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Quantum "solitons" which are SU(N) fermions*

M. B. Halpern
Department of Physics, University of California, Berkeley, California 94720
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In two dimensions, we find a construction for an SU(N) quark field in terms of N real Bose fields. Hence, equivalence is shown between certain massive SU(N) Thirring models and systems of quantum sine-Gordon–type equations. From the point of view of the bosons, the "soliton-quark" SU(N) is topological. To minimize guesswork in the development of such correspondences, we employ a systematic blend of Mandelstam's operator approach with the interaction picture.

I. INTRODUCTION

Recently Coleman[1] established the surprising result that the quantum sine-Gordon equation is equivalent to the massive Thirring model, at least in the zero-fermion sector. He also gave simple arguments which imply that the Thirring fermion is the soliton of the sine-Gordon equation. Soon afterward, Mandelstam[2] gave a nonperturbative proof of this by constructing the fermion out of the Bose operators. Such results are of obvious significance to particle theorists trying to perceive the medium in which they are immersed. Extensions may even lead to a "derivation" of dual models as extended solutions of local field theory.

It is my purpose in this paper to extend such correspondence to SU(N). I have in mind finding representations of SU(N) quarks in terms of Bose fields, and hence finding the quantum sine-Gordon–like (SGL) equations that correspond to certain interacting models of these fermions. Thus, SU(N) quarks will be quantum "solitons." As pure fermion examples, one seeks correspondence with massive SU(N) Thirring models, but it would be nice to have the results in a form whereby other interactions, such as $A_\mu^\alpha \bar{\psi}\frac{1}{2}\lambda^\alpha\gamma^\mu\psi$ (gluon), can be explored. We also have a disadvantage in not knowing ahead of time the form of the SGL equations.

I have chosen an approach which handles all this, and which circumvents, in an orderly fashion, a great deal of guesswork. The approach is based on the following observation: *If* one knows *free-field correspondences*, it is generally very easy to add *many* interactions in the *interaction picture*. Getting back to Heisenberg equations of motion is quite simple (at least when there are no coupling constant renormalizations). One can always check one's results directly in the Heisenberg picture, either by calculating matrix elements, or, preferably, by Mandelstam's operator manipulations.

The plan of the paper is then as follows: In Sec. II I give a brief discussion of the interaction picture for certain (soft-chiral-breaking) SU(N) Thirring models. Section III is a rapid rederivation of Coleman's correspondence, starting from the free correspondences. In Sec. IV I guess generalizations of Mandelstam's operators for the case of free SU(N) fermions. The representation involves N bosons. In Sec. V I leap quickly, via the interaction-picture method, to the corresponding SGL equations. The intricate form of these equations compliments the method; the structure seems quite difficult to guess (but easy to check) in a more direct approach. From the point of view of the bosons, the SU(N) of their "soliton-quarks" is topological. For example, the diagonal component of isospin (say I_3) is a matter of asymptotic behavior of the soliton, just as was baryon number in the U(1) case. The scalars themselves have no simple transformations under SU(N), and the SGL systems no obvious invariance. In Appendix A, however, I discuss the sense in which the scalars form an extraordinary nonlocal representation of SU(N). This may be interesting in its own right, but the connection with the quark isospin is highly indirect. In Sec. VI I make some preliminary remarks about the more difficult case of *hard*-chiral-breaking interactions. There is another appendix, B, in which I note that many of the manipulations and correspondences within two-dimensional field theories are known in dual models.

II. INTERACTION PICTURE

I will briefly review[3] the interaction-picture formulation for a baryon-number current-current interaction, thus a simple SU(N) Thirring model. We will be interested later in mass terms and other interactions. We seek correspondence then with the family of Dirac equations[4]

$$i\partial\!\!\!/\psi = g_B{:}J\!\!\!/\psi{:}\,,\quad J\!\!\!/ = J_\mu\gamma^\mu \tag{2.1}$$

where ψ_a, $a=1,\ldots,N$ is an SU(N) quark, and

$$J^\mu = \sum_{a=1}^{N} \bar{\psi}_a\gamma^\mu\psi_a = \bar{\psi}\gamma^\mu\psi \,. \tag{2.1}$$

SU(N) currents $J^\mu_\alpha = \bar{\psi}\gamma^\mu \frac{1}{2}\lambda^\alpha\psi$, $\mathrm{Tr}(\lambda^\alpha\lambda^\beta)=2\delta^{\alpha\beta}$, are interesting objects, but we shall not include these in the interaction until Sec. VI.

In the interaction picture, we deal only with free fields, and we expect to guarantee both Lorentz invariance and the usual Feynman series by studying $\theta^F_{0\mu}$ and $\theta^I_{0\mu}$, the free and interacting energy-momentum densities in the interaction picture (all functions of free fields, or free currents). We assume[5-7] for the free objects

$$\begin{aligned}
\theta^F_{00} &= -\frac{i}{2}\psi_D^\dagger\gamma_0\gamma^1\overleftrightarrow{\partial}_1\psi_D \equiv \theta^{FB}_{00}+\theta^{FV}_{00}\,,\\
\theta^{FB}_{00} &= \frac{1}{2C_B}:J^2_{0D}+J^2_{1D}:\,,\\
\theta^{FV}_{00} &= \frac{1}{2\tilde{C}_V}:J^\alpha_{0D}J^\alpha_{0D}+J^\alpha_{1D}J^\alpha_{1D}:\,,\\
\theta^F_{01} &= +\frac{i}{2}\psi_D^\dagger\overleftrightarrow{\partial}_1\psi_D \equiv \theta^{FB}_{01}+\theta^{FV}_{01}\,,\\
\theta^{FB}_{01} &= \frac{1}{C_B}:J_{0D}J_{1D}:,\quad \theta^{FV}_{01}=\frac{1}{\tilde{C}_V}:J^\alpha_{0D}J^\alpha_{1D}:\,.
\end{aligned} \tag{2.2}$$

The subscript D denotes all free quantities, and $C_B = N/\pi$, $\tilde{C}_V=(N+1)/2\pi$. Now for the interaction. We assume a form, $H_I=\int dx\,\theta^I_{00}$,

$$\theta^I_{00} = \tfrac{1}{2}g_B(J_{0D})^2 + b(J_{1D})^2\,,\quad \theta^I_{01}=0\,, \tag{2.3}$$

where b is to be determined and g_B will turn out to be that of Eq. (2.1). The form assumed is in general not a Lorentz scalar because there are Schwinger terms in the current algebra,[8]

$$\begin{aligned}
[J_{0D}(x),J_{1D}(y)] &= iC_B\partial_x\delta(x-y)\,,\\
[J^\alpha_{0D}(x),J^\beta_{1D}(y)] &= if^{\alpha\beta\gamma}J^\gamma_{1D}(x)\delta(x-y)\\
&\quad + i\delta^{\alpha\beta}C_V\partial_x\delta(x-y)\,,
\end{aligned} \tag{2.4}$$

where $C_V = 1/2\pi$. We will also need

$$\begin{aligned}
[J_{0D}(x),\psi_D(y)] &= -\psi_D(x)\delta(x-y)\,,\\
[J_{1D}(x),\psi_D(y)] &= -\gamma_0\gamma_1\psi_D(x)\delta(x-y)\,,\\
[J^\alpha_{0D}(x),\psi_D(y)] &= -\frac{\lambda^\alpha}{2}\psi_D(x)\delta(x-y)\,,\\
[J^\alpha_{1D}(x),\psi_D(y)] &= -\gamma_0\gamma_1\frac{\lambda^\alpha}{2}\psi_D(x)\delta(x-y)\,.
\end{aligned} \tag{2.5}$$

The way to determine b in terms of g_B is by requiring Lorentz invariance, that is, Schwinger's condition

$$[\theta_{00}(x),\theta_{00}(y)] = i[\theta_{01}(x)+\theta_{01}(y)]\partial_x\delta(x-y) \tag{2.6}$$

on *both* the free and the total $\theta^T_{00}=\theta^F_{00}+\theta^I_{00}$. Since θ^F_{00} already satisfies this, you may pick any other θ_{00}, with the result that

$$b = -\frac{g_B}{2}G\,,\quad G=(1+C_Bg_B)^{-1}\,. \tag{2.7}$$

As shown in Ref. 3, this guarantees a Lorentz-invariant S matrix (the Feynman series, if expanded in g_B).

We next turn our attention to the Lorentz-transformation properties of the currents. In the interaction picture (IP)

$$\begin{aligned}
[\theta^F_{00}(x),J_{0D}(y)] &= iJ_{1D}(x)\partial_x\delta(x-y)\,,\\
[\theta^F_{00}(x),J_{1D}(y)] &= iJ_{0D}(x)\partial_x\delta(x-y)\,,
\end{aligned} \tag{2.8}$$

and similarly for J^α_μ. These are useful in establishing that $J_{\mu D}$ is a two-vector in the IP.

$$\begin{aligned}
M^F_{01} &= \int dx[x_0\theta^F_{01}+x\theta^F_{00}]\,,\\
[M^F_{01},J_{0D}(x)] &= i(t\partial_x+x\partial_t)J_{0D}(x)-iJ_{1D}(x)\,,\\
[M^F_{01},J_{1D}(x)] &= i(t\partial_x+x\partial_t)J_{1D}(x)-iJ_{0D}(x)\,,
\end{aligned} \tag{2.9}$$

and similarly for J^α_μ. However, because there are Schwinger terms in the commutator (θ^I_{00},J_μ), this will not persist in the Heisenberg picture (HP). Define HP fields as $\psi_H=U^\dagger\psi_DU$, $\dot{U}=-iH_IU$, etc., and the full Lorentz generator $M^T_{01}(\theta^T_{00},\theta^T_{01})$. Then applying $U^\dagger\cdots U$, one easily establishes that J^μ_H is not a two-vector in the HP. However, in the same calculation, one sees that

$$\tilde{J}_\mu \equiv (J_{0H},GJ_{1H}) \tag{2.10}$$

is a vector. This result is well known in the Thirring model, and as noted above, it is a general phenomenon for any field which has Schwinger terms with θ^I_{00}.[3] $J^\alpha_{\mu H}$ remains a vector under this interaction.

We next make contact with the Thirring-Dirac equation. Applying $U^\dagger\cdots U$ to $i\partial\!\!\!/\psi_D=0$ and using (2.5) we obtain immediately that

$$i\partial\!\!\!/\psi_H = g_B:\tilde{J}\!\!\!/\psi_H:\,, \tag{2.11}$$

completing the correspondence. We also list the obvious relations

$$\begin{aligned}
[\tilde{J}_0(x),\psi_H(y)] &= -\psi_H(x)\delta(x-y)\,,\\
[\tilde{J}_1(x),\psi_H(y)] &= -G\gamma_0\gamma_1\psi_H\delta(x-y)\,,\\
[\tilde{J}_0(x),\tilde{J}_1(x)] &= iGC_B\partial_x\delta(x-y)\,,
\end{aligned} \tag{2.12}$$

all other commutators going over form-invariant to the HP.[9]

As a quick application of the method, we derive the Sugawara stress-tensor form for these theories.[5,6] Adding θ^F_{00} and θ^I_{00}, in either picture, and remembering to use $\tilde{J}_\mu$, we find

$$\theta_{\mu\nu}=\theta^{FV}_{\mu\nu}+\frac{1}{2C_B G}:2\tilde{J}_\mu\tilde{J}_\nu-g_{\mu\nu}\tilde{J}_\lambda\tilde{J}^\lambda:. \qquad (2.13)$$

A word about renormalizations is in order here. The IP methods suffice to define the unique one-parameter (g_B) unrenormalized Dyson S matrix of the Thirring models. However, we have ignored wave-function renormalizations in our passage to the HP. With an adiabatic limit, one can show that $[\psi_H(x),\psi^\dagger_H(y)]_+=Z\delta(x-y)$. The finite currents that we want are then $J^\mu_H=Z^{-1}:\bar{\psi}_H\gamma^\mu\psi_H$, $J^{\mu\alpha}_H=Z^{-1}:\bar{\psi}_H\gamma^\mu\frac{1}{2}\lambda^\alpha\psi_H:$, and (still) $\tilde{J}_\mu=(J_{0H},GJ_{1H})$. We shall understand this identification in Eqs. (2.10)–(2.13) and so on. No renormalization is needed for g_B (see, however, Sec. VI).

