PHYSICAL REVIEW D VOLUME 12, NUMBER 6 15 SEPTEMBER 1975

Quantum strings and the functional calculus

J. F. L. Hopkinson
Daresbury Laboratory, Daresbury, Warrington WA4 4AD, England

R. W. Tucker
University of Lancaster, Bailrigg, Lancaster LA1 4YB, England and Daresbury Laboratory, Daresbury, Warrington WA4 4AD, England

P. A. Collins
University of Lancaster, Bailrigg, Lancaster LA1 4YB, England
(Received 10 February 1975)

The field theory of strings is developed using the functional calculus as a practical technique. A Schrödinger-type equation and its eigensolutions are derived in light-cone coordinates. Vertices and amplitudes are represented entirely in a Fock space of second-quantized functionals. Two different open-string interactions are explicitly investigated. The three- and four-string vertex functionals are calculated for arbitrary times and compared with the asymptotic expressions obtained by other authors. We discuss how more exotic string configurations can be incorporated into the theory and emphasize the utility of the functional calculus in describing these topologies.

I. INTRODUCTION

In a number of recent papers considerable progress has been made in the development of a consistent field theory of relativistic strings. Such a development marks a novel approach to the study of quantum systems that are extended in space. The early analysis[1] of the dual resonance model was algebraic, and the operatorial techniques were developed after the construction of the dual scattering amplitudes themselves. Since the pioneering work of Goddard, Goldstone, Rebbi, and Thorn[2] (GGRT) it has become progressively clearer that the conventional dual model has an elegant formulation in terms of the scattering of relativistic strings—entities that move freely and sweep out an extremal area in space-time. Kaku and Kikkawa[3] (KK) have incorporated this idea in a second-quantized field theory and have indicated how amplitudes are constructed as light-cone expansions of ordered Feynman-type graphs. Cremmer and Gervais[4] (CG) have discussed certain processes, albeit in a formalism that mixes first- and second-quantized theories. Their amplitudes coincide with those proposed by Mandelstam[5] in his sum-over-string histories approach. In studying these methods we are conscious of certain differences between them and we have tried in this paper to unify string field theory into a coherent formalism, stressing those points at which we feel work still remains to be done. We are particularly concerned with developing techniques that will be as useful in describing arbitrarily complicted processes as the first-quantized operator formalism is. Although algebraic methods possess this virtue, they do not lend themselves to very great generalization. This is undesirable in view of the limitations of current dual models as physically acceptable theories. String theory has a rich variety of structure that can be visualized pictorially in terms of space-time surfaces. It is our hope that some of the methods to be discussed will assist in the realization of more general string models.

We attempt to identify our formalism as closely as possible with the methods of conventional quantum field theory. We shall work with second-quantized field operators that create and annihilate strings in specified states. Our tools will be those of the functional calculus and a considerable simplification in the formalism is attained if one adheres, wherever possible, to functions and kernels instead of their customary Fourier components. We believe that this approach has great generality and is able to incorporate different string topologies. We shall explicitly discuss open and closed string configurations.

The quantization of the free dynamical string is accomplished in Sec. II using the noncovariant canonical approach in the GGRT orthonormal transverse gauge. This exposes the independent degrees of freedom to be quantized and leads to a first-order Schrödinger-type equation for the wave functional.[6] We extract the Hamiltonian for this equation in a way which we believe is simple and unambiguous, using Dirac's method[7] of Lagrangian constraints in the functional phase space of canonical fields. In Sec. III we solve for the complete set of energy eigenfunctionals of the open string. From these eigenfunctionals the

Feynman propagator[8] is evaluated in a number of representations. The second-quantized formalism is constructed in Sec. IV and commutation relations postulated for the field operators. The question of three-string interactions is discussed and the difference between the vertices of KK and CG studied. Since we do not use a Fock space of first-quantized mode operators, general expressions for three-Reggeon amplitudes (i.e., excited couplings) are given. The four-Reggeon vertex is also written down and its evaluation from finite-time Neumann functions discussed. We conclude with some remarks on how the formalism can accommodate more exotic string topologies.

