Canonical quantization of nonlinear waves*

E. Tomboulis
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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By means of a canonical transformation we investigate nonlinear field theories that possess exact classical solutions. This transformation is equivalent to the method of collective coordinates recently applied to the same problem by Gervais and Sakita using functional techniques. It is pointed out that, because of the operator orderings, extra terms occur in the quantized theory which seem to be absent in the straightforward functional approach. Ordinary perturbation treatment of the resulting Hamiltonian reproduces results previously obtained by Goldstone and Jackiw using a different technique.

I. INTRODUCTION

Certain classical field theories have been known for a long time to possess exact nonlinear wave solutions. There is a large body of literature concerning these solitary waves (kinks) and their frequent occurrence in applied science.[1] They are of interest to particle physicists because they are believed to represent new states in the spectrum of the corresponding quantum field. These states might possess some of the features of extended hadrons.

The occurrence and interpretation of such solutions in various models have been examined by various authors in both the classical[2] and the quantum theories.[3-6] In particular, Goldstone and Jackiw[5] have interpreted the classical solution as the Fourier transform of the quantum field form factor of a baryon in a static limit, and have developed a complete calculational scheme in the one-baryon sector. They used a method based on the quantum equations of motion which is similar to techniques applied by Kerman and Klein to many-body problems.

A quite different approach to the problem was recently advocated by Gervais and Sakita.[7] These authors utilized the method of collective coordinates[8] for the functional quantization of the solitary solution in a two-dimensional field theory. In this approach the field $\Phi(x, t)$ is separated into two parts, ϕ_c and χ. ϕ_c is the classical static solution which is parametrized by the parameter X, the kink position. X is then promoted to a time-dependent dynamical variable. χ represents the meson part of Φ. The Hamiltonian can be expressed as the sum of a free meson and a kink (baryon) part plus the meson self-interaction and the kink-meson interaction part.

In this note we show that this development amounts to a canonical transformation of the original theory and give the corresponding classical and quantum Hamiltonian formalisms. In the quantum version we find that, as a result of the operator ordering, some terms of order $\hbar^2$ arise in the energy-momentum tensor, so that it is not given simply by the symmetrized classical expression. These terms are necessary for the Lorentz invariance of the theory. It is not obvious how they should be obtained in the functional approach.

Finally, ordinary perturbation theory can be used to obtain a systematic expansion in the coupling constant. In particular, the basic *Ansätze* and results of the calculational scheme developed in Ref. 5 are derived from the perturbation series.

II. THE CANONICAL TRANSFORMATION

We consider the class of field theories in one space and one time dimension described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - U(\Phi), \tag{1}$$

and which possess exact space-dependent solutions of the form $\phi_c(x-X)$. Here X is a parameter. These solutions are the well-known kinks or solitons and are believed to describe some stable particle.

To investigate these solutions further we introduce the transformation

$$\Phi(x, t) = \phi_c(x-X) + \chi(x-X, t). \tag{2}$$

We also decompose the canonical momentum $\Pi_0(x, t)$ of the theory described by (1) in the form

$$\Pi_0(x, t) = \pi(x-X, t) - \frac{p + \int \pi\chi'}{M_0(1+\xi/M_0)}\, \phi_c{}'(x-X), \tag{3}$$

where $\xi \equiv \int \chi' \phi_c{}'$ and $M_0 \equiv \int \phi_c{}'\phi_c{}'$. A prime denotes differentiation with respect to x. Unless we indicate otherwise, all integral signs denote one-dimensional unrestricted integrations over x where, for notational simplicity, the differential dx is suppressed. X is now considered as a new dynamical variable $X(t)$ conjugate to the new momentum variable $p(t)$. However, since we started with only two variables Φ and Π_0, conservation of degrees of freedom demands that the new variables be considered as functionals of Φ. This is achieved by imposing the conditions

$$\psi_1 = \int \chi(x,t)\phi_c'(x) = 0\,, \tag{4a}$$

$$\psi_2 = \int \pi(x,t)\phi_c'(x) = 0\,. \tag{4b}$$

The transformation (2) and (3) subject to the conditions (4) will be shown to be a canonical transformation.