As a warm-up for Secs. IV and V we will next apply the method to Coleman's correspondence.

III. A QUICK DERIVATION OF COLEMAN'S CORRESPONDENCE

Suppose we know the free-field correspondences[10] for U(1):

$$\begin{aligned}J^\mu_D&=-\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi_D\\&=-\sqrt{C_B}\,\epsilon^{\mu\nu}\partial_\nu\phi_D,\quad C_B=\frac{1}{\pi}\\:\psi^\dagger_{1D}\psi_{2D}:&=-\frac{C\mu}{2\pi}N_\mu e^{i2\sqrt{\pi}\phi_D},\\:\bar{\psi}_D\psi_D:&=-\frac{C\mu}{\pi}N_\mu\cos(2\sqrt{\pi}\,\phi_D),\end{aligned} \qquad (3.1)$$

where all fields are free (ψ = spinor field and ϕ = Bose field), and μ is the mass at which we choose to normal-order the ϕ field. For correspondence with a massless fermion, one should take $\mu\to 0$ (*after* operator manipulation or calculation of matrix elements). Later, however, we will follow Mandelstam's *convention*, and identify μ with the Bose quantum mass in the sine-Gordon equation. In (3.1) all expressions are in fact independent of μ, but, in general, the limit must be smooth because the Fermi side is smooth. The constant C is the same one that Mandelstam defines in his Eq. (2.3).

Now the IP is just the place to use these. We rewrite the U(1) theory of Sec. II. Equations (2.3) and (3.1) give

$$\theta^I_{00}=\frac{g_B C_B}{2}[(\partial_x\phi_D)^2-G(\partial_0\phi_D)^2]. \qquad (3.2)$$

This is quadratic, and when taken with $\theta^F_{00}=\frac{1}{2}[(\partial_0\phi)^2+(\partial_x\phi)^2]$, is easily seen to be only a finite wave-function renormalization on ϕ. I will first state the result and then prove it. The interaction (3.2) is equivalent to the HP Lagrangian

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}(1+\kappa)(\partial_\lambda\phi_H)^2\\&=\frac{1}{2G}(\partial_\lambda\phi_H)^2,\quad \kappa=C_B g_B\end{aligned} \qquad (3.3)$$

that is, a misnormalized free field. The way to see this is the following. Starting from (3.3) with κ unknown, go to Hamiltonian formalism, and perturb in small κ. The correct breakup is $\theta_{00}=\theta^F_{00}+\theta^I_{00}$,

$$\begin{aligned}\theta^F_{00}&=\tfrac{1}{2}[\pi_H{}^2+(\partial_x\phi_H)^2],\\\theta^I_{00}&=-\frac{\kappa}{2(1+\kappa)}\pi_H{}^2+\frac{\kappa}{2}(\partial_1\phi_H)^2,\end{aligned} \qquad (3.4)$$

where $\pi_H=\partial\mathcal{L}/\partial\dot{\phi}_H=(1+\kappa)\partial_0\phi_H$. Now go to the IP, $\phi_H=U^\dagger\phi_D U$. Because $U\pi_H U^\dagger=\partial_0\phi_D$, then θ^I_{00} in the IP is immediately compared with (3.2); it is the same when $\kappa=g_B C_B$.

The dynamics of (3.3) is entirely trivial. We write

$$\begin{aligned}U^\dagger\phi_D U&=\phi_H=\sqrt{G}\,\hat{\phi},\\U^\dagger\dot{\phi}_D U&=G^{-1}\dot{\phi}_H=\frac{1}{\sqrt{G}}\dot{\hat{\phi}},\end{aligned} \qquad (3.5)$$

where we have introduced the properly normalized free field $\hat{\phi}$. The second identity is immediately obtainable by differentiating the definition of ϕ_H in terms of ϕ_D. With the help of (3.5), we immediately transform our correspondences (3.1) to

$$\begin{aligned}:Z'\psi^\dagger_{1H}\psi_{2H}:&=-\frac{C\mu}{2\pi}N_\mu e^{i2(\pi G)^{1/2}\hat{\phi}},\\:Z'\bar{\psi}_H\psi_H:&=-\frac{C\mu}{\pi}N_\mu\cos[2(\pi G)^{1/2}\hat{\phi}],\\\tilde{J}^\mu&=(CG_B)^{1/2}\epsilon^{\mu\nu}\partial_\nu\hat{\phi}\\&=Z^{-1}:(\bar{\psi}_H\gamma^0\psi_H,G\bar{\psi}_H\gamma^1\psi_H):.\end{aligned} \qquad (3.6)$$

Here, as promised in Sec. II, I have left room for a wave-function renormalization, and a (different) mass renormalization Z'. The fermion fields solve the massless Thirring model, and the Bose field is free and massless ($\mu\to 0$ after calculation, as above).

The final step is the simplest: Add a mass term to the Thirring Lagrangian. In a "second" interaction picture, where the fields have the time dependence of the massless Thirring model, we use

$$\begin{aligned}+\theta^I_{00}&=+m':Z'\bar{\psi}\psi:\\&=-\frac{m'C\mu}{\pi}N_\mu\cos[2(\pi G)^{1/2}\hat{\phi}],\end{aligned} \qquad (3.7)$$

where m' is finite.

This interaction commutes with itself at equal times, so we need do no work. Going immediately to the final HP, we obtain the sine-Gordon Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\lambda\phi)^2+\frac{m'C\mu}{\pi}N_\mu\cos[2(\pi G)^{1/2}\phi]. \qquad (3.8)$$

Comparing this with the standard form, $(\alpha/\beta^2)N_\mu\cos\beta\phi$, we read off

$$2(\pi G)^{1/2}=\beta\,,\quad \left(\frac{4\pi}{\beta^2}=1+\frac{g_B}{\pi}\right). \qquad (3.9)$$

Following Mandelstam's convention, we *choose* μ to be the (tree-approximation) Bose mass. Then $\alpha=\mu^2$, $m'=\mu\pi/C\beta^2$. The identities (3.6) are form-invariant under this last transformation, with the understanding that ψ solves the massive Thirring model and ϕ solves the sine-Gordon equation. Taken together with the commutators (2.12), this completes the correspondence.

Mandelstam's operators[2] can also be boosted by these transformations from free to interacting fields:

$$\psi_D^1=\left(\frac{\mu C}{2\pi}\right)^{1/2}N_\mu\exp\left\{-i\sqrt{\pi}\left[\int_{-\infty}^{x}\dot{\phi}_D(\xi)+\phi_D(x)\right]\right\}. \qquad (3.10)$$

Then $\psi_H^1=U^\dagger\psi_D^1U$ is, using (3.5), the same form, but with $\dot{\phi}_D\to(1/\sqrt{G})\dot{\phi}$, $\phi_D\to\sqrt{G}\,\phi$.

With my machinery well oiled, I proceed to guess free-field correspondences for SU(N).

IV. FREE-FIELD CORRESPONDENCES FOR SU(N)

Begin with the case of SU(2). We need two quarks ψ_a^r, $r=1,2$ (Lorentz index), $a=1,2$ (isospin index). Mandelstam's representation[2] is immediately generalized to

$$\chi_a^1=\xi_1\psi_a^1=\left(\frac{\mu C}{2\pi}\right)^{1/2}N_\mu\exp\left\{-i\sqrt{\pi}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_a(\xi)+\phi_a(x)\right]\right\},$$
$$\chi_a^2=\xi_1\psi_a^2=+i\left(\frac{\mu C}{2\pi}\right)^{1/2}N_\mu\exp\left\{-i\sqrt{\pi}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_a(\xi)-\phi_a(x)\right]\right\}. \qquad (4.1)$$

Here[11] we have introduced two (Lorentz) scalar free Bose fields ϕ_a $(a=1,2)$. $\xi_1=(-1)^{N_1}$ is the Klein transformation operator, in terms of N_1, the number operator for the isospin-up quark. The fields χ_a anticommute when $a=b$ as they should, but commute when $a\neq b$. The Klein transformation corrects this for ψ_a^r; thus

$$[\psi_a^r(x),\psi_b^{s\dagger}(y)]_+=\delta_{rs}\delta_{ab}\delta(x-y), \qquad (4.2)$$

etc. is easily verified following Mandelstam. There are many other equivalent Klein transformations. For example, the scheme

$$\chi_a^1=\xi_1\cdots\xi_{a-1}\psi_a^1=\left(\frac{\mu C}{2\pi}\right)^{1/2}N_\mu\exp\left\{i\sqrt{\pi}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_a(\xi)+\phi_a(x)\right]\right\},$$
$$\chi_a^2=\xi_1\cdots\xi_{a-1}\psi_a^2=i\left(\frac{\mu C}{2\pi}\right)^{1/2}N_\mu\exp\left\{i\sqrt{\pi}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_a(\xi)-\phi_a(x)\right]\right\} \qquad (4.3)$$

generalizes to all SU(N). Here $\xi_j=(-1)^{N_j}$, $\xi_0=1$, and $a=1,\ldots,N$. I chose the one I did for SU(2) (a special case) because bilinears are so simple, $\chi_a^{\dagger r}\chi_b^s=\psi_a^{\dagger r}\psi_b^s$, and normal ordering with respect to ψ is the same as normal ordering with respect to χ. The normal-ordering equivalence persists for all SU(N), but, in general, the bilinears contain ξ's for $N\geq 3$. "Diagonal" bilinears, such as the mass term, etc., and SU(N) symmetric quartics are ξ free. In any case, the ξ's cause no trouble with the correspondences. As a matter of taste, I would leave them associated with the ψ's; they amount only to certain phases in a final calculation of Fermi Green's functions.

I am going to concentrate on working out the case of SU(2) in detail, with occasional remarks on general features of SU(N). At the end of Sec. V I will give a general discussion for SU(N). Moreover, I am going to assume familiarity with Mandelstam's manipulation of these operators, except when a step is somewhat unusual.