II. EQUATIONS OF MOTION

Although the general classical equations of motions for the free string are nonlinear, GGRT showed that a great simplification occurs if one adopts the transverse orthonormal gauge. Essentially this choice of coordinates enables one to use the fundamental constraints (arising from the gauge invariance of the action) to eliminate nondynamical degrees of freedom. Furthermore, a noncovariant canonical scheme can then be set up and a Schrödinger-type equation written down for the first-quantized theory. The procedure is well established in the literature.[9]

The classical motion of a free open string is derived from the action functional

$$S[X_\mu]=\int_{t_1}^{t_2} dt \int_{\sigma_1}^{\sigma_2} d\sigma(-\det g)^{1/2} \tag{2.1}$$

in terms of the metric tensor $g_{\alpha\beta}=\partial_\alpha X\cdot\partial_\beta X$ $(\alpha, \beta=\sigma, t)$ on the surface $X_\mu=X_\mu(\sigma, t)$ in space-time. The Lagrangian $L=\int_{\sigma_1}^{\sigma_2} d\sigma(-\det g)^{1/2}$ is singular and the constraints

$$\phi_1\equiv P_t\cdot X'=0,\quad \phi_2\equiv P_t^2+X'^2=0\ , \tag{2.2}$$

$$P_\sigma\cdot\dot X=0,\quad P_\sigma^2+\dot X^2=0 \tag{2.3}$$

follow from the covariant equations of classical motion $\delta S/\delta X_\mu=0$; i.e.,

$$\partial_t\frac{\partial L}{\partial\dot X_\mu}+\partial_\sigma\frac{\partial L}{\partial X'_\mu}=0\ , \tag{2.4}$$

$$P^\mu_\sigma(\sigma_1, t)=P^\mu_\sigma(\sigma_2, t)=0 \tag{2.5}$$

and the definitions

$$P^\mu_t=\frac{\partial L}{\partial\dot X_\mu},\quad P^\mu_\sigma=\frac{\partial L}{\partial X'_\mu}\ . \tag{2.6}$$

In these equations the overdot and prime refer to differentation with respect to t and σ, respectively. Quantum mechanically, following Feynman's formulation,[8] the probability that a string propagates from one configuration to another in space is given by a suitably weighted functional integral of $\exp\{(i/\hbar)S[X]\}$.

The orthonormal transverse gauge is defined in terms of the light-cone time and the component p_+ of the total string momentum:

$$X^+=p_+t,\quad P_t^+=p_+\equiv\eta\ , \tag{2.7}$$

where

$$X_\mu=(X^+, X^-, X^*)\ . \tag{2.8}$$

Solving for X^- and P_t^- [and denoting them by $X^-(*)$ and $P_t^-(*)$ to indicate the functional dependence] leaves X^*, P_t^*, and q as independent variables.[10] In terms of these variables the functional propagator in phase space is written

$$K[q(t_i), q(t_f), X_i^*(\sigma, t_i), P_t^*(\sigma, t_i), X_f^*(\sigma, t_f), P_t^*(\sigma, t_f), t_i-t_f]$$
$$=N\int DX_\mu DP_t^\nu \exp\left\{\frac{1}{\hbar}\int_{t_i}^{t_f} dt\int_{\sigma_i}^{\sigma_f} d\sigma\, S[X, P_t, \lambda_1, \lambda_2]\right\}\delta(P_t^+-p_+)\,\delta(P_t^--P_t^-(*))\,\delta(X^+-tp^+)\,\delta(X^--X^-(*))\ , \tag{2.9}$$

where

$$S=\dot X P_t-\lambda_1\phi_1-\lambda_2\phi_2\ .$$

After integrating out the redundant variables, we can identify the Hamiltonian,

$$H[X^*, P_t^*]=\tfrac{1}{2}\int d\sigma[P_t^{*2}+(X^{*\prime})^2]\ . \tag{2.10}$$

