mu \times \vec{p}_\mu}{m_\mu} \right), \tag{15}$$

$$H^{(0)} - U^{(0)} = \sum_\mu \frac{p_\mu^2}{2m_\mu}, \tag{16}$$

$$H^{(1)} - U^{(1)} = -\sum_\mu \frac{p_\mu^4}{8m_\mu^3 c^2}, \tag{17}$$

and, in addition, the relations

$$[\vec{P}, U^{(0)}] = [\vec{R}, U^{(0)}] = [\vec{J}, U^{(0)}] = 0, \tag{18}$$

$$[\vec{P}, U^{(1)}] = [\vec{J}, U^{(1)}] = 0, \tag{19}$$

$$[J_i, V_j^{(1)}] = i\epsilon_{ijk} V_k^{(1)}. \tag{20}$$

Our object is to find forms for $U^{(0)}$, $\vec{V}^{(1)}$, and $U^{(1)}$ such that Eqs. (12)–(14) and Eqs. (18)–(20) are satisfied and that furthermore they possess the separability property which we now define.

We shall modify very slightly the definition of separability given in Ref. 1. What was there required was that if the system of particles be separated in any way into two or more[10] subsystems I and II such that every particle of subsystem I is infinitely removed from every particle of subsystem II, then H and therefore U should each assume the form of the sum of two terms, $H_{\rm I} + H_{\rm II}$ and $U_{\rm I} + U_{\rm II}$, such that $H_{\rm I}$ and $U_{\rm I}$ involve only dynamical variables referring to particles in subsystem I (i.e., commute with all dynamical variables of particles belonging to subsystem II), and conversely for $H_{\rm II}$ and $U_{\rm II}$. This ensures that the dynamics of each of the two subsystems is independent of that of the other when they are infinitely separated. We shall now require further that the same shall be true of $\vec{K}$ and hence of $\vec{V}$ so that the Lorentz transformation properties of the two systems will also be independent under these same circumstances. Since this is already true for $\vec{P}$ and $\vec{J}$, this means that under the circumstances described, the representation of the Poincaré group becomes the direct product of two representations, one appropriate to each of the two infinitely separated subsystems. We remark that the noninteracting parts $H - U$ and $\vec{K} - \vec{V}$ possess this property since, like $\vec{P}$ and $\vec{J}$, they consist of the sum of one-body terms, but that U and $\vec{V}$ will generally involve two-body, and perhaps three- or more-body terms, and it is these that must be arranged to have the requisite separability properties.

III. SOLUTION OF THE PROBLEM TO ORDER $1/c^2$

Our first result will concern itself with the case where $U^{(0)}$ contains only two-body terms, which is the case of most immediate practical importance, and we shall show that in fact the solution to the problem we have posed for the case that the system is composed of N bodies can be divided into two parts: an appropriate solution of the problem for the case of two bodies, and a solution for a three-body piece arising from the intermingling of two two-body terms. We shall then explicitly construct a solution to the N-body problem and discuss the general solution.

At this point it is convenient to introduce the notation

$$M_{\mu\nu} = m_\mu + m_\nu, \quad \vec{r}_{\mu\nu} = \vec{r}_\mu - \vec{r}_\nu, \quad \vec{P}_{\mu\nu} = \vec{p}_\mu + \vec{p}_\nu,$$
$$\vec{R}_{\mu\nu} = (m_\mu \vec{r}_\mu + m_\nu \vec{r}_\nu)/M_{\mu\nu}, \tag{21}$$
$$\vec{p}_{\mu\nu} = (m_\nu \vec{p}_\mu - m_\mu \vec{p}_\nu)/M_{\mu\nu}.$$

These will be recognized as center-of-mass and relative coordinates and momentum for the nonrelativistic system composed of the μth and νth particles alone. Hence they have the familiar properties of these: The pair $\vec{R}_{\mu\nu}, \vec{P}_{\mu\nu}$ and $\vec{r}_{\mu\nu}, \vec{p}_{\mu\nu}$ are each canonically conjugate and the two pairs of variables commute. One has also

$$\frac{p_\mu^2}{2m_\mu} + \frac{p_\nu^2}{2m_\nu} = \frac{P_{\mu\nu}^2}{2M_{\mu\nu}} + \frac{p_{\mu\nu}^2}{2(m_\mu m_\nu/M_{\mu\nu})}, \tag{22}$$

$$\vec{J}_{\mu\nu} \equiv \vec{r}_\mu \times \vec{p}_\mu + \vec{r}_\nu \times \vec{p}_\nu + \vec{s}_\mu + \vec{s}_\nu$$
$$= \vec{R}_{\mu\nu} \times \vec{P}_{\mu\nu} + \vec{r}_{\mu\nu} \times \vec{p}_{\mu\nu} + \vec{s}_\mu + \vec{s}_\nu. \tag{23}$$

Let us now explicitly assume that $U^{(0)}$ is the sum of two-body terms

$$U^{(0)} = \tfrac{1}{2} \sum_{\mu,\nu} u_{\mu\nu}^{(0)}, \tag{24}$$

where $u_{\mu\nu}^{(0)} = u_{\nu\mu}^{(0)}$ and $u_{\mu\nu}^{(0)} \equiv 0$ for $\mu = \nu$. With $u_{\mu\nu}^{(0)}$ depending only on dynamical variables for the μth and νth particles, and using the fact that dynamical variables referring to different particles commute, Eqs. (18) require that

$$[\vec{P}_{\mu\nu}, u_{\mu\nu}^{(0)}] = [\vec{J}_{\mu\nu}, u_{\mu\nu}^{(0)}] = [\vec{R}_{\mu\nu}, u_{\mu\nu}^{(0)}] = 0. \tag{25}$$

This still permits $u_{\mu\nu}^{(0)}$ to be any rotationally invariant function of $\vec{r}_{\mu\nu}$, $\vec{p}_{\mu\nu}$, $\vec{s}_\mu$, and $\vec{s}_\nu$, but to satisfy the separability condition we must require that $u_{\mu\nu}^{(0)}$ go to zero sufficiently rapidly[11] as $r_{\mu\nu} \to \infty$.

We may now immediately write down a separable two-body form for $\vec{V}^{(1)}$:

$$\vec{V}^{(1)} = \frac{1}{2c^2} \sum_{\mu,\nu} \vec{R}_{\mu\nu} u_{\mu\nu}^{(0)}. \tag{26}$$

One can easily verify that Eqs. (12), (13), and (20) are satisfied and that furthermore $\vec{V}^{(1)}$ will be separable if $u_{\mu\nu}^{(0)}$ vanishes sufficiently rapidly with $r_{\mu\nu} \to \infty$.