A. Bilinears

The calculation of the bilinears follows Mandelstam closely. First, the "diagonal" currents (we will in general use : : for Fermi normal ordering),

$$J^\mu=:\bar{\psi}\gamma^\mu\psi:=-\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu(\phi_1+\phi_2)=-\sqrt{C_B}\,\epsilon^{\mu\nu}\partial_\nu\phi_+\,,$$
$$J_3^\mu=:\bar{\psi}\gamma^\mu\frac{\tau_3}{2}\psi:=-\frac{1}{2\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu(\phi_1-\phi_2)=-\frac{1}{(2\pi)^{1/2}}\epsilon^{\mu\nu}\partial_\nu\phi_-\,, \qquad (4.4)$$

where now $C_B=2/\pi$, and for later purposes, we have introduced the normalized combinations $\phi_\pm=(1/\sqrt{2})(\phi_1\pm\phi_2)$. For SU($N$), J^μ will be the sum over all ϕ_a, etc. Before going on, these forms are worth consideration. As in the Abelian model, the time components of these currents are total derivatives, so, e.g.,

$$I_3 = \int dx\, J_{03}$$

$$= \frac{1}{(2\pi)^{1/2}} \phi_- \Big|_{-\infty}^{+\infty} . \tag{4.5}$$

Thus, ϕ_a, $\dot{\phi}_a$ do not transform under I_3 (or B). This tells us that when we get to the SGL equations whose solitons are these quarks, then, from the point of view of the ϕ's, the isospin of their quark solitons will be essentially topological (a conservation of asymptotic properties). The ϕ's themselves must have no simple isospin properties. Parenthetically, a group theorist would partially presage these remarks; he knows that SU(N) cannot be represented (linearly) on N real fields. These remarks are explored further in Appendix A.

Proceeding, we give the charged currents,

$$\begin{aligned} J_+(\tau_+) &= J_+^\dagger(\tau_-) \\ &= 2 :\psi_1^{1\dagger}\psi_2^1: \\ &= \frac{\mu C}{\pi} N_\mu \\ &\quad \times \exp\left\{ i(2\pi)^{1/2}\left[\int_{-\infty}^{x} d\xi\, \dot{\phi}_-(\xi) + \phi_-(x)\right]\right\}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} J_-(\tau_+) &= J_-^\dagger(\tau_-) \\ &= 2 :\psi_1^{2\dagger}\psi_2^2: \\ &= \frac{\mu C}{\pi} N_\mu \\ &\quad \times \exp\left\{ i(2\pi)^{1/2}\left[\int_{-\infty}^{x} d\xi\, \dot{\phi}_-(\xi) - \phi_-(x)\right]\right\}. \end{aligned}$$

Here $J_\pm = J_0 \pm J_1$ and $2\tau_\pm = \tau_1 \pm i\tau_2$. These appear spatially nonlocal, in distinction to the neutral currents. This may be an interesting effect in a theory probed by W mesons and photons, or a gluon interaction. Note, however, that in an SU(2) symmetric theory, any *particular* component of J^μ_α can be made local. I remark also that every time I have studied some property that all J^α_μ should have in common, it all works out. For example, one easily calculates directly by differentiation and $\Box^2 \phi_\pm = 0$ that

$$\begin{aligned} &(\partial_0 \mp \partial_x) J_\pm = 0, \\ &(\partial_0 \mp \partial_x) J^\alpha_\pm = 0, \quad \alpha = 1,2,3. \end{aligned} \tag{4.7}$$

The simple calculations, however, go along different lines for charged and neutral currents. Equation (4.7) is equivalent to $\partial_\mu J^\mu = \partial_\mu J^\mu_5 = \partial_\mu J^\mu_\alpha = \partial_\mu J^\mu_{5\alpha} = 0$, and therefore *all* currents are proportional to gradients of free scalar fields, as expected. This was obvious for the neutrals (from our representation), but is extrémely indirect for the charged currents. It appears that, despite appearances, all J^α_μ are on equal footing. I also remark that J^α_μ (J_μ) are made up entirely of ϕ_- (ϕ_+); hence they commute. Commutators of $\phi_\pm$ with the isospin generators are discussed in Appendix A.

The "mass" term has the form

$$-:\bar{\psi}\psi: = \frac{\mu C}{\pi} N_\mu [\cos(2\sqrt{\pi}\,\phi_1) + \cos(2\sqrt{\pi}\,\phi_2)] \tag{4.8a}$$

$$= \frac{2\mu C}{\pi} N_\mu \cos[(2\pi)^{1/2}\phi_+]\cos[(2\pi)^{1/2}\phi_-] \tag{4.8b}$$

$$\begin{aligned} &= \frac{2C}{\pi}(\mu_+\mu_-)^{1/2} N_+ \cos[(2\pi)^{1/2}\phi_+] \\ &\quad \times N_- \cos[(2\pi)^{1/2}\phi_-]. \end{aligned} \tag{4.8c}$$

In the last step, we have used Coleman's identity[1] $N_\mu(\cos\beta\phi) = (\mu'/\mu)^{\beta^2/4\pi} N_{\mu'}(\cos\beta\phi)$ to re-normal-order $\phi_\pm$ at different masses $\mu_\pm$. I shall have more to say about this later. For those who understood Secs. II and III, Eq. (4.8c) is very close to the interaction in the forthcoming SGL equations. For SU(N), this term is a sum of cosines for each ϕ_a.

For reference, I will simply state the other bilinears. Defining

$$P \equiv :\bar{\psi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi:, \quad S_\alpha \equiv :\bar{\psi}\tau_\alpha\psi:,$$

and

$$P_\alpha \equiv :\bar{\psi}\tau_\alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi:,$$

we obtain

$$\begin{aligned} P &= \frac{2iC}{\pi}(\mu_+\mu_-)^{1/2} N_\pm \sin[(2\pi)^{1/2}\phi_+]\cos[(2\pi)^{1/2}\phi_-], \\ S_3 &= \frac{2C}{\pi}(\mu_+\mu_-)^{1/2} N_\pm \sin[(2\pi)^{1/2}\phi_+]\sin[(2\pi)^{1/2}\phi_-], \\ S_1 &= -\frac{2C}{\pi}(\mu_+\mu_-)^{1/2} N_\pm \sin[(2\pi)^{1/2}\phi_+] \\ &\qquad \times \cos\left[(2\pi)^{1/2}\int_{-\infty}^{x}\dot{\phi}_-(\xi)d\xi\right], \\ S_2 &= -\frac{2C}{\pi}(\mu_+\mu_-)^{1/2} N_\pm \sin[(2\pi)^{1/2}\phi_+] \\ &\qquad \times \sin\left[(2\pi)^{1/2}\int_{-\infty}^{x}\dot{\phi}_-(\xi)d\xi\right], \end{aligned} \tag{4.9}$$

$$P_3 = \frac{2iC}{\pi}(\mu_+\mu_-)^{1/2}N_\pm \cos[(2\pi)^{1/2}\phi_+]\sin[(2\pi)^{1/2}\phi_-],$$

$$P_1 = -\frac{2iC}{\pi}(\mu_+\mu_-)^{1/2}N_\pm \cos[(2\pi)^{1/2}\phi_+] \times\cos\left[(2\pi)^{1/2}\int_{-\infty}^{x}\dot\phi_-(\xi)d\xi\right],$$

$$P_2 = -\frac{2iC}{\pi}(\mu_+\mu_-)^{1/2}N_\pm \cos[(2\pi)^{1/2}\phi_+] \times\sin\left[(2\pi)^{1/2}\int_{-\infty}^{x}\dot\phi_-(\xi)d\xi\right].$$

Here, I am using the abbreviation $N_\pm$ for $N_{+,-}$. These equations all follow straightforwardly, with the observation that $\exp[i\beta\int_{-\infty}^{x} d\xi\,\dot\phi_\pm(\xi)]$ normal-orders just as $\exp(i\beta\phi_\pm)$. We now turn to some quartics. As examples, I will work these out for currents, and current algebra.

B. Some quartics and current algebra

Evidently

$$:J_0J_0: = N_\mu \frac{2}{\pi}(\partial_x\phi_+)^2,$$
$$:J_1J_1: = N_\mu \frac{2}{\pi}(\partial_0\phi_+)^2,$$
$$:J_\pm^3 J_\pm^3: = N_\mu \frac{1}{\pi}[(\partial_0 \pm \partial_x)\phi_-]^2, \qquad (4.10)$$
$$:J_+^3 J_-^3: = -N_\mu \frac{1}{2\pi}\partial_\lambda\phi_-\partial^\lambda\phi_-;$$

one must, however, take more care with products of charged currents. Following Mandelstam, we obtain ($\dot\phi \equiv \partial_0\phi$)

$$J_+(\tau_+,x)J_+(\tau_-,y) \underset{x\to y}{\simeq} -\frac{1}{(x-y-i\epsilon)^2}\frac{1}{\pi^2} - \frac{1}{\pi^2}\frac{i(2\pi)^{1/2}}{x-y-i\epsilon}[\dot\phi_-(y)+\partial_y\phi_-(y)]$$
$$-\frac{1}{2\pi^2}\{i(2\pi)^{1/2}[\partial_y\dot\phi_-(y)+\partial_y{}^2\phi_-(y)] - 2\pi[(\partial_0+\partial_y)\phi_-]^2\} + O(x-y), \qquad (4.11a)$$

$$J_+(\tau_-,y)J_+(\tau_+,x) \underset{x\to y}{\simeq} -\frac{1}{\pi^2}\frac{1}{(x-y+i\epsilon)^2} - \frac{1}{\pi^2}\frac{i(2\pi)^{1/2}}{x-y+i\epsilon}[\dot\phi_-(x)+\partial_x\phi_-(x)]$$
$$-\frac{1}{2\pi^2}\{-i(2\pi)^{1/2}[\partial_x\dot\phi_-(x)+\partial_x{}^2\phi_-(x)] - 2\pi[(\partial_0+\partial_x)\phi_-]^2\} + O(y-x) \qquad (4.11b)$$

Here we were careful to expand the exponentials to second order. (4.11b) is just (4.11a) with $x \leftrightarrow y$ and $\phi_- \to -\phi_-$. It is crucial to note now that reexpanding a first-order term at x, around y, can change second-order terms. I choose to compare the two expressions at y. Thus, I reexpress (4.11b) as

$$J_+(\tau_-,y)J_+(\tau_+,x) \underset{x\to y}{\simeq} -\frac{1}{\pi^2}\frac{1}{(x-y+i\epsilon)^2} - \frac{1}{\pi^2}\frac{i(2\pi)^{1/2}}{x-y+i\epsilon}[\dot\phi_-(y)+\partial_y\phi_-(y)]$$
$$-\frac{1}{2\pi^2}\{+i(2\pi)^{1/2}[\partial_y\dot\phi_-(y)+\partial_y{}^2\phi_-(y)] - 2\pi[(\partial_0+\partial_y)\phi_-]^2\} + O(y-x). \qquad (4.11b')$$

Note the sign change in the next-to-last term. Without this care, we would blunder into nonlocal current commutators. Now using $(z-i\epsilon)^{-1} = Pz^{-1} + i\pi\delta(z)$, together with (4.11a), (4.11b′), and (4.4), we obtain correctly

$$[J_+(\tau_+,x), J_+(\tau_-,y)] = 4J_+^3(x)\delta(x-y) + i\frac{2}{\pi}\partial_x\delta(x-y). \qquad (4.12)$$

The rest of current algebra follows smoothly: Commutators of charged currents with neutral currents are much easier.