From this classical Hamiltonian we first-quantize the free string by adopting $H[X^*, -i\hbar(\delta/\delta X^*)]$ as the operator Hamiltonian in the Schrödinger picture. Within operator reorderings (to be discussed below) we write the eigenequation for the wave functional $\Psi[X^*, t]$ as

$$H\left[X^*, -i\hbar\frac{\delta}{\delta X^*}\right]\Psi[X^*, t]=\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi[X^*, t]\ , \tag{2.11}$$

which after a trivial integration by parts and imposition of boundary conditions (open or closed string) gives ($\hbar=1$)

$$\int d\sigma\left[-\frac{1}{2}\frac{\delta^2}{\delta X^*(\sigma)^2}+\frac{1}{2}X^*(\sigma)(-\partial_\sigma^2)X^*(\sigma)\right]U_n[X^*]=E_n U_n[X^*] \tag{2.12}$$

for the "stationary" state functionals $\Psi=e^{-iE_nt}U_n[X^*]$.

Before proceeding to second-quantize the theory we shall solve this equation.

III. WAVE FUNCTIONALS IN THE FIRST-QUANTIZED THEORY

Kaku and Kikkawa observe[3] that Eq. (2.12) describes an infinite number of uncoupled quantum oscillators, each with its own characteristic potential. The solutions $U_n[X^*]$ then follow immediately in terms of the Fourier modes of X^*. It is our intention to solve Eq. (2.12) by functional methods. Not only do the fundamental kernels of the theory make their first appearance in the solution but considerable insight can be gained into the question of setting up equations for more general topologies. We discuss first the internal degrees of freedom by supposing that the center of mass x_0^* of the string is at rest.

It is convenient to parametrize σ in the range 0 to $\pi\eta$. To motivate the functional solution and establish notations (see Appendix A) consider the trial solution

$$U_0[X^*] = e^{-X^*\cdot G\cdot X^*/2}\ , \tag{3.1}$$

where

$$X^*\cdot G\cdot X^* = \int_0^{\pi\eta} d\sigma \int_0^{\pi\eta} d\sigma'\, X^*(\sigma)\, G(\sigma,\sigma')\, X^*(\sigma')\ . \tag{3.2}$$

Substituting Eq. (3.1) in Eq. (2.12) we find that U_0 will be an eigenfunctional with eigenvalue $\frac{1}{2}\,\mathrm{Tr}G = \frac{1}{2}\,d^*\int_0^{\pi\eta} d\sigma\, G(\sigma,\sigma)$ provided

$$X^*\cdot G\cdot G\cdot X^* = -X^*\cdot \partial_\sigma{}^2 X^*\ .$$

The function $X^*(\sigma)$ may be expanded in the set of basis functions $f_n(\sigma)$ on the interval $(0, \pi\eta)$ that satisfy the boundary conditions for the topology under consideration. We shall identify solutions with a particular topology by demanding that the kernel G be diagonal in that basis, i.e., have the form

$$G(\sigma,\sigma') = \sum_{n=1}^{\infty} g_n\, f_n(\sigma) f_n(\sigma')\ . \tag{3.3}$$

For the open string the fundamental orthonormal basis functions in the internal space of excitations are

$$f_n(\sigma) = \left(\frac{2}{\pi\eta}\right)^{1/2} \cos\frac{n\sigma}{\eta}\,, \quad n = 1, 2, \ldots\ . \tag{3.4}$$

Restricting G to have positive eigenvalues (so that U_0 exists) we readily obtain the open-string kernel, with $g_n = n/\eta$. We shall handle the infinite zero-point energy $\mathrm{Tr}G$ of the ground state[11] in a conventional manner. All excited levels of energy will be measured with reference to it. The proper rest mass of the ground state will be introduced below when we discuss the center-of-mass motion. Its value must be fixed by considerations outside the noncovariant framework.[12]

A convenient way to organize the general open-string eigensolutions is to introduce the functional operators

$$a^*[z(\sigma)]^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{\delta}{\delta z^*(\sigma)} + z^*(\sigma)\right]\ ,$$
$$a^*[z(\sigma)] = \frac{1}{\sqrt{2}}\left[\frac{\delta}{\delta z^*(\sigma)} + z^*(\sigma)\right] \tag{3.5}$$

which obey the rules[13]

$$[a^i[z(\sigma)], a^j[z(\sigma')]^\dagger] = \delta(\sigma-\sigma')\,\delta_{ij}\,. \tag{3.6}$$