The central problem is now to determine $U^{(1)}$ from Eq. (14) subject both to the conditions of Eqs. (19) and to the condition that it too be separable. To this end let us first take note of the fact that, with our solutions for $U^{(0)}$ and $\vec{V}^{(1)}$ together with Eqs. (15) and (16), the second and third commutators in (14) contain only two-body terms. On the other hand, as we shall shortly demonstrate, the last commutator in (14) contains only three-body terms. This suggests that we write $U^{(1)}$ as the sum of two parts,

$$U^{(1)} = U_2^{(1)} + U_3^{(1)}, \tag{27}$$

where the first term is the sum of two-body terms only while the second is the sum of three-body terms only. When inserted into the first commutator of Eq. (14), the first term is then to cancel the second and third commutators of that equation while the second term is to cancel the last commutator.

Let us now consider the last commutator:

$$[\vec{V}^{(1)}, U^{(0)}] = \frac{1}{4c^2} \sum_{\mu,\nu} \sum_{\sigma,\tau} [\vec{R}_{\mu\nu} u_{\mu\nu}^{(0)}, u_{\sigma\tau}^{(0)}]. \tag{28}$$

The important observation is that each commutator in the sum vanishes unless one of the following four conditions is satisfied:

$$\begin{aligned} &\text{(i) } \mu = \sigma, \quad \nu \neq \tau; \\ &\text{(ii) } \mu = \tau, \quad \nu \neq \sigma; \\ &\text{(iii) } \nu = \sigma, \quad \mu \neq \tau; \\ &\text{(iv) } \nu = \tau, \quad \mu \neq \sigma. \end{aligned} \tag{29}$$

Hence on using the condition that $u_{\mu\nu}^{(0)}$ is symmetric in its subscripts, one may write

$$[\vec{V}^{(1)}, U^{(0)}] = \frac{1}{c^2} \sum_{\mu\nu\sigma} [\vec{R}_{\mu\nu} u^{(0)}_{\mu\nu}, u^{(0)}_{\mu\sigma}], \tag{30}$$

which is the sum of three-body terms only. One now finds that this sum is canceled by $U_3^{(1)}$ in Eq. (14) provided[12]

$$U_3^{(1)} = \frac{i}{2c^2} \sum_{\mu,\nu,\sigma} \{\vec{P}_{\mu\nu\sigma} \cdot [\vec{R}_{\mu\nu} u^{(0)}_{\mu\nu}, u^{(0)}_{\mu\sigma}] + [\vec{R}_{\mu\nu} u^{(0)}_{\mu\nu}, u^{(0)}_{\mu\sigma}] \cdot \vec{P}_{\mu\nu\sigma}\}/M_{\mu\nu\sigma}, \tag{31}$$

where

$$\vec{P}_{\mu\nu\sigma} \equiv \vec{p}_\mu + \vec{p}_\nu + \vec{p}_\sigma, \quad M_{\mu\nu\sigma} = m_\mu + m_\nu + m_\sigma. \tag{32}$$

Now if we write for $U_2^{(1)}$

$$U_2^{(1)} = \tfrac{1}{2} \sum_{\mu,\nu} u^{(1)}_{\mu\nu}, \tag{33}$$

and require that all of the two-body terms in Eq. (14) referring to a particular pair of particles separately satisfy the equation we have

$$M_{\mu\nu}[\vec{R}_{\mu\nu}, u^{(1)}_{\mu\nu}] + \frac{1}{2c^2}\left[\vec{r}_\mu \frac{p_\mu{}^2}{2m_\mu} + \frac{p_\mu{}^2}{2m_\mu}\vec{r}_\mu + \vec{r}_\nu \frac{p_\nu{}^2}{2m\nu} + \frac{p_\nu{}^2}{2m_\nu}\vec{r}_\nu - \frac{\vec{s}_\mu \times \vec{p}_\mu}{m_\mu} - \frac{\vec{s}_\nu \times \vec{p}_\nu}{m_\nu}, u^{(0)}_{\mu\nu}\right] + \frac{1}{c^2}\left[\vec{R}_{\mu\nu} u^{(0)}_{\mu\nu}, \frac{p_\mu{}^2}{2m_\mu} + \frac{p_\nu{}^2}{2m_\nu}\right] = 0, \tag{34}$$

and clearly if this is satisfied for every pair then Eq. (14) will be satisfied. But determining $u^{(1)}_{\mu\nu}$ from (34) is nothing more than requiring the solution of our original problem in the case that there are only two particles; thus the N-body problem has been reduced to solving the two-body problem, summing over all pairs, and then adding the three-body term $U_3^{(1)}$.

The solution of Eq. (34) for $u^{(1)}_{\mu\nu}$ is readily obtained by standard commutator algebra as used in Ref. 1; a particular separable solution, which is derived in the Appendix, yields for $U_2^{(1)}$

$$U_2^{(1)} = \frac{1}{2c^2} \sum_{\mu,\nu} \left\{ -\frac{P_{\mu\nu}{}^2}{2M_{\mu\nu}{}^2} u^{(0)}_{\mu\nu} + \frac{i(m_\nu - m_\mu)}{4m_\mu m_\nu M_{\mu\nu}} [(\vec{r}_{\mu\nu} \cdot \vec{P}_{\mu\nu}) p_{\mu\nu}{}^2 + p_{\mu\nu}{}^2 (\vec{P}_{\mu\nu} \cdot \vec{r}_{\mu\nu}), u^{(0)}_{\mu\nu}] + \frac{i}{4M_{\mu\nu}{}^2}[(\vec{r}_{\mu\nu} \cdot \vec{P}_{\mu\nu})(\vec{p}_{\mu\nu} \cdot \vec{P}_{\mu\nu}) + (\vec{P}_{\mu\nu} \cdot \vec{p}_{\mu\nu})(\vec{P}_{\mu\nu} \cdot \vec{r}_{\mu\nu}), u^{(0)}_{\mu\nu}] - \frac{i}{2M_{\mu\nu}}\left[\left(\frac{\vec{s}_\mu}{m_\mu} - \frac{\vec{s}_\nu}{m_\nu}\right) \times \vec{p}_{\mu\nu} \cdot \vec{P}_{\mu\nu}, u^{(0)}_{\mu\nu}\right] \right\}. \tag{35}$$

This is identical with the two-body interaction of ZPS, though expressed somewhat differently.