With the help of (4.11a), and (4.11b′) we can also construct the point quartic

$$:J_+(\tau_+,x)J_+(\tau_-,x): = :\sum_{\alpha=1}^{2} J_+^\alpha(x)J_+^\alpha(x):.$$

This object is symmetric under $\tau_+ \leftrightarrow \tau_-$, so we define

$$J_+(\tau_+,x)J_+(\tau_-,x) \equiv \lim_{x\to y}\tfrac{1}{2}[J_+(\tau_+,x)J_+(\tau_-,y) + J_+(\tau_-,x)J_+(\tau_+,y)]. \qquad (4.13)$$

You may also further symmetrize with respect to x and y, at no cost. After an ordinary *Fermi* normal ordering to remove the c-number vacuum expectation value, we obtain [just from the last term of (4.11a) and (4.11b′)]

$$:J_\pm(\tau_+,x)J_\pm(\tau_-,x): = N_\mu \frac{1}{\pi}[(\partial_0 \pm \partial_x)\phi_-]^2 . \quad (4.14)$$

We combine (4.10) and (4.14) into some more familiar forms:

$$\begin{aligned} \theta_{00}^{FB} &= \frac{1}{2C_B}:J_0J_0+J_1J_1: \\ &= N_\mu \tfrac{1}{2}[(\dot\phi_+)^2+(\partial_x\phi_+)^2] \\ &= \theta_{00}^F(\phi_+), \\ \theta_{00}^{FV} &= \frac{1}{2\tilde C_V}:J_0^\alpha J_0^\alpha + J_1^\alpha J_1^\alpha: \\ &= \frac{1}{4\tilde C_V}:J_+^\alpha J_+^\alpha + J_-^\alpha J_-^\alpha: \\ &= N_\mu \tfrac{1}{2}[(\dot\phi_-)^2+(\partial_x\phi_-)^2] \\ &= \theta_{00}^F(\phi_-). \end{aligned} \quad (4.15)$$

Remember that for SU(2), $C_B=2/\pi$, $\tilde C_V = 3/2\pi$. We see that the free Fermi and Bose Hamiltonians are identical, a result which had to be true. It is also true for the entire free stress tensor. On the other hand, the forms (4.10) and (4.14) caused me some consternation. For example, I can write

$$:J_\pm^\alpha J_\pm^\alpha: = 3:J_\pm^3 J_\pm^3: , \quad (4.16)$$

where α is summed as usual from 1 to 3. This is a surprise, as the left-hand side is isoscalar and the right a tensor. Is there trouble, or is (4.16) true, plus many other relations (by isospin rotations, commutators, etc.)? The answer is that it *is* true, and it is a simple property of Fermi statistics. The reader is invited to write out (4.16) explicitly in terms of free fermions and see for himself. Such identities are also true in SU(N), where $J_\pm^\alpha J_\pm^\alpha$ equals a weighted sum of "diagonal" currents squared. Out of many identities of this sort, I mention also

$$:J_+^\alpha \tau^\alpha \psi^1: = 3:J_+^3\tau_3\psi^1: , \quad (4.17)$$

and a similar equation for ψ^2 with J_-. These follow immediately on commutation of ψ with (4.16), and we will make mention of (4.16) and (4.17) again in Sec. VI.

None of this will directly help us simplify a $J_\mu^\alpha J_\alpha^\mu$ interaction, however; to construct this, we need to calculate

$$\tfrac{1}{2}:J_+(\tau_+)J_-(\tau_-)+J_+(\tau_-)J_-(\tau_+): \\ = -\left(\frac{C\mu}{\pi}\right)^2 N_\mu \cos[2(2\pi)^{1/2}\phi_-] , \quad (4.18)$$

or, with (4.9),

$$:J_\lambda^\alpha J_\alpha^\lambda: = :J_+^\alpha J_-^\alpha: \\ = -\left(\frac{C\mu}{\pi}\right)^2 N_\mu \cos[2(2\pi)^{1/2}\phi_-] - \frac{1}{2\pi}N_\mu(\partial_\lambda\phi_-)^2 . \quad (4.19)$$

We see manifest in this representation the basis of the statement that such an interaction breaks (naive) conformal invariance. Although the right-hand side of (4.19) is independent of μ, it still provides a mass scale. We will return to this interaction in Sec. VI. I will not write out the other quartics ($S^\alpha S^\alpha$, etc.). They are, however, quite local. I have also checked a representative selection of the quark algebra among the J^μ's, S's, and P's.

Before going on to introduce interaction, I want to use a little hindsight to introduce a slightly different free-field representation. We shall see that $\phi_\pm = (1/\sqrt2)(\phi_1 \pm \phi_2)$ are the natural eigenstates of the theory, and they are not in general treated symmetrically. (The masses of the $\phi_\pm$ quanta are not the same.) It is therefore convenient to have also

$$\begin{aligned} \chi_1^1 &= \xi_1\psi_1^1 \\ &= \left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4}N_+\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\,\dot\phi_+(\xi)+\phi_+(x)\right]\right\}N_-\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\,\dot\phi_-(\xi)+\phi_-(x)\right]\right\}, \\ \chi_2^1 &= \xi_1\psi_2^1 \\ &= \left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4}N_+\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\,\dot\phi_+(\xi)+\phi_+(x)\right]\right\}N_-\exp\left\{+i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\,\dot\phi_-(\xi)+\phi_-(x)\right]\right\}, \end{aligned} \quad (4.20)$$

and similarly for ψ_a^2 (with $\phi_\pm \to -\phi_\pm$, $\dot\phi_\pm \to \dot\phi_\pm$). What we have done here relative to (4.1) is to rewrite in terms of $\phi_\pm$, and normal-order each separately at $\mu_\pm$. I believe (4.1) and (4.20) are equivalent (certainly for the free theory), but the latter is more convenient for using Mandelstam's formalism, in which, in the end, one is normal ordering directly at the Bose quanta masses. I will therefore switch to (4.20) for the remainder of the paper.

For this representation, it is easily checked that there are only a few minor modifications in all our correspondences. All forms are the same, with the understanding that expressions involving pure ϕ_+ (or ϕ_-) are normal ordered at μ_+ (μ_-). Our mixed expressions, Eqs. (4.8) and (4.9), for S's and P's, come out directly as (4.8c) and (4.9). We are now ready for interaction.

V. INTERACTION AND THE SINE-GORDON-LIKE EQUATIONS

Armed with Secs. II and IV, introduction of interaction is mechanical. We first take the simple $J^\mu J_\mu$ interaction set up in Sec. II. Following Sec. III, we reexpress Eq. (2.3), using (4.10), as

$$\begin{aligned}\theta^I_{00} &= \tfrac{1}{2} g_B (J_{0D}{}^2 - G J_{1D}{}^2)\\ &= \tfrac{1}{2} g_B C_B [(\partial_x \phi_{+D})^2 - G(\partial_0 \phi_{+D})^2],\end{aligned} \qquad (5.1)$$

where $G=(1+C_B g_B)^{-1}$, $C_B = 2/\pi$. Everything sweeps through as before, including $\kappa = C_B g_B$, $\tilde{J}_\mu = (J_0, GJ_1)$, $U^\dagger \phi_{+D} U = \sqrt{G}\,\hat{\phi}_+$, $U^\dagger \dot{\phi}_{+D} U = (1/\sqrt{G})\dot{\hat{\phi}}_+$, where $\hat{\phi}_+$ is the correctly normalized free field. Thus (all in HP)

$$\begin{aligned}\tilde{J}^\mu &= -(C_B G)^{1/2} \epsilon^{\mu\nu} \partial_\nu \phi_+\\ &= :Z^{-1}(\bar{\psi}\gamma^0\psi, G\bar{\psi}\gamma^1\psi):,\end{aligned} \qquad (5.2a)$$

$$\begin{aligned}\tilde{J}_3^\mu &= J_3^\mu\\ &= -\frac{1}{(2\pi)^{1/2}} \epsilon^{\mu\nu}\partial_\nu \phi_-\\ &= :Z^{-1}\left(\bar{\psi}\gamma^0 \frac{\tau_3}{2}\psi, \bar{\psi}\gamma^1 \frac{\tau_3}{2}\psi\right):,\end{aligned}$$

$$\begin{aligned}&[\tilde{J}_0(x), \psi(y)] = -\psi(x)\delta(x-y),\\ &[\tilde{J}_1(x), \psi(y)] = -\gamma_0\gamma_1 G\psi(x)\delta(x-y),\\ &[J_0^\alpha(x), \psi(y)] = -\tfrac{1}{2}\lambda_\alpha \psi(x)\delta(x-y),\\ &[\tilde{J}_0(x), \tilde{J}_1(y)] = iC_B \partial_x \delta(x-y),\end{aligned} \qquad (5.2b)$$

and so on. The isospin scale is not changed because ϕ_- is undisturbed. Indeed, all the isospin current relations of Sec. IV are completely unchanged for this reason. There are only the usual expected changes $\phi_+ \to \sqrt{G}\,\hat{\phi}_+$, $\dot{\phi}_+ \to (1/\sqrt{G})\dot{\hat{\phi}}_+$ everywhere. Thus, e.g., (4.20) is changed to

$$\begin{aligned}\chi_1^1 &= \xi_1 \psi_1^1\\ &= \left(\frac{C}{2\pi}\right)^{1/2} (\mu_+\mu_-)^{1/4} N_\pm \exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi \frac{1}{\sqrt{G}} \dot{\hat{\phi}}_+ + \sqrt{G}\,\hat{\phi}_+(x)\right]\right\} \exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\, \dot{\phi}_- + \phi_-(x)\right]\right\},\end{aligned} \qquad (5.3)$$

etc., and from (4.8c)

$$-:Z'\bar{\psi}\psi: = \frac{2C}{\pi}(\mu_+\mu_-)^{1/2} N_\pm \cos[(2\pi G)^{1/2}\hat{\phi}_+]\cos[(2\pi)^{1/2}\phi_-]. \qquad (5.4)$$

Still following Sec. III, we next add in the mass term, $-m':Z'\bar{\psi}\psi:$, obtaining the boson Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\nu \phi_+)^2 + \tfrac{1}{2}(\partial_\nu \phi_-)^2 + \frac{2m'C}{\pi}(\mu_+\mu_-)^{1/2} N_\pm \cos[(2\pi G)^{1/2}\phi_+]\cos[(2\pi)^{1/2}\phi_-]. \qquad (5.5)$$

This implies two coupled SGL equations for $\phi_\pm$. I will discuss the system presently.

Before that, however, it is wise to check our results directly. Starting from the SGL equations implied by (5.5), and our final fermions

$$\begin{aligned}\chi_1^1 &= \xi_1\psi_1^1\\ &= \left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4} N_\pm \exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi \frac{1}{\sqrt{G}}\dot{\phi}_+(\xi) + \sqrt{G}\,\phi_+(x)\right]\right\}\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\, \dot{\phi}_-(\xi) + \phi_-(x)\right]\right\},\\ \chi_2^1 &= \xi_1\psi_2^1\\ &= \left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4} N_\pm \exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi \frac{1}{\sqrt{G}}\dot{\phi}_+(\xi) + \sqrt{G}\,\phi_+(x)\right]\right\}\exp\left\{+i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\, \dot{\phi}_-(\xi) + \phi_-(x)\right]\right\},\\ \chi_1^2 &= \xi_1\psi_1^2\\ &= i\left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4} N_\pm \exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi \frac{1}{\sqrt{G}}\dot{\phi}_+(\xi) - \sqrt{G}\,\phi_+(x)\right]\right\}\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\, \dot{\phi}_-(\xi) - \phi_-(x)\right]\right\},\\ \chi_2^2 &= \xi_1\psi_2^2\\ &= i\left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4} N_\pm \exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi \frac{1}{\sqrt{G}}\dot{\phi}_+(\xi) - \sqrt{G}\,\phi_+(x)\right]\right\}\exp\left\{+i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^x d\xi\, \dot{\phi}_-(\xi) - \phi_-(x)\right]\right\},\end{aligned} \qquad (5.6)$$

one can wend one's way back through all the identities to the massive-Thirring-model Dirac equation.