In terms of the kernel

$$G^{1/2}(\sigma,\sigma') = \sum_{n=1}^{\infty}\left(\frac{n}{\eta}\right)^{1/2} f_n(\sigma) f_n(\sigma')$$

introduce the variable $z^*(\sigma) = G^{1/2}\cdot X^*$ and write Eq. (2.12) in the form (see Appendix B)

$$(a^*\cdot G\cdot a^*)\, U_n[z^*] = \epsilon_n\, U_n[z^*]\ . \tag{3.7}$$

The functional

$$U_{\{\lambda_n^*\}}[z^*] = \prod_n \frac{(f_n\cdot a^{*\dagger})^{\lambda_n^*}}{(\lambda_n^*!)^{1/2}}\, e^{-z^{*2}/2} \tag{3.8}$$

satisfies Eq. (3.7) with eigenvalue

$$\epsilon_{\{\lambda_n^*\}} = \eta \sum_n g_n \lambda_n^*$$
$$= \sum_n n\lambda_n^*\ . \tag{3.9}$$

$\{\lambda_n^*\}$ specifies the set of integers n, λ_n^* that occur in Eq. (3.9). These eigenfunctionals are orthonormal since

$$\int Dz^*\, U_{\{\lambda_n^*\}}[z^*]\, U_{\{\mu_n^*\}}[z^*] = \delta_{\{\lambda_n^*\}\{\mu_n^*\}}\ . \tag{3.10}$$

We have made these solutions precise since they play a fundamental role in constructing the string propagator. In Appendix B we express these solutions in terms of Hermite functionals

$$U_{\{\lambda_n^*\}}[X^*] = \prod_n H_{\lambda_n^*}(f_n\cdot X^*)\, e^{-X^*\cdot G^R\cdot X^*/2}$$
$$= \prod_n i^{\lambda_n^*} H_{\lambda_n^*}\left(f_n\cdot (G^R)^{-1}\cdot \frac{\delta}{\delta X^*}\right)$$
$$\times e^{-X^*\cdot G^R\cdot X^*/2}. \tag{3.11}$$

This last form is particularly useful for the evaluation of excited string amplitudes. We shall show in Sec. V that these may also be written in terms of the Hermite operators, there being one

$H_{\lambda_n^*}(f_n \cdot G^{-1} \cdot \delta)$ for each λ^*-fold excitation of the nth string harmonic.

We observe that a string with definite energy does not have a specific shape. $|U_{\{\lambda_n^*\}}[X^*]|^2 DX^*$ is a measure of the probability of finding the string in the state specified by the set $\{\lambda_n^*\}$ with a configuration between $X^*(\sigma)$ and $X^*(\sigma)+DX^*(\sigma)$. (In practice X^*+DX^* can be replaced by those Fourier coefficients which significantly differ from those that synthesize X^*.) It is possible, of course, to construct string wave packets from these energy eigensolutions. Such packets will disperse in time since they are superpositions of functionals, each with its own time dependence.

In the foregoing discussion we have deliberately written the eigenfunctionals in terms of the field $Z^*(\sigma)$. When considering a more general topology satisfying Eq. (2.12) we can apply the same techniques as in the open-string topology. Once a basis appropriate to the boundary conditions is adopted then the kernel G is defined by demanding that it be diagonal (with positive eigenvalues g_n) in that basis.

Up to this point we have discussed only the internal motion of the quantum string. The use of light-cone coordinates enables one to isolate the internal dynamical degrees of freedom. It is useful, once the transverse variables have been selected, to rewrite the Schrödinger equation (2.12) (including x_0^*) in a form reminiscent of a Klein-Gordon equation for a system with internal structure. To this end, consider a string with $X_\mu(\sigma) = (t, q, X^*(\sigma))$ described by the complete functional $\Psi[q, t, x_0^*, x^*]$, satisfying in the GGRT gauge

$$\{\Box + m_0^2 + \eta a^{*\dagger}[Z]\cdot G\cdot a^*[Z]\}\Psi[q, t, x_0^*, x^*] = 0 . \tag{3.12}$$