We have thus far obtained a particular solution $U^{(1)}$ of the commutation relations which is also separable. We are now in a position to examine the question of how to determine a general solution. For this purpose we return to Eqs. (12) and (13) and ask what form of an addition $\Delta\vec{V}^{(1)}$ to $\vec{V}^{(1)}$ as given in Eq. (26) is permitted. We see that the essential conditions on it are

$$[\Delta\vec{V}^{(1)}, P_j] = 0, \tag{36}$$

$$[R_i, \Delta V_j^{(1)}] - [R_j, \Delta V_i^{(1)}] = 0. \tag{37}$$

The second of these implies that $\Delta\vec{V}^{(1)}$ be of the form

$$\Delta\vec{V}^{(1)} = \frac{i}{c^2}[\vec{R}, O^{(0)}], \tag{38}$$

where $O^{(0)}$ is rotationally invariant to satisfy (20). Combining this with (36) tells us that $O^{(0)}$ must be translationally invariant. If we also assume that $O^{(0)}$ is a sum of two-body terms only (with appropriate mass factors) then $\Delta\vec{V}^{(1)}$ can be written in the form

$$\Delta\vec{V}^{(1)} = \frac{i}{2c^2} \sum_{\mu,\nu} [\vec{R}_{\mu\nu}, o^{(0)}_{\mu\nu}], \tag{39}$$

with $o^{(0)}_{\mu\nu}$ a rotationally invariant function of $\vec{P}_{\mu\nu}$, $\vec{r}_{\mu\nu}$, $\vec{p}_{\mu\nu}$, $\vec{s}_\mu$, and $\vec{s}_\nu$, symmetric in μ and ν and zero if $\mu = \nu$. The effect of the addition of such a term to $\vec{V}^{(1)}$ in Eq. (14) is (in disagreement[13] with Ref. 8) to induce a change in $U^{(1)}$ of the form

$$\Delta U^{(1)} = \frac{i}{2c^2} \sum_{\mu,\nu} \left[\frac{p_{\mu\nu}{}^2}{2m_\mu m_\nu}, o^{(0)}_{\mu\nu}\right] + \frac{i}{c^2} \sum\nolimits_{\mu,\nu,\sigma} [u^{(0)}_{\mu\nu}, o^{(0)}_{\mu\sigma}]/M_{\mu\sigma}, \tag{40}$$

where the first sum consists only of two-body terms while the second generally contains both two- and three-body terms. In addition one can add to $U^{(1)}$ the sum of two-, three-, or more-body terms (of order $1/c^2$) which are rotationally invariant functions of internal variables only (i.e., commute with $\vec{P}$ and $\vec{R}$) and which are separable (see last paragraph of the appendix); for such terms (examples of which are easily constructed) the commutator with $\vec{R}$ in Eq. (14) vanishes by hypothesis and Eqs. (19) are automatically satisfied.

It should also be clear at this point that had one included in $\Delta\vec{V}^{(1)}$ of Eq. (39) three-body terms of the form

$$\frac{i}{8c^2}\sum_{\mu,\nu,\sigma}[\vec{R}_{\mu\nu\sigma}, o^{(0)}_{\mu\nu\sigma}], \tag{41}$$

with $\vec{R}_{\mu\nu\sigma}=(m_\mu\vec{r}_\mu+m_\nu\vec{r}_\nu+m_\sigma\vec{r}_\sigma)/(m_\mu+m_\nu+m_\sigma)$, one would in general generate additional two-, three-, and four-body terms in $U^{(1)}$. Finally, one could drop the restriction to two-body terms in $U^{(0)}$. The required generalization appears straightforward. Arguments parallel to those already given would indicate then that, in general, terms of higher particle number than three-body terms would be generated in $U^{(1)}$.

We can now summarize our results of this section. We have shown that Shirokov[8] has found a particular solution of the Poincaré commutation algebra corresponding to $\vec{V}^{(1)}_{\mu\nu}=\vec{R}_{\mu\nu}u^{(0)}_{\mu\nu}=RU^{(0)}$ for the case of two interacting particles. However, the N-body extension of ZPS[9] is only correct provided $U^{(1)}_3$ vanishes. Consequently, except in special cases, two-body nonrelativistic interactions require at least the presence of three-body terms in the interaction of order $1/c^2$. The exceptions occur when all the two-body interaction terms commute with each other. This will be the case if they depend on particle coordinates only, or, for example, involve isotopic spin in such a way that the above condition is satisfied; the Coulomb interaction between nucleons in isospin notation is such an example. It is an academic one in view of the fact that the spin dependence of nuclear forces is such as to cause a violation of the condition in any case. (If one avoids isospin notation, which one can always do, then this case is subsumed under the earlier exception.)

Furthermore, we have shown that one can easily generalize the results of Shirokov,[8] and consequently the N-body extension as well, by adding to $\vec{V}^{(1)}$ an arbitrary term $\Delta\vec{V}^{(1)}$ such that the resulting $\Delta U^{(1)}$ can depend upon the total momentum of a subsystem of particles, and we have given some explicit two-body as well as three-body examples. We also note that one can easily modify these results by adding an "integration constant" of internal variables to $U^{(1)}_2$ and $U^{(1)}_3$, or by adding a three-body term, or more-body term to $U^{(0)}$.

IV. A NOTE ON UNITARY TRANSFORMATIONS

Examination of the arbitrary terms in $\Delta\vec{V}^{(1)}$ as given in Eqs. (39) and (41) shows that (to order $1/c^2$) they can be readily eliminated from $\vec{K}$ by a unitary transformation on the representation of the Poincaré group, for, with $\mathcal{U}$ defined by

$$\mathcal{U}=\exp(i\mathcal{S}),$$
$$\mathcal{S}=\frac{1}{2c^2}\sum_{\mu\nu} o^{(0)}_{\mu\nu}/M_{\mu\nu}+\frac{1}{8c^2}\sum_{\mu\nu\sigma} o^{(0)}_{\mu\nu\sigma}/M_{\mu\nu\sigma}, \tag{42}$$

one has to order $1/c^2$

$$\begin{aligned}\mathcal{U}\vec{K}\mathcal{U}^{-1}&=\vec{K}+i[\mathcal{S},\vec{K}]=\vec{K}-i[M\vec{R},\mathcal{S}]\\&=\vec{K}-\frac{i}{2c^2}\sum_{\mu\nu}[\vec{R}_{\mu\nu},o^{(0)}_{\mu\nu}]\\&\quad-\frac{i}{8c^2}\sum_{\mu\nu\sigma}[\vec{R}_{\mu\nu\sigma},o^{(0)}_{\mu\nu\sigma}]\\&=\vec{K}-\Delta\vec{V}^{(1)}.\end{aligned} \tag{43}$$

This would also eliminate terms in $\Delta U^{(1)}$ from $U^{(1)}$ which depend upon the total momentum of any subsystem of particles. Thus for the two-particle case this would again yield a result for $U^{(1)}$ identical with that obtained by Shirokov,[14] though apparently for different reasons.[13]