Indeed, following Mandelstam, we find the acknowledged wave-function renormalizations Z [in the form $(x-y)^{\sigma}$] the same for baryon number and SU(2) currents. The renormalizations for all the S's and P's are the same among themselves, but different from the currents, hence Z'. Indeed, we recover *all* the results of the interaction picture, i.e., all[12] the results of Sec. IV, with the simple map $\phi_+ \to \sqrt{G}\,\phi_+$, $\dot\phi_+ \to (1/\sqrt{G})\dot{\phi}_+$. In going all the way back to the Thirring-Dirac equation, one calculates derivatives of (5.6). The useful identities analogous to Mandelstam's[2] Eq. (4.6) are of the form

$$\pi N'_{\pm}\int_{-\infty}^{x} d\xi\,(\{\cos[(2\pi G)^{1/2}\phi_+(\xi)]\sin[(2\pi)^{1/2}\phi_-(\xi)]+\sin[(2\pi G)^{1/2}\phi_+(\xi)]\cos[(2\pi)^{1/2}\phi_-(\xi)]\}\chi_1^1(x))$$
$$=\int_{-\infty}^{+\infty} d\xi\,[N_{\pm}\cos[(2\pi G)^{1/2}\phi_+(\xi)]\cos[(2\pi)^{1/2}\phi_-(\xi)],\chi_1^1(x)]\,, \quad (5.7)$$

where N' is Mandelstam's "block" normal ordering, and the second line is a commutator. Recognizing $\bar\psi\psi$ in the last line, one obtains the Dirac equation for χ. Multiplying in a ξ_1 from the left results in the expected equation for ψ. Everything goes through smoothly, so we turn out attention back to the SGL system (5.5).

We transform (5.5) into a "standard" form, say

$$\mathcal{L}_I=\frac{\alpha}{\beta_+^{\,2}}N_{\pm}\cos(\beta_+\phi_+)\cos[(2\pi)^{1/2}\phi_-]\,. \quad (5.8)$$

Again following Mandelstam's convention, we choose to set $\mu_{\pm}$ equal to the Bose quantum masses. Thus $\alpha=\mu_+^{\,2}$, $(m'2C/\pi)(\mu_+\mu_-)^{1/2}=\mu_+^{\,2}\beta_+^{\,-2}$, and

$$\beta_+=(2\pi G)^{1/2}\,,$$
$$\frac{2\pi}{\beta_+^{\,2}}=1+g_BC_B$$
$$=1+\frac{2g_B}{\pi}\,. \quad (5.9)$$

We also notice the curious fact that the "mass" μ_- of the ϕ_- quantum is *fixed* in terms of μ_+ (mass of ϕ_+),

$$\mu_-^{\,2}=\frac{2\pi}{\beta_+^{\,2}}\mu_+^{\,2}\,, \quad (5.10)$$

and thus $m'=(\mu_+/4C)G^{-3/4}=\mu_+\beta_+^{\,-3/2}(\pi/2C)(2\pi)^{-1/4}$. This fixed mass ratio is easily understood: The corresponding massive Thirring model has only one dimensional parameter m'. The fact that $\mu_+\neq\mu_-$ is also the last blow to any hope that the SGL system would exhibit a (linear) SU(2) symmetry. It apparently has no (linear) continuous symmetry at all. In Appendix A, however, we discuss the sense in which the sine-Gordon–like equations may be thought of as providing a (spatially) nonlocal representation of isospin. Of course, (5.10) is not a reliable prediction as large (calculable) higher-order corrections are expected from the interaction (even for small β_+).

What about soliton-like extended solutions of the classical SGL equations? I have not tried to find them analytically, but, by our very construction, they do exist [and will be free—no scattering—when $\beta_+=(2\pi)^{1/2}$]. The qualitative features of the solutions can be seen through the discrete symmetries of the system:

$$\phi_+\to\phi_++\frac{\pi n}{\beta_+}\ \textit{and}\ \phi_-\to\phi_-+\frac{\pi m}{(2\pi)^{1/2}}\quad\text{(type I)}\,,$$
$$\phi_+\to\phi_++\frac{2\pi n}{\beta_+}\ \textit{or}\ \phi_-\to\phi_-+\frac{2\pi m}{(2\pi)^{1/2}}\quad\text{(type II)}\,, \quad (5.11)$$

where n, m are odd for type I. Thus, we expect solutions with such (say one-sided) asymptotic behavior. Consulting Eqs. (5.2a), now in the form

$$\tilde{J}^{\mu}=-\frac{\beta_{\pm}}{\pi}\epsilon^{\mu\nu}\partial_\nu\phi_+,\quad J_3^{\mu}=-\frac{1}{(2\pi)^{1/2}}\epsilon^{\mu\nu}\partial_\nu\phi_-, \quad (5.2a')$$

that is, $B=(\beta_+/\pi)\phi_+|_{-\infty}^{+\infty}$, $I_3=[1/(2\pi)^{1/2}]\phi_-|_{-\infty}^{+\infty}$, we easily correlate asymptotic behaviors with quark content. Assuming $\phi_{\pm}(-\infty)=0$, then a solution for which $(x\to+\infty)$

$$\phi_+\to\frac{\pi}{\beta_+},\quad \phi_-\to\left(\frac{\pi}{2}\right)^{1/2} \quad (5.12)$$

is a quark with $I_3=+\frac{1}{2}$ (plus perhaps $qq\bar{q}$ in $I_3=\frac{1}{2}$, etc.). In general, states with asymptotic behavior of type I [Eq. (5.11)] have half-integer isospin correlated with odd quark number. States of type II have even quark number and integer isospin. It is not clear to me whether there is a simple way of reading *total* isospin from the classical solutions.

Finally, I will mention that, having repeated Coleman's[1] variational vacuum calculation on the Hamiltonian corresponding to (5.8), I find $\beta_+^{\,2}<6\pi$ ($g_B>-\frac{1}{3}\pi$), or else the energy is unbounded below. This is to be compared with the "obvious" bound $g_B>-\frac{1}{2}\pi$.

General remarks on SU(N). So much for SU(2). What can we say about SU(N)? In general, the isoscalar interaction

$$J^\mu = -\sqrt{C_B}\,\epsilon^{\mu\nu}\partial_\nu \phi_+, \quad \phi_+ \equiv \frac{1}{\sqrt{N}}\sum_a \phi_a, \quad C_B = \frac{N}{\pi}$$

$$\tilde{J}^\mu = -(C_B G)^{1/2})\epsilon^{\mu\nu}\partial_\nu \phi_+, \quad G = (1+g_B C_B)^{-1} \tag{5.13}$$

scale shifts only ϕ_+, leaving undisturbed the other $N-1$ orthogonal combinations Φ_a, $a=2,\dots,N$. It is, of course, a matter of taste how to choose the Φ_a. A choice for SU(3) might be

$$\begin{aligned}
\phi_+ &= \frac{1}{\sqrt{3}}(\phi_1+\phi_2+\phi_3),\\
\Phi_2 &= \frac{1}{\sqrt{2}}(\phi_1-\phi_2),\\
\Phi_3 &= \frac{1}{\sqrt{6}}(\phi_1+\phi_2-2\phi_3),\\
\phi_1 &= \frac{1}{\sqrt{3}}\phi_+ + \frac{1}{\sqrt{2}}\Phi_2 + \frac{1}{\sqrt{6}}\Phi_3,\\
\phi_2 &= \frac{1}{\sqrt{3}}\phi_+ - \frac{1}{\sqrt{2}}\Phi_2 + \frac{1}{\sqrt{6}}\Phi_3,\\
\phi_3 &= \frac{1}{\sqrt{3}}\phi_+ - \left(\frac{2}{3}\right)^{1/2}\Phi_3,
\end{aligned} \tag{5.14}$$

but $\Phi_{2,3}$ can be orthogonally mixed. In general

$$\phi_a = \frac{1}{\sqrt{N}}\phi_+ + D_{ab}\Phi_b, \quad \sum_a D_{ab} = 0,$$

$$\sum_a D_{ab}D_{ab'} = \delta_{bb'}.$$

We expect and will note later that the masses of the Φ_a are degenerate [a bosonic $O(N-1)$ symmetry?] under the isoscalar interaction, so the convenient quark representation [analogous to (4.19)] is constructed as follows: Start from (4.1); break ϕ_a up into ϕ_+, Φ_a, and shift $\dot{\phi}_+ \to (1/\sqrt{G})\dot{\phi}_+$, $\phi_+ \to \sqrt{G}\,\phi_+$. Normal-order ϕ_+ at μ_+ and Φ_a at μ_-. The breakup of the normalization factor is

$$\left(\frac{\mu C}{2\pi}\right)^{1/2} \to \left(\frac{\mu_+ C}{2\pi}\right)^{1/2N}\left(\frac{\mu_- C}{2\pi}\right)^{(1-1/N)/2}. \tag{5.15}$$

The SU(N) currents are functions of Φ_a only. The Fermi-mass term generates the SGL interaction

$$\begin{aligned}
\mathcal{L}_I &= -m':Z'\bar\psi\psi:\\
&= 2m'\left(\frac{\mu_+ C}{2\pi}\right)^{1/N}\left(\frac{\mu_- C}{2\pi}\right)^{1-1/N}\\
&\quad\times\sum_a \cos\left[2\left(\frac{\pi G}{N}\right)^{1/2}\phi_+ + 2\sqrt{\pi}\,D_{ab}\Phi_b\right].
\end{aligned} \tag{5.16}$$

Expanding to second order in the fields, we obtain the boson mass matrix $\mathfrak{m}^2$,

$$\begin{aligned}
-\tfrac{1}{2}\mathfrak{m}^2 &= \mathcal{L}_{(2)}\\
&\cong -2m'\left(\frac{\mu_+ C}{2\pi}\right)^{1/N}\left(\frac{\mu_- C}{2\pi}\right)^{1-1/N}\\
&\quad\times\left(2\pi G\phi_+^2 + 2\pi\sum_a \Phi_a^2\right).
\end{aligned} \tag{5.17}$$

Thus $\mathfrak{m}^2$ has the promised $O(N-1)$ symmetry. Calling μ_+ the mass of ϕ_+ and μ_- all the others, we find $\mu_-^2 = G^{-1}\mu_+^2$, as above. Normalizing to μ_+, then $m' = (\mu_+/4C)G^{-1/2-1/2N}$ We also remark that as in Sec. IV this peculiar $\mu_\pm$ pattern in the mass term (5.20) can be obtained directly from the representation (4.1) by re-normal-ordering in the free theory before interaction, as in (4.8c).

Do we have a real bosonic (linear) $O(N-1)$? The answer is no. The rest of the interaction completely ruins it. As a simple example of what is going on, consider a toy Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi_1)^2 + \tfrac{1}{2}(\partial\phi_2)^2 + \mu^2(\cos\phi_1 + \cos\phi_2).$$

The mass and kinetic energy terms have a U(1) symmetry, but not the interaction. We could make a U(1) transformation on the Lagrangian, thus introducing a parameter, but it is not a symmetry transformation, only a change of variables. This is precisely the situation among the Φ_a in our SU(N) models, except that in the SGL equations the symmetry is *required* and then broken. Its breaking must then be *calculable*. [For that matter, the deviation of μ_-^2 from $\mu_+^2 G^{-1}$ is also calculable in our SU(2) model.] It is certainly not surprising that there seems to be no direct connection between this $O(N-1)$ and the quark SU(N).