The old variables Φ and Π_0 have the usual canonical Poisson brackets,

$$\begin{aligned}&\{\Phi(x,t),\ \Phi(y,t)\} = \{\Pi_0(x,t),\Pi_0(y,t)\} = 0\,,\\ &\{\Phi(x,t),\ \Pi_0(y,t)\} = \delta(x-y)\,.\end{aligned} \tag{5}$$

Consider now the new set χ, π, X, p. Let us, for the time being, assume canonical brackets for χ and π:

$$\{\chi(x,t),\ \pi(y,t)\} = \delta(x-y)\,.$$

Then the conditions (4) do not commute; they are in Dirac's terminology[9] second-class constraints. Indeed the matrix of the brackets of ψ_1 and ψ_2 does not vanish weakly:

$$(\{\psi_i,\psi_j\}) = \begin{pmatrix} 0 & M_0 \\ -M_0 & 0 \end{pmatrix}, \quad i,j = 1,2\,. \tag{6}$$

Constraints of this kind can be made strong equations by a modification of the conventional bracket. According to the well-known Hamiltonian formalism for constrained systems,[9] the new canonical bracket is

$$\{\chi(x,t),\pi(y,t)\} = \delta(x-y) - \{\chi(x,t),\psi_i\}(\{\psi_i,\psi_j\})^{-1}\{\psi_j,\pi(y,t)\}$$

or

$$\{\chi(x,t),\pi(y,t)\} = \delta(x-y) - \frac{1}{M_0}\,\phi_c'(x)\phi_c'(y)\,. \tag{7}$$

We also take

$$\{X,p\} = 1\,, \tag{8}$$

with all remaining Poisson brackets vanishing.

If we now express Φ and Π_0 on the left-hand side of Eq. (5) in terms of χ, π, X, and p we can show by straightforward computation using the brackets (7) and (8) as well as the constraints (4) that the right-hand sides of Eq. (5) are reproduced.

The components of the energy-momentum tensor in terms of the new variables are

$$\begin{aligned}T_{00} &= \tfrac{1}{2}\Pi_0^2(x,t) + \tfrac{1}{2}\Phi'^2(x,t) + U(\Phi)\\ &= \tfrac{1}{2}\pi^2(x-X,t) - \pi(x-X,t)\frac{p+\int \pi\chi'}{M_0(1+\xi/M_0)}\,\phi_c'(x-X)\\ &\quad + \frac{1}{2}\,\frac{(p+\int\pi\chi')^2}{M_0^2(1+\xi/M_0)^2}\,\phi_c'^2(x-X) + \tfrac{1}{2}\chi'^2(x-X,t) + \tfrac{1}{2}\phi_c'^2(x-X) + \chi'(x-X,t)\phi_c'(x-X) + U(\phi_c+\chi)\,,\end{aligned} \tag{9a}$$

$$T_{11} = \tfrac{1}{2}\Pi_0^2(x,t) + \tfrac{1}{2}\Phi'^2(x,t) - U(\Phi) = T_{00} - 2U(\phi_c+\chi)\,, \tag{9b}$$

$$\begin{aligned}T_{01} &= \Pi_0(x,t)\Phi'(x,t)\\ &= \pi(x-X,t)[\phi_c'(x-X) + \chi'(x-X,t)] - \frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)}[\phi_c'(x-X)\chi'(x-X,t) + \phi_c'^2(x-X)]\,.\end{aligned} \tag{9c}$$

The Hamiltonian is then given by

$$H = \int T_{00} = M_0 + \frac{1}{2M_0}\,\frac{(p+\int\pi\chi')^2}{(1+\xi/M_0)^2} + \int \mathcal{H}_f\,, \tag{10}$$

where

$$\mathcal{H}_f(x,t) = \tfrac{1}{2}\pi^2(x,t) + \tfrac{1}{2}\chi'^2(x,t) + V(\chi,\phi_c)\,, \tag{11}$$

$$V(\chi,\phi_c) = U(\phi_c+\chi) - \chi(x,t)U'(\phi_c) - U(\phi_c)\,, \tag{12}$$

and primes denote differentiation of U with respect to the argument. In obtaining (10) we used (4), the fact that ϕ_c satisfies the classical static equations

$$\phi_c'' = U'(\phi_c), \tag{13}$$

$$\tfrac{1}{2}(\phi_c')^2 = U(\phi_c), \tag{14}$$

and shifted the variable of integration $x \to x+X$. Equation (10) is the result of Gervais and Sakita.[7]

Similarly the total momentum of the system is found to be

$$P=\int T^{01}=p; \tag{15}$$

X, the variable conjugate to p, can then be viewed as a center-of-mass coordinate.