Furthermore, one can easily show that *any* $\Delta\vec{V}^{(1)}$ added to $\vec{V}^{(1)}$ of Eq. (26) can be removed by a unitary transformation. Suppose one has the operators

$$\vec{P}'=\vec{P}, \tag{44}$$

$$\vec{J}'=\vec{J}, \tag{45}$$

$$\vec{K}'=\vec{K}+\Delta\vec{V}^{(1)}, \tag{46}$$

$$H'=H+\Delta U^{(1)}, \tag{47}$$

where, to order $1/c^2$, $\vec{P}$, $\vec{J}$, $\vec{K}$, and H are defined by Eqs. (1)–(4), (24), (26), (27), (31), and (35), $\Delta\vec{V}^{(1)}$ satisfies the equations

$$[\Delta V^{(1)}_i, P_j]=0, \tag{48}$$

$$[\Delta V^{(1)}_i, J_j]=i\epsilon_{ijk}\Delta V^{(1)}_k, \tag{49}$$

$$[R_i,\Delta V^{(1)}_j]-[R_j,\Delta V^{(1)}_i]=0, \tag{50}$$

$\Delta U^{(1)}$ is determined from

$$[M\vec{R},\Delta U^{(1)}]+[\Delta\vec{V}^{(1)},H^{(0)}]=0, \tag{51}$$

and both $\{\vec{P},\vec{J},\vec{K},H\}$ and $\{\vec{P}',\vec{J}',\vec{K}',H'\}$ satisfy the usual commutation relations of the Poincaré group. One can then always find an operator $S^{(1)}$ with the following properties:

$$S^{(1)\dagger}=S^{(1)}, \tag{52}$$

$$[S^{(1)},\vec{P}]=[S^{(1)},\vec{J}]=0, \tag{53}$$

$$\Delta\vec{V}^{(1)}=i[M\vec{R},S^{(1)}]. \tag{54}$$

The last equation can be solved for $S^{(1)}$ provided the $\vec{P}$ curl of $\Delta\vec{V}^{(1)}$ vanishes. But this is just Eq. (50). Consequently, with $\mathcal{U}_1$ defined by

$$\mathcal{U}_1=e^{iS^{(1)}},$$

we have to order $1/c^2$

$$\mathcal{U}_1\vec{K}'\mathcal{U}_1^{-1} = \vec{K} + \Delta\vec{V}^{(1)} - i[M\vec{R}, S^{(1)}] = \vec{K}, \tag{55}$$

$$\mathcal{U}_1 H'\mathcal{U}_1^{-1} = H + \Delta U^{(1)} + i[S^{(1)}, H^{(0)}] \equiv H + \Delta^{(1)}, \tag{56}$$

where from Eqs. (51) and (54), we have

$$[\vec{R}, \Delta^{(1)}] = 0, \tag{57}$$

and trivially that

$$[\vec{P}, \Delta^{(1)}] = [\vec{J}, \Delta^{(1)}] = 0. \tag{58}$$

Thus $\Delta^{(1)}$ is at most an arbitrary translationally and rotationally invariant function of internal variables only, so for the two-particle case the result for $U^{(1)}$ is again identical with that of Shirokov.[14] ($\Delta^{(1)}$ is essentially the "integration constant" for $\Delta U^{(1)}$.)

We will not consider $\Delta^{(1)}$ further, except to note that $\vec{V}^{(2)}$, $U^{(1)}$, and $\Delta^{(1)}$ are related by

$$[V_i^{(2)}, P_j] = i\delta_{ij}(U^{(1)} + \Delta^{(1)})/c^2, \tag{59}$$

where higher-order terms in $1/c^2$ such as $\vec{V}^{(2)}$ will be discussed in the next section. It suffices to say that by a previous argument[15] one may construct a "standard representation" of the Poincaré group to any order in $1/c^2$ from any particular solution. In any case the simplicity and reduction in arbitrariness introduced by such a transformation is sufficiently attractive to demand a discussion of its significance.

Of course unitary changes of representation are always possible in quantum mechanics. The only point on which care is required is the recognition that the operator representatives such as $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ after the transformation now represent different physical observables from before. The observables represented by these symbols before the transformation presumably possessed some operational significance defined by some particular though unspecified measurement process. After the transformation the same symbols represent observables defined by a different measurement process. In the case at hand one has no real conception as to what the difference between the two may be, and since for example $\mathcal{S}$ vanishes in Eq. (42) except when two or more particles are within their range of interaction, the physical significance of the observables differs only in these regions and not when the particles or subsystems are asymptotically separated. Hence the S matrix for scattering and the energy eigenvalues of bound subsystems in the asymptotic region are unchanged by the transformation. Thus, for many purposes it could be argued that the transformation has no physical consequence unless one can indeed identify the operational significance of the fundamental observables—a viewpoint which would be espoused at least by pure S matricists. To those who aspire to more detailed knowledge of particle interactions, one can interpret the unitary transformations as those of a class which in particular carry two-particle potentials into "phase-shift equivalent" potentials. These have different off-shell extensions of their scattering amplitudes from the directly measurable on-shell amplitudes, and would hence be of great significance for predicting multiparticle interactions from knowledge of two-body interactions. Of course, such a significance is diminished when there are many-body direct interactions as well. Also, for those who wish to determine the off-shell scattering amplitudes by the use of bremsstrahlung or other electromagnetic interactions, it is essential thay they have prior knowledge of electromagnetic interactions in terms of fundamental dynamical variables of specific observational significance, and hence must resist losing this information through a unitary transformation containing arbitrary elements such as the $o_{\mu\nu}^{(0)}$.

Finally, as a specific example of how a unitary transformation may be misused in the name of simplicity, consider the following. Suppose one had chosen as the simplest form for $\vec{V}^{(1)}$ the expression $\vec{V}^{(1)} = \vec{R}U^{(0)}$ for a system of N particles, where $N > 2$. This represents a particular solution to the Poincaré commutation relations to order $1/c^2$ and by the argument presented at the beginning of this section, one could then find a unitary transformation which would remove any $\Delta\vec{V}^{(1)}$ added to this particular solution, and so, for the sake of simplicity, argue that this particular solution and the resulting $U^{(1)}$ be accepted as the "standard representation" of the Poincaré group to order $1/c^2$. However, this "standard representation" would not satisfy the condition of separability if $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ were considered as having their usual meaning.[16]

At this point it is somewhat futile to argue the relative merits of simplifying the representation by a unitary transformation on the one hand, and, on the other hand, adhering to a rather tenuous insistence that one indeed knows what the operational significance of the fundamental dynamical variables must be. Rather, we point out the possibility of such a transformation. but emphasize that discretion must be exercised in its use. Along these lines, it is of interest to note the close parallel of this situation with that of axiomatic field theory, in which the asymptotic fields are presumed to have physical significance from an S-matrix viewpoint, but the arbitrariness in interpolating fields makes a case for the observational significance of the latter somewhat tenuous.