As a closing remark in this section, I mention that although we have worked in terms of an SU(N) quark representation for the fermions, one can as easily take quarks with color, octets (baryons), etc. As in Sec. IV, one simply takes one ϕ for each independent Fermi field.

VI. REMARKS ON HARD-CHIRAL-SYMMETRY-BREAKING INTERACTIONS

We have previously concerned ourselves only with interactions of the form $g_B J^\mu J_\mu$ (baryon number current-current interaction), and mass terms. Collectively, these imply at most a soft breaking of the chiral SU(N). Other hard-breaking interactions, such as $g_V J_\mu^\alpha J^\mu_\alpha$ are presently under investigation, and I will confine myself here to a few preliminary remarks.

In the first place, we are on much more treacherous grounds with these interactions: It is known[13] that g will require a renormalization, and it may even be necessary to include at least one other interaction (say $g_S\bar\psi\psi\bar\psi\psi$) for a consistent renormalization.[14] We expect our interaction picture approach to be less useful, but still suggestive.

Following Sec. II, we can write

$$\theta_{00}^{I}=\frac{g_V}{2}J_0^\alpha J_0^\alpha+b_V J_1^\alpha J_1^\alpha+\frac{g_B}{2}J_0J_0+b_B J_1J_1$$
$$=A_V\theta_{00}^{FV}+B_V J_+^\alpha J_-^\alpha+A_B\theta_{00}^{FB}+B_B J_+J_-\,, \quad (6.1)$$

where $J_\pm=J_0\pm J_1$, $A_V=(\frac{1}{8}g_V+\frac{1}{4}b_V)4\tilde{C}_V$, $B_V=\frac{1}{4}g_V-\frac{1}{2}b_V$, $A_B=(\frac{1}{8}g_B+\frac{1}{4}b_B)4C_B$, $B_B=\frac{1}{4}g_B-\frac{1}{2}b_B$, and we have used (2.2). Also $J_+J_-=J_\mu J^\mu$. These are convenient for evaluating Schwinger's condition. We obtain

$$A_V{}^2+2A_V=4B_V{}^2C_V\tilde{C}_V\,,$$
$$A_B{}^2+2A_B=4B_B{}^2C_B{}^2\,. \quad (6.2)$$

The second restriction is essentially that given in Sec. II; the vector restriction is new, and bears comment. Its solution has a two-sheeted structure, only one sheet of which is perturbative ($b_V\to 0$ as $g_V\to 0$). Presumably, this sheet defines ordinary formal unrenormalized perturbation theory for the vector interaction. If we further require the Dirac equation in terms of a simple rescaling $\tilde{J}_\mu^\alpha=(J_0^\alpha,\lambda J_1^\alpha)$, we need $g_V\lambda=-2b_V$. The requirement that $\tilde{J}_\mu$ be a two-vector in the Heisenberg picture is $\lambda-(A_V-2B_VC_V)=1$, $\lambda^{-1}-(A_V+2B_VC_V)=1$. The entire set of requirements is, in general, *inconsistent* (except at $g_V=0$), indicating that our scheme of passage to the Heisenberg picture is too naive: g_V needs a renormalization.

Curiously enough, there is one nontrivial value of the coupling, on the *second* sheet, for which all the equations can be satisfied. That value is

$$g_V=-\frac{4\pi}{n+1},$$

and $B_V=0$, $A_V=-2$, $\lambda=-1$. The value of g_V is the negative of the Dashen-Frishman[6] value, and $B_V=0$ means conserved axial-vector currents. $\lambda=-1$ is a change of sign in the axial-vector current commutators with the field. This was suggestive enough to pursue. For correspondence with the Dashen-Frishman equations, it turns out that one must map $\psi(-x,-t)|_{\text{here}}=\psi_{\text{DF}}(x,t)$, and similarly for all the currents. This changes the sign of g_V (but reverses g_B). We obtain, in the notation of Dashen and Frishman, precisely their Dirac equations, and $\delta=-1$, $C_1=1/2\pi$, $\tilde{C}_1=(n+1)/2\pi$, $\bar{a}=[1-g_B(n/\pi)]^{-1}$, $a=1$, $C_0=-(n/\pi)[1-g_B(n/\pi)]^{-1}$. Unfortunately, plugging into their Eq. (17), we get -1, i.e., an "antispinor" (with an interchange of ψ^1 and ψ^2). Presumably then, this is *not* their solution. Indeed, a glance at their equations (15) shows that a solution for $\delta=-1$ is the $\psi^1\leftrightarrow\psi^2$ interchange of the $\delta=+1$ solution.

We turn now to a few remarks about the $\phi_\pm$ representation. Using (4.19), we rewrite the vector part of (6.1) as

$$\theta_{00}^{I}=\tfrac{1}{2}(\dot{\phi}_-)^2\left(A_V-\frac{B_V}{\pi}\right)+\tfrac{1}{2}(\partial_x\phi_-)^2\left(A_V+\frac{B_V}{\pi}\right)$$
$$-B_V\left(\frac{C\mu_-}{\pi}\right)^2 N\cos[2(2\pi)^{1/2}\phi_-]\,. \quad (6.3)$$

We see immediately that, in general, g_V renormalization will be tangled with normal-ordering. Note that we can sweep out the quadratic form into a covariant wave-function renormalization (as in Sec. III),

$$\mathcal{L}=\tfrac{1}{2}(1+\kappa)(\partial_\mu\phi_-)^2-B_V\left(\frac{C\mu_-}{\pi}\right)^2 N\cos[2(2\pi)^{1/2}\phi_-]\,, \quad (6.4)$$

if $\kappa=A_V+B_V/\pi$ and $-\kappa/(1+\kappa)=A_V-B_V/\pi$. This form is suggestive of the requisite SGL equation, but I do not trust it. These conditions on κ are inconsistent with (6.2) (being instead the conditions on λ above). I think the point is that the cosine interaction is quite bizarre. Remember that it represents $\sum_{\alpha=1}^{2}J_\mu^\alpha J_\alpha^\mu$ which certainly does *not* commute with itself at equal time. As a matter of fact, the same problem exists in the Abelian Thirring model, where $[\bar{\psi}(1+\gamma_5)\psi,\bar{\psi}(1-\gamma_5)\psi]\sim[e^{+i\beta\phi},e^{-i\beta\phi}]$ appears to be zero, and yet, from the ψ's, must be proportional to $\delta(x-y)J_1(x)\sim\delta(x-y)\partial_0\phi$. To see such things, one must do (at least) a smearing in time.

Our final remark in this section concerns the explicit $\phi_\pm$ construction of our solution to the Dashen-Frishman Dirac equation. One can work through the interaction picture, as described above, but I will just state the result directly. Take their equations in the form

$$i(\partial_0+\partial_x)\psi_{\text{DF}}^2=:g_B\tilde{J}_+^B\psi_{\text{DF}}^2+g_V\left[\tfrac{1}{4}\int_{-\infty}^{+\infty}dy\,\tilde{J}_+^\alpha(y)\tilde{J}_+^\alpha(y),\psi_{\text{DF}}^2(x)\right]:\,, \quad (6.5)$$

where the last term is a commutator, and similarly for ψ_{DF}^1. By $\tilde{J}$ I mean Heisenberg currents that transform as two-vectors. When $g_V=4\pi/n+1$, this is their case $\delta=-1$. Now for the solution: Consider our $g_V=0$ solutions (Secs. IV and V), e.g.,

$$\chi_1^1=\xi_1\psi_1^1$$
$$=\left(\frac{C}{2\pi}\right)^{1/2}(\mu_+\mu_-)^{1/4}N\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^{x}d\xi\,\frac{1}{\sqrt{\mathcal{G}}}\dot{\phi}_+ +\sqrt{\mathcal{G}}\,\phi_+(x)\right]\right\}N\exp\left\{-i\left(\frac{\pi}{2}\right)^{1/2}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_-(\xi)+\phi_-(x)\right]\right\},\tag{6.6}$$

and so on. Leave $\mathcal{G}$ undetermined, however, for the moment. Construct the Dashen-Frishman currents out of these, *as usual* [$\mathcal{J}^\mu=-(2\mathcal{G}/\pi)^{1/2}\epsilon^{\mu\nu}\partial_\nu\phi_+$, $\mathcal{J}_3^\mu=J_3^\mu=-(1/2\pi)^{1/2}\epsilon^{\mu\nu}\partial_\nu\phi_-$, etc.], *but* take the Dashen-Frishman fields to be $\psi_{\rm DF}^{1,2}=\psi^{2,1}$. Thus in terms of *our* fields (6.6), Eq. (6.5) reads

$$i(\partial_0+\partial_x)\psi^1=:g_B\mathcal{J}_+^B\psi^1+g_V\left[\tfrac{1}{4}\int_{-\infty}^{+\infty}dy\,J_+^\alpha J_+^\alpha,\,\psi^1(x)\right]:.\tag{6.7}$$

Recall our identity (4.16). Since ϕ_- is not excited, it is true for $\mathcal{G}\neq 1$ and the last term in (6.7) is merely

$$:g_V 3\frac{\tau_3}{2}J_+^3(x)\psi^1:.$$

Now, it is a simple matter to differentiate our ψ's. We obtain, e.g.,

$$i(\partial_0+\partial_1)\psi_1^1=N_\mu\left\{\left[\frac{\pi}{2}\left(1+\frac{1}{\mathcal{G}}\right)J_+ +2\pi J_+^3\right]\psi_1^1\right\}$$

and hence $g_B=\frac{1}{2}\pi(1+1/\mathcal{G})$, $g_V=\frac{4}{3}\pi$, as required. All this is as I reasoned it from the interaction picture, and it is no surprise that our solution $\psi_{\rm DF}$ is an "antispinor."

NOTE ADDED IN PROOF

By a simple extension of the approach in Sec. VI, I have been able also to construct the "correct" Dashen-Frishman (DF) solution. The solution has the form

$$(\chi_1^2)_{\rm DF}=\xi_1(\psi_1^2)_{\rm DF}=\chi_1^1=\xi_1\psi_1^1=\kappa N\exp\left\{-i\left[\alpha\int_{-\infty}^{x}d\xi\,\dot{\phi}_+(\xi)+\beta\phi_+(x)\right]\right\}$$
$$\times\exp\left\{-i(\pi/2)^{1/2}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_-(\xi)+\phi_-(x)\right]\right\},$$

$$(\chi_2^2)_{\rm DF}=\xi_1(\psi_2^2)_{\rm DF}=\chi_2^1=\xi_1\psi_2^1=\kappa N\exp\left\{-i\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_+(\xi)+\beta\phi_+(x)\right]\right\}$$
$$\times\exp\left\{+i(\pi/2)^{1/2}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_-(\xi)+\phi_-(x)\right]\right\},$$

$$(\chi_1^1)_{\rm DF}=\xi_1(\psi_1^1)_{\rm DF}=\chi_1^2=\xi_1\psi_1^2=i\kappa N\exp\left\{-i\left[\alpha\int_{-\infty}^{x}d\xi\,\dot{\phi}_+(\xi)-\beta\phi_+(x)\right]\right\}$$
$$\times\exp\left\{-i(\pi/2)^{1/2}\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_-(\xi)-\phi_-(x)\right]\right\},$$

$$(\chi_2^1)_{\rm DF}=\xi_1(\psi_2^1)_{\rm DF}=\chi_2^2=\xi_1\psi_2^2=i\kappa N\exp\left\{-i\left[\int_{-\infty}^{x}d\xi\,\dot{\phi}_+(\xi)-\beta\phi_+(x)\right]\right\}$$
$$\times\exp\left\{+i(\pi/2)^{1/2}\left[\int_{-\infty}^{x}d\xi\,\phi_-(\xi)-\phi_-(x)\right]\right\}.$$

Here $\kappa=(C\mu/2\pi)^{1/2}$, and we have generalized (6.6), leaving two parameters α,β; this will allow the proper "spin." Following Sec. VI, we take the isospin currents as in (4.4) and (4.6). Indeed, we do not tamper with the ϕ_- (isospin) structure at all, so current algebra follows, with $C_1=1/2\pi$, $\tilde{C}_1=3/2\pi$. Because we maintain the $\psi^1\leftrightarrow\psi^2$ interchange of the text, $\delta=-1$, and the equation to be solved is still (6.7), with its companion remarks.