The equations of motion for χ, π, X, and p are

$$\dot{\chi}(x,t)=\{\chi(x,t),H\}$$
$$=\pi(x,t)+\frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)^2}\left[\chi'(x,t)-\frac{1}{M_0}\xi\phi_c'(x)\right], \tag{16a}$$

$$\dot{\pi}(x,t)=\{\pi(x,t),H\}$$
$$=\frac{p+\int\chi'\pi}{M_0(1+\xi/M_0)^2}\left[\pi'(x,t)+\frac{1}{M_0}\phi_c'(x)\int\pi\phi_c''-\frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)}\phi_c''(x)\right]$$
$$+\chi''(x,t)-V'(\chi,\phi_c)+\frac{1}{M_0}\phi_c'(x)\left(\int\chi'\phi_c''+\int V'\phi_c'\right), \tag{16b}$$

where

$$V'(\chi,\phi_c)\equiv\frac{\partial V(\chi,\phi_c)}{\partial\chi(x,t)},$$

and

$$\dot{X}=\{X,H\}=\frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)^2},\quad \dot{p}=\{p,H\}=0. \tag{16c}$$

One can verify that Eqs. (16), together with (2) and (3), are equivalent to the equations of motion of the original variables Φ and Π_0 that are derived from (1).

Finally the Lorentz boost generator L at $t=0$, obtained by integrating xT_{00} over x and shifting, is given by

$$L=\int xT_{00}=XH+\left[1+\frac{(p+\int\chi'\pi)^2}{2M_0^2(1+\xi/M_0)^2}\right]\int x\phi_c'\phi_c'-\frac{p+\int\pi\chi'}{M_0(1+\xi/M_0)}\int x\pi\phi_c'+\int x\mathcal{H}_f. \tag{17}$$

In explicit examples the integral $\int x\phi_c'\phi_c'$ vanishes by symmetry. However, we do not use this fact.

Lorentz invariance is shown by proving that H, P, and L form the Poincaré algebra,

$$\{H,p\}=0, \tag{18a}$$

$$\{L,p\}=H, \tag{18b}$$

$$\{L,H\}=P. \tag{18c}$$

The relations (18a) and (18b) are trivial. Equation (18c) follows after a somewhat lengthy calculation.

III. QUANTIZATION

We now proceed to quantize the theory by postulating the usual correspondence between classical brackets and commutators. The only nonvanishing ones are

$$i[\pi(x,t),\chi(y,t)]=\delta(x-y)-\frac{1}{M_0}\phi_c'(x)\phi_c'(y), \tag{19}$$

$$i[p,X]=1. \tag{20}$$

We must also order the noncommuting factors in the canonical transformation (2) and (3). We take

$$\Phi(x,t)=\phi_c(x-X)+\chi(x-X,t), \tag{21}$$

$$\Pi_0(x,t)=\pi(x-X,t)-\frac{1}{2M_0}\left[\phi_c'(x-X)\frac{1}{1+\xi/M_0}\left(p+\int\chi'\pi\right)+\left(p+\int\pi\chi'\right)\frac{1}{1+\xi/M_0}\phi_c'(x-X)\right]. \tag{22}$$

A straightforward calculation shows that, just as in the classical case, Eqs. (19) and (20) ensure that Φ and Π_0 satisfy conventional commutation relations.