V. HIGHER-ORDER TERMS IN $1/c^2$

Turning our attention now to second-order terms in $1/c^2$, we note that the critical equations are Eqs. (8) and (7). Using the result that $\vec{K}^{(0)}=M\vec{R}-t\vec{P}$, the latter takes the form

$$M[R_i,V_j^{(2)}]-M[R_j,V_i^{(2)}]+[K_i^{(1)}-V_i^{(1)},V_j^{(1)}]_2 -[K_j^{(1)}-V_j^{(1)},V_i^{(1)}]_2+[V_i^{(1)},V_j^{(1)}]_{2,3}=0. \quad (60)$$

The subscript on a commutator bracket indicates the order in particle number of each bracket if we assume $U^{(0)}$ and $\vec{V}^{(1)}$ are two-body operators, while $U^{(1)}$ is the sum of a two-body and three-body operator. Equation (60) then clearly demands that $\vec{V}^{(2)}$ is generally a three-body operator. The condition arising from Eq. (8) can then be written

$$M[\vec{R},U^{(2)}]+[\vec{K}^{(1)}-\vec{V}^{(1)},U^{(1)}]_{2,3}+[\vec{K}^{(2)}-\vec{V}^{(2)},U^{(0)}]_2+[\vec{V}^{(1)},H^{(1)}-U^{(1)}]_2+[\vec{V}^{(2)},H^{(0)}-U^{(0)}]_{2,3} +[\vec{V}^{(1)},\vec{U}^{(1)}]_{2,3,4}+[\vec{V}^{(2)},U^{(0)}]_{2,3,4}=0. \quad (61)$$

This requires that $U^{(2)}$ contains two-, three-, and four-body operators in general. Even if $U^{(0)}$ is of the special form such that $U^{(1)}$ contains only two-body operators, then still the momentum dependence of these according to Eq. (35) would require at least three-body terms in $U^{(2)}$. Thus as one goes to higher and higher order in $1/c^2$ one will generate interaction terms containing increasing particle number operators. Eventually, then, for an N-body system there will be N-body forces.

Thus far we have not established that Eqs. (60) and (61) can in fact be integrated for $\vec{V}^{(2)}$ and $U^{(2)}$, nor whether separable solutions exist should Eqs. (60) and (61) be integrable. Furthermore, by considering $\vec{V}^{(2)}$ in more detail we can emphasize another point: A necessary condition that Eq. (60) have a solution for $\vec{V}^{(2)}$ is obtained by commuting Eq. (60) with R_k and contracting with ϵ_{ijk} on all indices to yield

$$\epsilon_{ijk}[2[K_i^{(1)}-V_i^{(1)},V_j^{(1)}]+[V_i^{(1)},V_j^{(1)}],R_k]=0. \quad (62)$$

Equation (62) is just the statement that the $\vec{P}$ divergence of the $\vec{P}$ curl of $\vec{V}^{(2)}$ must vanish. The interesting question is whether Eq. (62) is identically satisfied with our general solution for $\vec{V}^{(1)}$ or whether it imposes further conditions on $\vec{V}^{(1)}$ so as to reduce or remove the arbitrariness found to be associated with $\vec{V}^{(1)}$ in Sec. III. More generally, one may ask whether any solution exists for $\vec{V}^{(n)}$, whether any separable solution exists for $\vec{V}^{(n)}$, and whether the existence of $\vec{V}^{(n)}$ imposes conditions upon $\vec{V}^{(n-1)}$, where $n>1$. We will answer these questions directly and provide a heuristic outline of the associated calculations. The calculations are somewhat lengthy and will be reserved for a succeeding publication.[17]

One can indeed show that $\vec{V}^{(n)}$ exists[18] for all $n>1$, the important point being that $\vec{V}^{(n)}$ can be shown to exist by finding a closed form expression for it. To see how the construction goes, expand Eq. (8) in powers of $1/c^2$ and note that the nth order expression containing $\vec{V}^{(n)}$ can be written in the form

$$\epsilon_{ijk}[MR_i,V_j^{(n)}]=\epsilon_{ijk}g_{ij}^{(n)}. \quad (63a)$$

This is a vector equation of the form

$$\vec{\nabla}_P\times\vec{V}^{(n)}=\vec{G}^{(n)}/M, \quad (63b)$$

where $\vec{\nabla}_P$ is the gradient with respect to $\vec{P}$. To prove that a solution to this equation exists for $\vec{V}^{(n)}$, one need only show that $\vec{\nabla}_P\cdot\vec{G}^{(n)}=0$.

To find a solution for $\vec{V}^{(n)}$ which satisfies both Eq. (63) and the boundary condition given by Eq. (6) above, first express the solution to Eq. (63) in a closed form and then substitute this solution into Eq. (6). This will yield an equation which is a function of the arbitrary parameters in the solution of Eq. (63). This equation can in turn be solved for the parameters of Eq. (63) to yield a closed form expression for $\vec{V}^{(n)}$ which satisfies both Eqs. (6) and (8) to order $1/c^{2n}$. Furthermore, from this expression for $\vec{V}^{(n)}$, one can prescribe a procedure by which separable solutions might be constructed to all orders in $1/c^2$. Briefly, this procedure takes advantage of the fact that, for an N-particle system, the operator

$$[M\vec{R}]_{\mu(1)\mu(2)\cdots\mu(k)}\equiv M_{\mu(1)\mu(2)\cdots\mu(k)}\vec{R}_{\mu(1)\mu(2)\cdots\mu(k)} \quad (64a)$$

is canonically conjugate to the operator

$$[\vec{P}/M]_{\mu(1)\mu(2)\cdots\mu(l)}\equiv\frac{\vec{P}_{\mu(1)\mu(2)\cdots\mu(l)}}{M_{\mu(1)\mu(2)\cdots\mu(l)}}, \quad (64b)$$

where

$$\vec{R}_{\mu(1)\mu(2)\cdots\mu(k)}\equiv\frac{1}{M_{\mu(1)\mu(2)\cdots\mu(k)}}\sum_{\nu=\mu(1)}^{\mu(k)}m_\nu\vec{r}_\nu, \quad (64c)$$

$$\vec{P}_{\mu(1)\mu(2)\cdots(l)} \equiv \sum_{\nu=\mu(1)}^{\mu(l)} \vec{p}_\nu, \tag{64d}$$

$$M_{\mu(1)\mu(2)\cdots\mu(m)} \equiv \sum_{\nu=\mu(1)}^{\mu(m)} m_\nu, \tag{64e}$$

$$M \equiv \sum_{\nu=\mu(1)}^{\mu(N)} m_\nu, \tag{64f}$$

and $k, l, m \leq N$. Consequently, by using only separable functions in the closed form expression for $\vec{V}^{(n)}$ and "integrating" these separable functions by replacing $M\vec{R}$ with $[M\vec{R}]_{\mu(1)\mu(2)\cdots\mu(k)}$ and $\vec{P}/M$ with $[\vec{P}/M]_{\mu(1)\mu(2)\cdots\mu(l)}$, one can produce another function which is again formally separable. What is somewhat disappointing is that the existence of $\vec{V}^{(n)}$ does not remove or reduce the arbitrariness in the choice of $\vec{V}^{(n-1)}$.