Anticommutativity of the Fermi fields requires $\alpha\beta/\pi=\frac{1}{2}+2j$ with j an integer ($j=0$ is the solution of the text). The baryon-number current (normalized to $a=1$) is easily seen to be $\tilde{J}_\mu = -\alpha^{-1}\epsilon_{\mu\nu}\partial^\nu\phi_+$; then $\bar{a} = -\beta\alpha^{-1}$ and $C_0 = \alpha^{-2}$. Differentiating $\psi(\Box^2\phi_\pm = 0)$, and comparing with (6.7) we identify $g_V = 4\pi/3$, $g_B = \alpha(\alpha+\beta)$. Finally, one calculates the spin [either directly via the Lorentz generator in terms of $\phi_\pm$, or just by substituting into DF(17)]. The result is $s = -\frac{1}{2}[2j+1]$. The solution of the text is $j=0$, $s=-\frac{1}{2}$; now we choose $j=-1$, $s=+\frac{1}{2}$. The final algebra is trivial, and we record ($s=+\frac{1}{2}$)

$$\bar{a} = \left(1 + \frac{2g_B}{3\pi}\right)^{-1}, \quad C_0 = \left(g_B + \frac{3\pi}{2}\right)^{-1},$$

$$\alpha^2 = g_B + \frac{3\pi}{2}, \quad \beta = -\frac{3\pi}{2\alpha},$$

thus completing the DF solution.

It is easy to add a Fermi mass term. Following the method of the text, we calculate $[(\bar{\psi}\psi)_{\rm DF} = \bar{\psi}\psi]$ the equivalent SGL system,

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi_+)^2 + \tfrac{1}{2}(\partial\phi_-)^2 - \frac{2C\mu m'}{\pi}\cos(2\beta\phi_+)\cos[(2\pi)^{1/2}\phi_-]$$

with $\beta = -(3\pi/2)(g_B + 3\pi/2)^{-1/2}$. The over-all sign of the interaction can be changed via a redefinition $\psi \to \gamma_5\psi$. Comparing with (5.8) we see that the boson systems for $\delta = \pm 1$ are the same, with the identification $g_B^- = 3\pi + 9g_B^+$ ($\pm$ are $\delta = \pm 1$). In this sense, the $\delta = \pm 1$ solutions are themselves equivalent. This does not, however, imply equality of particular Fermi Green's functions.

I wish to thank M. Kaku for bringing to my attention recent similar work on this model by Dashen and Frishman, and by Bhattacharya and Roy.

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APPENDIX A: THE SCALARS AS A (SPATIALLY) NONLOCAL REPRESENTATION OF ISOSPIN

As discussed in Sec. IV it is formally true that

$$[I_3, \phi_\pm] = [B, \phi_\pm] = [I_3, \dot{\phi}_\pm] = [B, \dot{\phi}_\pm] = 0, \quad [I_\pm, \phi_+] = [I_\pm, \dot{\phi}_+] = 0. \tag{A1}$$

Here, $I_\pm$ are isospin-raising and -lowering operators. Thus, although one may say that ϕ_+ is an isoscalar, the isospin transformation of ϕ_- cannot be simple. Indeed, as discussed in Sec. V, the quark isospin arises as a topological property of ϕ_-, so there can be no *direct* connection between quark isospin and ϕ_- transformation properties. Yet, there is a curious sense in which the scalar system provides an eldritch (spatially) nonlocal representation of isospin; I will now sketch how this goes.

Using Eqs. (4.6), it is not hard to show that

$$\begin{aligned}
&[J_\pm(\tau_\pm, x), \phi_-(y)] = \pm(2\pi)^{1/2}\theta(x-y)J_\pm(\tau_\pm, x),\\
&[J_+(\tau_\pm, x), \dot{\phi}_-(y)] = \mp(2\pi)^{1/2}\delta(x-y)J_+(\tau_\pm, x),\\
&[J_-(\tau_\pm, x), \dot{\phi}_-(y)] = \pm(2\pi)^{1/2}\delta(x-y)J_-(\tau_\pm, x),
\end{aligned} \tag{A2}$$

where the $\pm$ in front of $(2\pi)^{1/2}$ always goes with $\tau_\pm$. Note that the first equation of (A2) is nonlocal. Such commutators are easy to find in these constructions, even in the original U(1) case. I do not think any noncausality is implied. Defining $I_\pm = \int_{-\infty}^{+\infty} dx\,J_0(\tau_\pm, x)$, we obtain the isospin transformation properties of ϕ_-:

$$\begin{aligned}
&[I_\pm, \phi_-(x)] = \pm(2\pi)^{1/2}\int_x^\infty dx\,J_0(\tau_\pm, x),\\
&[I_\pm, \dot{\phi}_-(x)] = \mp(2\pi)^{1/2}J_1(\tau_\pm, x),
\end{aligned} \tag{A3}$$

The right-hand sides of (A3) are of course wild (spatially) nonlocal functions of ϕ_-. I have checked the Jacobi identities among (A1) and (A3) and they are satisfied. ϕ_- appears to be a legitimate representation.

Let us see how such transformations can be invariances of the SGL equations (5.5), and (5.8),

$$\Box^2\phi_+ = \frac{\alpha}{\beta_+}\sin(\beta_+\phi_+)\cos[(2\pi)^{1/2}\phi_-], \tag{A4a}$$

$$\Box^2\phi_- = \frac{\alpha(2\pi)^{1/2}}{\beta_+^{\,2}}\cos(\beta_+\phi_+)\sin[(2\pi)^{1/2}\phi_-]. \tag{A4b}$$

I will concentrate on the left-hand sides, and sketch the result for the right-hand sides. Define infinitesimal transformations $\delta_\pm\phi_\pm$, $\delta_\pm\dot{\phi}_\pm$ by the commutators (A1), (A3). Here $\delta_\pm$ means the change due to $I_\pm$. One shows directly from (A3) that $\delta_\pm\dot{\phi}_\pm = (d/dt)\delta_\pm\phi_\pm$, and thus

$$\delta_\pm\Box^2\phi_+ = 0, \tag{A5a}$$

$$\delta_\pm\Box^2\phi_- = \mp(2\pi)^{1/2}[\partial_0 J_1(\tau_\pm) - \partial_x J_0(\tau_\pm)]. \tag{A5b}$$

The right-hand side of (A5b) is proportional to the divergence of the charged axial-vector currents. *Using the equations of motion,* these vanish for the free theory and are proportional to $P(\tau_\pm)$ in the interacting theory. Thus, for the free theory, we already see that our transformation is an invariance. I find it intriguing that a single free massless Bose field can formally support an isospin in thus way. With interaction, we need also transform the right-hand sides of (A4). For brevity, I will ignore β_+, $(2\pi)^{1/2}$'s and normal-ordering in sketching the transformation for (A4b):

$$\cos\phi_+\delta_\pm \sin\phi_- \sim \cos\phi_+(x)\cos\phi_-(x)\int_x^\infty d\xi\, J_0(\tau_\pm,\xi)$$
$$\sim \int_{-\infty}^{+\infty} dy[\cos\phi_+(x)\sin\phi_-(x), J_0(\tau_\pm, y)]. \tag{A6}$$

The commutator is proportional [see Eqs. (4.9)] to $[P_3, J_0(\tau_\pm)]$, which is proportional to $P(\tau_\pm)$. This exactly parallels the change (A5b) in the left-hand side of (A4b), and again the transformation is an invariance. For the right-hand side of Eq. (A4a), a chain of identities parallel to (A6) leads to $[P, J_0(\tau_\pm)]\sim 0$, so this equation is also invariant.

Is this invariance an observable symmetry in, say, a perturbative approach to the Bose system? I do not think so; our manipulations, though formal, suggest why. Notice that $I_\pm$, though well defined with respect to the fermions, fail to annihilate the boson vacuum. In fact, $I_\pm$ are quite poorly defined with respect to that vacuum (infinite expectation value, for example.) From the point of view of the bosons then, the isospin is (something like) spontaneously broken; it is thus not observable in any ordinary sense until the soliton-fermions are obtained.

APPENDIX B: CONNECTIONS WITH DUAL MODELS

As I was going through these two-dimensional theories, and the relevant correspondences, I noticed that much of the field-theoretic work has a direct map onto past work in dual models. I will begin by discussing free-field connections, and return to interactions later. Consider a free SU(N) Fermi field in two dimensions. Introduce $u=t+x$, $v=t-x$; then $\psi_a^1(u), \psi_a^2(v)$ form two independent spaces. I will focus on $\psi_a^1(u)$ alone, remembering that the complete system is a doubling. The claim is that $\psi_a^1(u)$ is the Bardakci-Halpern[8] dual quark field[15] $\psi_a(\theta)$.

Towards this result, an appropriate map is the projective transformation

$$e^{i\theta}=\frac{u-i}{u+i}, \quad 0<\theta<2\pi, \quad -\infty<u<\infty. \tag{B1}$$

Useful identities are $d\theta/du = 2/(u^2+1)$, and

$$\delta(u-u')=2\sin^2(\tfrac{1}{2}\theta)\delta(\theta-\theta'),$$
$$\frac{\partial}{\partial u}\delta(u-u')=[2\sin^2(\tfrac{1}{2}\theta)][2\sin^2(\tfrac{1}{2}\theta')]\partial_\theta\delta(\theta-\theta'). \tag{B2}$$

Then it is easy to see from

$$[\psi_a^1(u), \psi_b^{1\dagger}(u')]_+=\delta_{ab}\delta(u-u'),$$
$$[\psi_a(\theta), \psi_b^\dagger(\theta')]_+=2\pi\delta_{ab}\delta(\theta-\theta') \tag{B3}$$

that

$$\frac{1}{\sqrt{\pi}}\sin(\tfrac{1}{2}\theta)\psi_a(\theta)=\psi_a^1(-u). \tag{B4}$$

Actually the $(-u)$ is only a convention to keep dual-model creation (and annihilation) operators in correspondence with field-theory creation (and annihilation) operators—with $(+u)$, they are anticorrelated. Let us see how this goes. In the dual model one expands [drop SU(N) labels, with immediate generalization]

$$\psi(\theta)=\sum_{n=1}^{\infty}(e^{i(n+1/2)\theta}b_{1/2+n}+e^{-i(n+1/2)\theta}d_{1/2+n}^\dagger), \tag{B5}$$

whereas

$$\psi^1(u)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{0}dp[d^\dagger(p)e^{ip_0u}+b(p)e^{-ip_0u}]. \tag{B6}$$

We can calculate, say $b_{n+1/2}$ in terms of the field-theoretic operators,

$$b_{1/2+n}=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, e^{-i\theta(n+1/2)}\psi(\theta)$$
$$=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}du\frac{2}{1+u^2}(1+u^2)^{1/2}\left(\frac{u+i}{u-i}\right)^{n+1/2}\times\psi^1(-u)$$
$$=\frac{1}{\pi\sqrt{2}}\int_{-\infty}^{0}b(p)\int_{-\infty}^{+\infty}\frac{du}{u-i}\left(\frac{u+i}{u-i}\right)^n e^{ip_0u}$$
$$=\sqrt{2}\,i\int_{-\infty}^{0}dp\, b(p)e^{-p_0}L_n(2p_0). \tag{B7}$$

In the last step, we have recognized the Laguerre polynomials $L_n(Z)=e^Z d^n/dZ^n(Z^n e^{-Z})$. A similar calculation gives

$$b_{n+1/2}^\dagger=-i\sqrt{2}\int_{-\infty}^{0}dp\, b^\dagger(p)e^{-p_0}L_n(2p_0).$$

With the usual orthonormality properties of L_n, one easily verifies $[b_{n+1/2}, b_{m+1/2}^\dagger]_+=\delta_{n,m}$ from $[b(p), b^\dagger(\kappa)]_+=\delta(p-\kappa)$. Evidently, 2 conformal spin $\frac{1}{2}$ dual quarks correspond to one Lorentz spinor.