We now proceed to evaluate Π_0^2 as a preliminary step in obtaining the quantum expression for $T_{\mu\nu}$. Defining

$$a \equiv \left[p + \int \pi\chi'\right]\frac{1}{1+\xi/M_0}, \quad a^\dagger \equiv \frac{1}{1+\xi/M_0}\left[p + \int \chi'\pi\right], \quad A \equiv \tfrac{1}{2}(a+a^\dagger), \tag{23}$$

we have

$$\Pi_0^2 = \pi^2 - \frac{1}{M_0}[\pi\phi_c', A]_+ + \frac{1}{2M_0^2}[\phi_c'^2, A^2]_+ - \frac{i}{2M_0}\left[\pi, \frac{1}{1+\xi/M_0}\right]\phi_c''$$
$$+ \frac{1}{4M_0^2}\left\{\tfrac{1}{2}[a^2, \phi_c'^2] + \tfrac{1}{2}[\phi_c'^2, a^{\dagger 2}] - 2ia\frac{\phi_c''\phi_c'}{1+\xi/M_0} + 2i\frac{\phi_c''\phi_c'}{1+\xi/M_0}a^\dagger + \frac{i}{2}[\phi_c', a^\dagger]\frac{\phi_c''}{1+\xi/M_0} - \frac{i}{2}\frac{\phi_c''}{1+\xi/M_0}[a, \phi_c']\right\},$$

where $[\ ,\ ]_+$ denotes an anticommutator, and we reorderd all factors in the expression that results from squaring (22) so that a and $a^\dagger$ are the rightmost or leftmost factor in each term. Evaluating the commutators we finally obtain

$$\Pi_0^2(x,t) = \pi^2(x-X,t) - \frac{1}{M_0}[\pi(x-X,t)\phi_c'(x-X), A]_+ + \frac{1}{2M_0^2}[\phi_c'^2(x-X), A^2]_+ - \frac{1}{4M_0^2}\frac{\phi_c''^2(x-X)}{(1+\xi/M_0)^2}$$
$$- \frac{1}{4M_0^3(1+\xi/M_0)^3}\left(\frac{d}{dx}\phi_c'^2(x-X)\right)\int\chi'\phi_c'' + \frac{1}{4M_0^2}\frac{1}{(1+\xi/M_0)^2}\frac{d^2}{dx^2}\phi_c'^2(x-X). \tag{24}$$

In evaluating integrated expressions in the classical theory we shifted the variable of integration $x \to x+X$. In the quantum case we can do this only if every factor under the integral sign commutes with X. It is for this purpose that we brought Π_0^2 in the form (24), the last three terms of which give quantum contributions absent in the classical expression. Using (24) and proceeding as before we obtain

$$H = \int T_{00} = M_0 + \frac{1}{2M_0}A^2 + \int \mathcal{H}_f - \frac{1}{8M_0^2}\frac{\int\phi_c''\phi_c''}{(1+\xi/M_0)^2}, \tag{25}$$

$$P = -\int T_{01} = p, \tag{26}$$

$$L = \int xT_{00}$$
$$= \tfrac{1}{2}[X, H]_+ + \left(1 + \frac{1}{2M_0^2}A^2\right)\int x\phi_c'\phi_c' - \frac{1}{2M_0}\left[A, \int x\pi\phi_c'\right]_+ + \int x\mathcal{H}_f - \frac{1}{8M_0^2}\frac{\int x\phi_c''\phi_c''}{(1+\xi/M_0)^2}$$
$$+ \frac{1}{8M_0^2}\frac{1}{(1+\xi/M_0)^3}\int\chi'\phi_c''. \tag{27}$$

Comparing with the classical expressions we note that apart from symmetrizations the last term in (25) and the last two terms in (27) are the new quantum contributions.

The equations of motion now are

$$\dot\chi(x,t) = i[H, \chi(x,t)]$$
$$= \pi(x,t) + \frac{1}{2M_0}\left[\frac{1}{1+\xi/M_0}\left(\chi'(x,t) - \frac{1}{M_0}\xi\phi_c'(x)\right), A\right]_+, \tag{28a}$$

$$\dot\pi(x,t) = i[H, \pi(x,t)]$$
$$= -\frac{1}{4M_0^2}\left[\left[\frac{1}{1+\xi/M_0}\phi_c''(x), A\right]_+, A\right]_+ + \frac{1}{4M_0}\left[\left[\frac{1}{1+\xi/M_0}, \left(\pi'(x,t) + \frac{1}{M_0}\phi_c'(x)\int\pi\phi_c''\right)\right]_+, A\right]_+$$
$$+ \chi''(x,t) - V'(\chi, \phi_c) + \frac{1}{M_0}\phi_c'(x)\left(\int\chi'\phi_c'' + \int V'\phi_c'\right) + \frac{1}{4M_0^3}\int\phi_c''\phi_c''\frac{1}{(1+\xi/M_0)^3}\phi_c''(x), \tag{28b}$$

$$\dot{X}=i[H,X]=\frac{1}{2M_0}\left[\frac{1}{1+\xi/M_0},A\right]_+ . \tag{28c}$$

By a slightly tedious calculation one can verify that Eqs. (28), together with (21) and (22), are equivalent to the quantum equations of motion for the original variables Φ and Π_0. The presence of the last term in (28b), which is absent in the classical expression, is important in this connection.