The operator $U^{(n)}$ is determined from an expansion of Eq. (7) in powers of $1/c^2$. The nth order expression containing $U^{(n)}$ can be written in the form

$$\vec{\nabla}_P U^{(n)} = \vec{G}^{(n)}/M. \tag{65}$$

To show that a solution exists,[18] one need only establish that the condition $\vec{\nabla}_P \times \vec{G}^{(n)} = 0$ is satisfied. One can then show that $[G_i^{(n)}, P_j] = 0$, so that a closed form expression for $U^{(n)}$ can be found which satisfies Eq. (7) and the boundary condition given by Eq. (5), $[U^{(n)}, \vec{P}] = 0$. Separable solutions might then be constructed by using the same procedure as for $\vec{V}^{(n)}$.

One other point should be made with regard to the solutions for $\vec{V}^{(n)}$ and $U^{(n)}$. Note that the expression for $\vec{V}^{(n)}$ will in general involve lower-order terms in both $U^{(k)}$ and $\vec{V}^{(k)}$, where $k < n$. The expression for $U^{(n)}$ on the other hand will depend in general upon lower-order terms in $U^{(k)}$, where $k < n$, but upon $\vec{V}^{(m)}$ such that $m \leq n$. So the above procedure, by which separable $\vec{V}^{(n)}$ and $U^{(n)}$ might be constructed, has a "bootstrap" feature: Given a separable $U^{(0)}$, a separable $\vec{V}^{(1)}$ may be found; given $U^{(0)}$ and $\vec{V}^{(1)}$, a separable $U^{(1)}$ may be found; given $U^{(0)}$, $\vec{V}^{(1)}$, and $U^{(1)}$, a separable $\vec{V}^{(2)}$ may be found; and so on.

VI. APPLICATIONS

Interest in the construction of directly interacting relativistic particle systems stems from two sources:

(a) Interest in whether such a theory can be made self-consistent irrespective of the accuracy with which it might describe nature. In this respect it is a theoretical alternative to field theories and their problems.

proach to practical calculations on many-body systems where real or virtual particle production is not expected to play an essential role and it would be expected that other relativistic effects are small. Examples where such approaches are useful, in spite of their limitations, include the physics of atoms and nuclei at moderate excitation energies and primitive quark models of elementary particles. This type of approach could, we hope, allow one to gain useful insight into a variety of interesting physical phenomena and theoretical principles. An example of the last is the general need for many-body forces that has been emphasized above and earlier.

We shall dwell on only one aspect of an application of the latter kind. Nuclear structure calculations are achieving an accuracy where relativistic corrections are significant insofar as selecting from among alternative expressions for the phenomenological nuclear interaction. To consider such corrections was the motivation of the work of Zhivopistsev, Perolomov, and Shirokov.[9] Further work in this direction has been carried out by Bhakar,[19] and more recently by Coester, Pieper, and Serduke.[20] Except in the last work no consideration is given to either the necessary presence of many-body, particularly three-body forces, nor to possible ambiguities in the relativistic corrections to two-body forces owing to the arbitrariness in $U_2^{(1)}$. In Ref. 19, on the basis of consistency arguments, the three-body forces pointed out here are neglected for the purpose of calculating the binding energy and equilibrium density of infinite nuclear matter since three-body correlations are also excluded from consideration. On the other hand, apart from the problem of phase-shift equivalent potentials, the question of the ambiguities in two-body forces does not appear to arise explicitly. We can only conjecture at this point that the ambiguities referred to above may be transferred to the off-shell behavior of the two-body potential or that different possible connections[21] between single-particle and center-of-mass variables may correspond to generating phase-shift equivalent potentials from any one phenomenological potential. Further clarification of this point is clearly needed.

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APPENDIX: SOLUTION FOR TWO-BODY TERMS

We explicitly solve for the function $u_{\mu\nu}^{(1)}$ of Eq. (34). We note first that the spin-independent terms of the second commutator in (34) can be rewritten as the sum of two terms $\vec{C}_1^{\mu\nu}$ and $\vec{C}_2^{\mu\nu}$ by writing for $\vec{r}_\mu$ and $\vec{r}_\nu$

$$\vec{r}_\mu = \vec{R}_{\mu\nu} + \frac{m_\nu}{M_{\mu\nu}}\vec{r}_{\mu\nu}, \quad \vec{r}_\nu = \vec{R}_{\mu\nu} - \frac{m_\mu}{M_{\mu\nu}}\vec{r}_{\mu\nu}, \tag{A1}$$

and using (21), to yield

$$\begin{aligned}\vec{C}_1^{\mu\nu} &= \frac{1}{2c^2}\left[\vec{R}_{\mu\nu}\left(\frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right)+\left(\frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right)\vec{R}_{\mu\nu}, u_{\mu\nu}^{(0)}\right]\\ &= \frac{1}{2c^2}\vec{R}_{\mu\nu}\left[\frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}, u_{\mu\nu}^{(0)}\right]+\frac{1}{2c^2}\left[\frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}, u_{\mu\nu}^{(0)}\right]\vec{R}_{\mu\nu},\end{aligned} \tag{A2}$$

$$\vec{C}_2^{\mu\nu} = \frac{1}{2M_{\mu\nu}c^2}\left[m_\nu\vec{r}_{\mu\nu}\frac{p_\mu^2}{2m_\mu}+\frac{p_\mu^2}{2m_\mu}m_\nu\vec{r}_{\mu\nu}-m_\mu\vec{r}_{\mu\nu}\frac{p_\nu^2}{2m_\nu}-\frac{p_\nu^2}{2m_\nu}m_\nu\vec{r}_{\mu\nu}, u_{\mu\nu}^{(0)}\right]. \tag{A3}$$