We can go much further. We turn for a moment to scalar fields. It is known that if $\Box^2\phi=0$, then $\phi=f(u)+g(v)$, and, for example,

$$f(u)=\int_{-\infty}^{0}\frac{dk}{(2\pi 2k_0)^{1/2}}[e^{-iuk_0}a(k)+e^{+iuk_0}a^\dagger(k)], \tag{B8a}$$

$$[\partial_u f(u), f(u')]=-\frac{i}{2}\delta(u-u'). \tag{B8b}$$

Comparing with the dual model (fifth dimension, no four-vector index) conformal scalar-vector system,[15]

$$[Q_5(\theta), \pi_5(\theta')] = -2\pi\delta(\theta-\theta'), \qquad \pi_5(\theta) = i\partial_\theta Q_5(\theta) \tag{B9}$$

we identify

$$Q_5(\theta) = -i2\sqrt{\pi} f(-u), \qquad \pi_5(\theta) = \frac{\sqrt{\pi}\,\partial_{-u} f(-u)}{\sin^2(\frac{1}{2}\theta)}. \tag{B10}$$

The troubles in normal ordering the exponential of a free massless scalar field are thus the same as those of the dual vertex. In dual models, it was known[15] how to construct a $\pi_5(\theta)$ from the quark field,

$$i\partial_\theta Q_5 = \pi_5(\theta) = :\psi^\dagger(\theta)\psi(\theta):. \tag{B11}$$

This dual identity is then the familiar field-theoretic statement that vector currents can be written as gradients of free massless scalar fields. Something that was *not* realized in dual models is the analog of Mandelstam's representation in the field theory. Using that fact that for free fields $\int_{-\infty}^{x} d\xi\, \dot{\phi}(\xi) = f(u) - g(v)$, we easily show the inverse of (B11), namely that

$$\psi(\theta) \sim :e^{iQ_5(\theta)}: \sim :e^{\sqrt{2}k_5 Q_5(\theta)}: \tag{B12}$$

when $k_5^2 = -\frac{1}{2}$. That is, the spinor field can be expressed in terms of the scalar when k_5 is chosen so that the exponential is a conformal spinor.

We can go still further. The field-theoretic identity[5,6] [back to SU(N), $\theta_\pm = \theta_{00} \pm \theta_{01}$]

$$\theta_+(u) = i\psi^{1\dagger}\overleftrightarrow{\partial}_u\psi^1 = :\frac{1}{2\tilde{C}_V}(J_+^\alpha J_+^\alpha) + \frac{1}{2C_B}J_+J_+: \tag{B13}$$

[$\tilde{C}_V = (n+1)/2\pi$, $C_B = n/\pi$] maps directly onto[15]

$$\mathcal{L}(\theta) = -\frac{i}{2}\psi^+\overleftrightarrow{\partial}_\theta\psi = :\frac{1}{n+1}J^\alpha(\theta)J^\alpha(\theta) + \frac{1}{2n}J(\theta)J(\theta):. \tag{B14}$$

Here the dual quantities need some comment for nonexperts. $\mathcal{L}(\theta)$ is the conformal density, i.e., the object from which we build the conformal algebra $L_m = (1/2\pi)\int_0^{2\pi} e^{-im\theta}\mathcal{L}(\theta)$. The currents $J(\theta), J^\alpha(\theta)$ are formed as

$$J^\alpha(\theta) = :\psi^\dagger\frac{\lambda^\alpha}{2}\psi:, \qquad J(\theta) = :\psi^\dagger\psi: \tag{B15}$$

and satisfy

$$[J^\alpha(\theta), J^\beta(\theta')] = 2\pi i f^{\alpha\beta\gamma}J^\alpha(\theta)\delta(\theta-\theta') - \pi i\delta^{\alpha\beta}\partial_\theta\delta(\theta-\theta'), \tag{B16}$$

$$[J(\theta), J(\theta')] = -2\pi i n\partial_\theta\delta(\theta-\theta').$$

Thus, comparing with Refs. 5 and 6, we identify

$$J_+^\alpha(-u) = \frac{2}{\pi}\sin^2(\tfrac{1}{2}\theta)J^\alpha(\theta), \qquad J_+(-u) = \frac{2}{\pi}\sin^2(\tfrac{1}{2}\theta)J(\theta). \tag{B17}$$

Identity (B14) appears explicitly in Ref. 15 for the case of SU(3). (Caution: Here I have normalized the currents with $\frac{1}{2}\lambda^\alpha$. In Ref. 15, they are λ^α.) It is clear then that the dual method of defining normal-ordered quartics $[(J)^2]$ is the same as the field-theoretic method.

Going on toward interaction, we note some more general correspondences. The fact that {$\psi(\theta)$ is a conformal spinor} ⟷ {$\psi(x,t)$ is a Lorentz spinor}. The fact that {$\pi(\theta)$ is a conformal vector} ⟷ {$\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector}, and so on. The most important of these is the fact that under the correspondence

$$\theta_+(-u) = \frac{4}{\pi}\sin^4(\tfrac{1}{2}\theta)\mathcal{L}(\theta) \tag{B18}$$

then, the "conformal-Schwinger condition" [Eq. (11) of Ref. 6] is equivalent to requiring that $\mathcal{L}$ is indeed a conformal density,

$$(\mathcal{L}(\theta), \mathcal{L}(\theta')) = -2\pi i[\partial_\theta\mathcal{L}(\theta)\delta(\theta-\theta') + 2\mathcal{L}(\theta)\partial_\theta\delta(\theta-\theta')]. \tag{B19}$$

I believe these correspondences provide the opportunity for further flow between the two fields. E.g., any conformally invariant two-dimensional field theory [hence a $\theta_+(u)$] provides us with the conformal algebra of a dual model: Just construct $\mathcal{L}(\theta)$ via (B18). Conversely, any dual $\mathcal{L}(\theta)$ provides us with a θ_+. From the dual-model viewpoint, the Dashen-Frishman phenomenon corresponds to reversing the sign of the Schwinger term in $J^\alpha(\theta)$. This corresponds to taking a $b|0\rangle = 0$ vacuum for J, but a $b^+|0\rangle = 0$ vacuum for J^α. Hence the difficulty in constructing a proper ψ. Much work has been done on $\mathcal{L}$ in dual models, most of which has involved four-Lorentz indices which would be superimposed above the two-Lorentz indices of the field theories. One application for which this feature will not appear is *discretely broken* SU(N). Such an application has been discussed[15] in the dual models, and hence one may be able to do the same in a conformal field theory. Quantized couplings occur all the time in dual $\mathcal{L}$'s, and, in general, I believe this is the same as the

Dashen-Frishman phenomenon: Conformal invariance quantizes couplings.

In general, however, because there is a doubling of two dual operators to one "Thirring operator," or, more simply, a u and v, the field theories correspond more closely to *Virasoro-Shapiro models*. Indeed it takes only a moment to show that the Thirring model Green's functions [for $\bar{\psi}(1 \pm \gamma_5)\psi$ at $\beta^2 = 8\pi$] are the Virasoro-Shapiro N-point functions, with a fifth dimension to bring the ground state up to $k^\mu k_\mu = 0$, all evaluated at $k_\mu = 0$. The sum of these functions, $Z(J)$ for the massive Thirring model, is then the function $W(J)$ useful in the dual-model spontaneous-breakdown approach of Bardakci and Halpern.[16]

I finally note that the work of the present paper is equivalent to constructing an SU(N) out of dual fifth-, sixth- (etc.) dimension orbital operators, even though the orbital operators have no simple transformation properties under SU(N). In the dual model, of course, $N \leq 22$ for orbital models and $N \leq 6$ for models with spin.

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[7]See also Appendix B.

[8]The formal procedure here is the same in four dimensions. However, in two dimensions, the Schwinger terms are finite and calculable. For such reasons, a two-dimensional interaction picture has greater reliability than a four-dimensional one.

[9]It is a curious fact that, with our methods, we calculate zero anomalous dimension for ψ_H: Using as dilation operator the standard form $D = \int dx (t\theta_{00} + x\theta_{01})$, and assuming no anomalous dimension in the IP, the result follows immediately because $\theta^I_{01} = 0$. I checked through Refs. 5 and 6, and discovered that their generators are slightly different. Translating back from their (u, v) language, these references use $D \mp M = \int dx\, x\theta_\mp(0, \mp x)$ and $H \mp P = \int dx\, \theta_\mp (0, \mp x)$, whereas "standard" forms are $D \mp M = \mp \int dx\, x\theta_\mp (0, x)$ and $H \mp P = \int dx \theta_\mp (0, x)$. There are evidently certain advantages in these unconventional forms. (Caution: These are *only* for conformal models, whereas the "standard" forms work more generally.) Notice that with these unconventional forms, P has an interacting part $H_I(t) = \frac{1}{2}\int dx[\theta^I_{00}(x) + \theta^I_{00}(-x)]$, $P_I(t) = \frac{1}{2}\int dx[\theta^I_{00}(x) - \theta^I_{00}(-x)]$. The path is clear to the intriguing development of a "conformal interaction picture" in both H_I, P_I.

[10]The "current" identities among (3.1) are extremely well known. See also Appendix B. According to J. Kogut and L. Susskind, Phys. Rev. D 11, 3594 (1975), the others were known as well.

[11]See the remarks about $\mu \to 0$ of Sec. III. I have also suppressed the subscripts D on all the free fields of Sec. III.

[12]Except of course (4.7). It is not hard to show by direct differentiation that the interacting currents have the expected divergences. As in (4.7), however, the charged and neutral calculations go different routes, the charged calculation being somewhat tricky. For example, one finds

$$\partial \cdot J_5(\tau_+) \sim N' \left[\int_{-\infty}^{x} d\xi \cos[(2\pi G)^{1/2}\phi_+(\xi)] \times \sin[(2\pi)^{1/2}\phi_-(\xi)] J_1(\tau_+, x) \right]$$
$$\sim \int_{-\infty}^{+\infty} d\xi [S(\xi), J_1(\tau_+, x)]$$
$$\sim P(\tau_+).$$

For $\partial \cdot J(\tau_+)$, $J_1 \to J_0$, and the vector currents are still conserved.

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