Again it follows trivially that

$$i[p,H]=0$$

and

$$i[p,L]=H .$$

The proof of the relation

$$i[H,L]=P ,$$

which establishes Lorentz invariance, is very lengthy and will not be reproduced here. The existence of the new quantum terms in H and L is necessary for obtaining the result.

IV. PERTURBATION THEORY

The Hamiltonian (25) can be separated into a free and an interacting part

$$H=H_0+H_I , \tag{29a}$$

$$H_0=M_0+\int\left[\tfrac{1}{2}\pi^2(x,t)+\tfrac{1}{2}\chi'^2(x,t)+\tfrac{1}{2}\chi^2(x,t)U''(\phi_c)\right] . \tag{29b}$$

H_0 is the Hamiltonian of a static particle of mass M_0 plus that of the free meson field χ. The quantization of the free χ is performed in the standard fashion by expanding

$$\chi(x,t)=\sum_k \frac{1}{(2\omega_k)^{1/2}}\left[b_k\psi_k(x)e^{-i\omega_k t}+b_k^\dagger\psi_k^\dagger(x)e^{i\omega_k t}\right] \tag{30a}$$

and

$$\pi(x,t)=\sum_k(-i)\left(\frac{\omega_k}{2}\right)^{1/2}\left[b_k\psi_k(x)e^{-i\omega_k t}-b_k^\dagger\psi_k^\dagger(x)e^{i\omega_k t}\right] , \tag{30b}$$

where $\omega_k=(k^2+\mu^2)^{1/2}$, with μ the meson mass. The expansion is in terms of the complete set of solutions of the free field equation

$$-\chi''+U''(\phi_c)\chi=\omega^2\chi . \tag{31}$$

This set includes the zero-frequency mode $\psi_0 =(1/\sqrt{M_0})\phi_c'$ (see Ref. 5) which must be excluded from (30) because of (4). Therefore, the expansion (30) is made in the set $\{\psi_k\}$ with the completeness property

$$\sum_k \psi_k(x)\psi_k^\dagger(y)=\delta(x-y)-\frac{1}{M_0}\phi_c'(x)\phi_c'(y) . \tag{32}$$

Furthermore, there are two possible sets, $\{\psi_k^{\text{in}}\}$ and $\{\psi_k^{\text{out}}\}$, corresponding to "in" and "out" states as determined by the appropriate boundary conditions for (31). We can work in terms of either of these two and in the following we shall not exhibit the "in" or "out" label explicitly. The commutation relations (19) are then satisfied by

$$[b_k,b_{k'}^\dagger]=\delta_{kk'} ,\qquad [b_k,b_{k'}]=[b_k^\dagger,b_{k'}^\dagger]=0 . \tag{33}$$

One can set up a Hilbert space of baryon-multimeson states $|P',\{k_i\}\rangle$ characterized by a total momentum P' and a set of meson moments $\{k_i\}$. There is no difference between "in" and "out" for the no-meson state which we call the baryon state.

We can now do ordinary perturbation theory by going into the Schrödinger picture where we use the expansion (30) for the full operators χ and π with $t=0$. Though, of course, the development is completely general, it will be convenient in the following to refer to a specific model. We take the Φ^4 model examined in Refs. 4, 5, and 7. Then the potential is given by

$$U(\Phi)=\frac{m^4}{2\lambda}-m^2\Phi^2+\tfrac{1}{2}\lambda\Phi^4 , \tag{34}$$

which breaks the symmetry $\Phi\to-\Phi$ in the vacuum states $\Phi_{1,2}=\pm m/\lambda^{1/2}$. The classical solution is