On the other hand, the third commutator in (34) can be reduced:

$$\begin{aligned}\vec{C}_3^{\mu\nu} &= \frac{1}{c^2}\left[\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right]\\ &= \frac{1}{2c^2}\left[\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right]+\frac{1}{2c^2}\left[u_{\mu\nu}^{(0)}\vec{R}_{\mu\nu}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right]\\ &= \frac{1}{2c^2}\vec{R}_{\mu\nu}\left[u_{\mu\nu}^{(0)}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right]+\frac{1}{2c^2}\left[u_{\mu\nu}^{(0)}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right]\vec{R}_{\mu\nu}+\frac{1}{2c^2}\left[\vec{R}_{\mu\nu}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right]u_{\mu\nu}^{(0)}\\ &\quad+\frac{1}{2c^2}u_{\mu\nu}^{(0)}\left[\vec{R}_{\mu\nu}, \frac{p_\mu^2}{2m_\mu}+\frac{p_\nu^2}{2m_\nu}\right].\end{aligned} \tag{A4}$$

The first pair of terms in the last expression for $\vec{C}_3^{\mu\nu}$ simply cancels $\vec{C}_1^{\mu\nu}$. Substituting from Eq. (22) into the second pair yields the results

$$\begin{aligned}\frac{1}{2c^2}\left[\vec{R}_{\mu\nu}, \frac{P_{\mu\nu}^2}{2M_{\mu\nu}}\right]u_{\mu\nu}^{(0)}+\frac{1}{2c^2}u_{\mu\nu}^{(0)}\left[\vec{R}_{\mu\nu}, \frac{P_{\mu\nu}^2}{2M_{\mu\nu}}\right] &= \frac{i}{M_{\mu\nu}c^2}\vec{P}_{\mu\nu}u_{\mu\nu}^{(0)}\\ &= \left[M_{\mu\nu}\vec{R}_{\mu\nu}, \frac{P_{\mu\nu}^2}{2M_{\mu\nu}^2c^2}u_{\mu\nu}^{(0)}\right].\end{aligned} \tag{A5}$$

Thus with a term

$$\Delta_1 u_{\mu\nu}^{(1)} = -\frac{P_{\mu\nu}^2}{2M_{\mu\nu}^2c^2}u_{\mu\nu}^{(0)} \tag{A6}$$

in $u_{\mu\nu}^{(1)}$ we will cancel this term from Eq. (34).

Now let us turn our attention to $\vec{C}_2^{\mu\nu}$. By substitution of

$$\vec{p}_\mu = \vec{p}_{\mu\nu}+\frac{m_\mu}{M_{\mu\nu}}\vec{P}_{\mu\nu}, \quad \vec{p}_\nu = -\vec{p}_{\mu\nu}+\frac{m_\nu}{M_{\mu\nu}}\vec{P}_{\mu\nu}, \tag{A7}$$

it splits into three terms which we now evaluate:

$$\begin{aligned}\vec{C}_{21}^{\mu\nu} &= \frac{1}{2M_{\mu\nu}c^2}\left[\frac{m_\nu}{2m_\mu}(\vec{r}_{\mu\nu}p_{\mu\nu}^2+p_{\mu\nu}^2\vec{r}_{\mu\nu})-\frac{m_\mu}{2m_\nu}(\vec{r}_{\mu\nu}p_{\mu\nu}^2+p_{\mu\nu}^2\vec{r}_{\mu\nu}), u_{\mu\nu}^{(0)}\right]\\ &= \frac{m_\nu-m_\mu}{4m_\mu m_\nu c^2}[\vec{r}_{\mu\nu}p_{\mu\nu}^2+p_{\mu\nu}^2\vec{r}_{\mu\nu}, u_{\mu\nu}^{(0)}]\\ &= -i\left[M_{\mu\nu}\vec{R}_{\mu\nu}, \frac{m_\nu-m_\mu}{4m_\mu m_\nu M_{\mu\nu}c^2}[\vec{r}_{\mu\nu}p_{\mu\nu}^2+p_{\mu\nu}^2\vec{r}_{\mu\nu}, u_{\mu\nu}^{(0)}]\cdot\vec{P}_{\mu\nu}\right].\end{aligned} \tag{A8}$$

Hence, by including a term

$$\Delta_2 u_{\mu\nu}^{(1)} = \frac{i(m_\nu - m_\mu)}{4m_\nu m_\mu M_{\mu\nu} c^2}[\vec{r}_{\mu\nu} p_{\mu\nu}{}^2 + p_{\mu\nu}{}^2 \vec{r}_{\mu\nu}, u_{\mu\nu}^{(0)}]\cdot\vec{P}_{\mu\nu} \tag{A9}$$

in $u_{\mu\nu}^{(1)}$ we will cancel this term from Eq. (34). In addition we have

$$C_{22}^{\mu\nu} = \frac{1}{2c^2}\left[\frac{m_\nu}{M_{\mu\nu}}\left(\frac{m_\mu}{M_{\mu\nu}{}^2}\vec{r}_{\mu\nu}P_{\mu\nu}{}^2\right) - \frac{m_\mu}{M_{\mu\nu}}\left(\frac{m_\nu}{M_{\mu\nu}{}^2}\vec{r}_{\mu\nu}P_{\mu\nu}{}^2\right), u_{\mu\nu}^{(0)}\right] = 0\ , \tag{A10}$$

and lastly,

$$\begin{aligned} C_{23}^{\mu\nu} &= \frac{1}{2M_{\mu\nu}{}^2c^2}[m_\nu\{\vec{r}_{\mu\nu}(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu}) + (\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})\vec{r}_{\mu\nu}\} + m_\mu\{\vec{r}_{\mu\nu}(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu}) + (\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})\vec{r}_{\mu\nu}\}, u_{\mu\nu}^{(0)}] \\ &= \frac{1}{4M_{\mu\nu}c^2}[\vec{r}_{\mu\nu}(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu}) + (\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})\vec{r}_{\mu\nu} + (\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})\vec{p}_{\mu\nu} + \vec{p}_{\mu\nu}(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu}), u_{\mu\nu}^{(0)}] \\ &\quad - \frac{1}{4M_{\mu\nu}c^2}[(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})\vec{p}_{\mu\nu} - \vec{r}_{\mu\nu}(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu}) - (\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})\vec{r}_{\mu\nu} + \vec{p}_{\mu\nu}(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu}), u_{\mu\nu}^{(0)}]\ . \end{aligned} \tag{A11}$$

The first commutator of (A11) can be canceled from Eq. (34) by a term

$$\Delta_3 u_{\mu\nu}^{(1)} = \frac{i}{4M_{\mu\nu}{}^2c^2}[(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu}) + (\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu}), u_{\mu\nu}^{(0)}] \tag{A12}$$

in $u_{\mu\nu}^{(1)}$. The second commutator of (A11) can be rewritten as

$$-\frac{1}{4M_{\mu\nu}c^2}[(\vec{r}_{\mu\nu}\times\vec{p}_{\mu\nu})\times\vec{P}_{\mu\nu} - (\vec{p}_{\mu\nu}\times\vec{r}_{\mu\nu})\times\vec{P}_{\mu\nu}, u_{\mu\nu}^{(0)}] = -\frac{1}{2M_{\mu\nu}c^2}[(\vec{r}_{\mu\nu}\times\vec{p}_{\mu\nu})\times\vec{P}_{\mu\nu}, u_{\mu\nu}^{(0)}]\ . \tag{A13}$$

If we use the fact that $\vec{R}_{\mu\nu}$ and $\vec{P}_{\mu\nu}$ commute with $u_{\mu\nu}^{(0)}$ and that

$$[\vec{J}_{\mu\nu}, u_{\mu\nu}^{(0)}] = [\vec{r}_{\mu\nu}\times\vec{p}_{\mu\nu} + \vec{s}_\mu + \vec{s}_\nu, u_{\mu\nu}^{(0)}] = 0\ , \tag{A14}$$

we can rewrite the preceding expression

$$\frac{1}{2M_{\mu\nu}c^2}[(\vec{s}_\mu + \vec{s}_\nu)\times\vec{P}_{\mu\nu}, u_{\mu\nu}^{(0)}]\ . \tag{A15}$$

Finally the spin-dependent terms in the second commutator in Eq. (34) are

$$\begin{aligned} -\frac{1}{c^2}\left[\frac{\vec{s}_\mu\times\vec{p}_\mu}{2m_\mu} + \frac{\vec{s}_\nu\times\vec{p}_\nu}{2m_\nu}, u_{\mu\nu}^{(0)}\right] &= -\frac{1}{c^2}\left[\frac{\vec{s}_\mu\times[\vec{p}_{\mu\nu} + (m_\mu/M_{\mu\nu})\vec{P}_{\mu\nu}]}{2m_\mu} - \frac{\vec{s}_\nu\times[\vec{p}_{\mu\nu} - (m_\nu/M_{\mu\nu})\vec{P}_{\mu\nu}]}{2m_\nu}, u_{\mu\nu}^{(0)}\right] \\ &= -\frac{1}{2c^2}\left[\left(\frac{\vec{s}_\mu}{m_\mu} - \frac{\vec{s}_\nu}{m_\nu}\right)\times\vec{p}_{\mu\nu}, u_{\mu\nu}^{(0)}\right] - \frac{1}{2M_{\mu\nu}c^2}[(\vec{s}_\mu + \vec{s}_\nu)\times\vec{P}_{\mu\nu}, u_{\mu\nu}^{(0)}]\ . \end{aligned} \tag{A16}$$

The second term cancels the expression (A15) while the first term will be canceled from Eq. (34) if we include a contribution

$$\Delta_4 u_{\mu\nu}^{(1)} = -\frac{i}{2M_{\mu\nu}c^2}\left[\left(\frac{\vec{s}_\mu}{m_\mu} - \frac{\vec{s}_\nu}{m_\nu}\right)\times\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu}, u_{\mu\nu}^{(0)}\right] \tag{A17}$$

in $u_{\mu\nu}^{(1)}$. Thus with

$$u_{\mu\nu}^{(1)} = \Delta_1 u_{\mu\nu}^{(1)} + \Delta_2 u_{\mu\nu}^{(1)} + \Delta_3 u_{\mu\nu}^{(1)} + \Delta_4 u_{\mu\nu}^{(1)}\ , \tag{A18}$$

we have obtained a particular separable solution to Eq. (34), for $u_{\mu\nu}^{(1)}$, and hence have achieved our objective.

With respect to the generality of this solution we may remark that any other solution must be such that its difference from the above commutes with $\vec{R}_{\mu\nu}$. Hence it will consist of a rotationally invariant function of the nonrelativistic internal variables[16] $\vec{r}_{\mu\nu}$, $\vec{p}_{\mu\nu}$, $\vec{s}_\mu$, and $\vec{s}_\nu$. It thus has the same structure as $u_{\mu\nu}^{(0)}$ and, like it, for separability, must vanish sufficiently rapidly as $r_{\mu\nu}\to\infty$, but will be of order $1/c^2$.

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[10]We state the separability condition in terms of separation into *two* subsystems, but in general one should state the requirement valid for any subdivision. The latter form of the condition will follow from the former if the representations of all separated subsystems are themselves separable with respect to further subdivision. The last condition will hold for the cases described below.

[11]We take this to be the condition that the strong limit

$$\lim_{a\to\infty} \|\exp(i\vec{a}\cdot\vec{p}_{\mu\nu})u^{(0)}_{\mu\nu}\exp(-i\vec{a}\cdot\vec{p}_{\mu\nu})\Psi\| = 0$$

for a dense set of vectors Ψ of the Hilbert space. The same definition is to hold as well for other operators such as $\vec{V}^{(1)}_{\mu\nu}$ although not necessarily on the same dense set. In this paper, we will not further consider problems associated with the domain of an operator.

[12]Despite the presence of $\vec{R}_{\mu\nu}$ in $U^{(1)}_3$, the latter is translationally invariant since the commutator of $\vec{P}$ with $U^{(1)}_3$ results in a sum over a pair of indices in which the summand is antisymmetric.

[13]Shirokov (Ref. 8) seems to find no arbitrariness with respect to the total momentum dependence of the interaction in the two-body system. His result appears to arise from an implicit assumption that indeed $\vec{V}^{(1)}_{\mu\nu} = \vec{R}_{\mu\nu}u^{(0)}_{\mu\nu} = \vec{R}U^{(0)}$ for the two-body system, and not from an apparent neglect of an interaction term in $\vec{K}^{(1)}$ (which he designates $\vec{N}^{(1)}$).

[14]In the two-body case of Ref. 8, Shirokov considered only those correction terms which were a function of total momentum.

[15]See p. 285 of Ref. 1.

[16]R. A. Krajcik and L. L. Foldy, Phys. Rev. D 10, 1777 (1974). See Sec. IV for further discussion.

[17]R. A. Krajcik and L. L. Foldy (unpublished).

[18]It has been observed by Foldy and Coester that the formal existence of $\vec{V}^{(n)}$ and $U^{(n)}$ can be established simply because the $\vec{V}^{(n)}$ and $U^{(n)}$ description of a collection of particles (e.g., Refs. 1 and 16) is equivalent to the center-of mass description of Bakamjian and Thomas (Ref. 3).

[19]B. S. Bhakar, Phys. Rev. C 2, 905 (1970).

[20]F. Coester, S. C. Pieper, and F. J. D. Serduke, Phys. Rev. C 11, 1 (1975).

[21]See Sec. II of Ref. 16.