$$\phi_c(x-X)=\frac{m}{\lambda^{1/2}}\tanh m(x-X) , \tag{35}$$

with $\phi_c(x)\to\Phi_{1,2}$ for $x\to\mp\infty$. The corresponding classical energy

$$M_0=\int\phi_c'^2=\frac{4}{3}\frac{m^3}{\lambda} \tag{36}$$

is of order λ^{-1} and the meson mass equals $2m$. The interaction Hamiltonian in the theory (34) is

$$\begin{aligned}H_I&=2\lambda\int\phi_c\chi^3+\tfrac{1}{2}\lambda\int\chi^4\\&\quad+\frac{1}{2M_0}\left[\left(p+\int\pi\chi'\right),\frac{1}{2(1+\xi/M_0)}\right]_+^2\\&\quad-\frac{1}{8M_0^2}\frac{1}{(1+\xi/M_0)^2}\int\phi_c''\phi_c''\\&\equiv H_{I1}+H_{I2}+H_{I3}+H_{I4} .\end{aligned} \tag{37}$$

H_{I1} is of order $\lambda^{1/2}$; the other three terms in (37) are of order λ.

The baryon energy $E(P)$ to zeroth order, i.e., the matrix element of H_0 between no-meson states, is easily found to be

$$\langle P|H_0|P'\rangle = E_0(P)\delta(P-P') ,$$
$$E_0(P) = M_0 + \tfrac{1}{4}\int\sum_k 2\omega_k{}' \psi_k^\dagger(x)\psi_k(x) , \tag{38}$$

where we used (31) to simplify the expression. This is the result given in Refs. 3–5. It is the sum of the classical energy plus the zero-point energy of the meson field. In the explicit evaluation of (38) we have, after a vacuum energy subtraction, a remaining ultraviolet divergence which can be removed by meson mass renormalization.[4] This renormalization is actually sufficient to remove all divergences from the theory.

The next correction to $E_0(P)$, to which all four terms in H_I contribute, is obviously of order λ and includes the first kinematic corrections to the static baryon.

To lowest order the matrix element of $\Phi(x)$ between baryon states is

$$\begin{aligned}\langle P'|\Phi(x)|P\rangle &= \langle P'|\phi_c(x-X)|P\rangle \\ &= \int \langle P'|X'\rangle\langle X'|\phi_c(x-X)|X''\rangle\langle X''|P\rangle \\ &\quad \times dX'\,dX'' \\ &= \frac{1}{2\pi}\int e^{i(P-P')X'}\phi_c(x-X')\,dX' . \end{aligned}\tag{39}$$

The classical solution is then the leading term in the expansion of the Fourier transform of the form factor which is the Ansatz used in Ref. 5. The next correction is of order $\lambda^{1/2}$ and is given by

$$\begin{aligned}\langle P'|\Phi(x)|P\rangle^{(1)} &= \sum_m \langle P'|\Phi(x)|m\rangle \frac{\langle m|H_{I1}|P\rangle}{E_0-E_m} + \sum_m \frac{\langle P'|H_{I1}|m\rangle}{E_0-E_m}\langle m|\Phi(x)|P\rangle \\ &= -\int dX'\, e^{i(P-P')X'}\int dy\, 6\lambda\,\phi_c(y)\sum_{kk'}\left\{\frac{1}{2\omega_k{}^2}\left[\psi_k(x-X')\psi_k^\dagger(y)+\psi_k(y)\psi_k^\dagger(x-X')\right]\frac{1}{2\omega_{k'}}\psi_{k'}(y)\psi_{k'}^\dagger(y)\right\} ,\end{aligned}\tag{40}$$

which again agrees with the result found in Ref. 5. In the computational scheme developed in Ref. 5 the main object of interest is the connected matrix element of $\Phi(x)$ between m and n ("in" or "out") mesons. The no-meson to one-meson matrix element to lowest order is clearly given by

$$\langle P'|\Phi(x)|P,k\rangle = \frac{1}{2\pi}\int e^{i(P-P')X'}\frac{1}{(2\omega_k)^{1/2}}\psi_k(x-X')\,dX' . \tag{41}$$

For the general connected m-n element it is not hard to show from the perturbation series by a counting argument that the leading order in its expansion in powers of λ is $\lambda^{(m+n-1)/2}$. This is the essential assumption in Ref. 5.

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