In the absence of internal excitation this describes a spinless particle with rest mass m_0. The internal structure is attributed entirely to the transverse excitations in this gauge. Defining the generalized Fourier transform

$$\Psi = \sum_{\{\lambda_n^*\}} \int d^4p\, e^{ip\cdot x} U_{\{\lambda_n^*\}}[Z]\lambda(p_+, p_-, p_0^*, \{\lambda_n^*\}) , \tag{3.13}$$

where $p_\mu \equiv (p_+, p_-, p_0^*)$ in light-cone coordinates, we have a general solution to Eq. (3.12) provided

$$\begin{aligned}\lambda &= \delta(p^2 - m_0^2 - \eta\,\epsilon_{\{\lambda_n^*\}})\Lambda_{\{\lambda_n^*\}}(p_+, p_-, p_0^*)\\ &= \frac{1}{2p_+}\delta\left(p_- - \frac{1}{2\eta}(p_0^{*2} + m_0^2 + \eta\,\epsilon_{\{\lambda_n^*\}})\right)\\ &\quad\times\Lambda_{\{\lambda_n^*\}}(p_+, p_-, p_0^*).\end{aligned} \tag{3.14}$$

We observe that the upper (lower) mass hyperboloid $p_0 \geq m_0$ $(p_0 \leq -m_0)$, $|\vec{p}| \geq 0$ now lies in the positive (negative) p_-, p_+ quadrant. For the present discussion we must assume that m_0^2 is positive. Identifying antistrings as those with $p_+ \leq 0, p_- \leq 0$ we write the solutions

$$\Psi[q, t, x_0^*, x^*] = \sum_{\{\lambda_n^*\}} \int_{-\infty}^{\infty} dp_0^* \int_0^{\infty} \frac{dp_+}{(2p_+)^{1/2}} U_{\{\lambda_n^*\}}[Z^*][e^{ip\cdot x}\Lambda^+_{\{\lambda_n^*\}}(p_+, p_0^*) + e^{-ip\cdot x}\Lambda^-_{\{\lambda_n^*\}}(p_+, p_0^*)], \tag{3.15}$$

where

$$\Lambda^{\pm}_{\{\lambda_n^*\}}(p_+, p_0^*) = \frac{1}{(2\eta)^{1/2}}\Lambda_{\{\lambda_n^*\}}(\pm p_+, \pm p_-, \pm p_0^*)\,\theta(\pm p_-), \quad p_- = \frac{1}{2p_+}(p_0^{*2} + m_0^2 + \eta\,\epsilon_{\{\lambda_n^*\}}), \quad p\cdot X = p_+ q + p^- t - p_0^* x_0^* .$$

The positive-frequency part of Ψ breaks up naturally into a part depending on the internal modes and a part describing the barycentric light-cone variables,

$$\Psi^\dagger[q, t, x_0^*, x^*] = \int dp_0^* \int_0^\infty \frac{dp_+}{(2p_+)^{1/2}} \exp\left[i(p_0^* x_0^* - p_+ q) - \frac{ip_0^{*2} t}{2\eta}\right] \sum_{\{\lambda_n^*\}} e^{-(it\epsilon\{\lambda_n^*\}/2p_+)} U_{\{\lambda_n^*\}}[Z^*]\Lambda^+_{\{\lambda_n^*\}}(p_+, p_0^*) . \tag{3.16}$$

As usual with a theory expressed in these coordinates, once the "longitudinal" modes are isolated the barycentric coordinates appear to describe a nonrelativistic particle with momentum p_0^* and mass p_+.[14]

IV. SECOND QUANTIZATION

The free string is second-quantized in the orthonormal transverse gauge by postulating for the Fourier coefficients in Eq. (3.15) the commutation rules

$$[\Lambda^-_{\{\lambda_n^*\}}(p_0^*, \eta), \Lambda^+_{\{\mu_m^*\}}(p_0^{*\prime}, \eta')] = \delta(\eta - \eta')\,\delta(p_0^* - p_0^{*\prime})\,\delta_{\{\lambda_n^*\}\{\mu_m^*\}} . \tag{4.1}$$

We regard $\Lambda^+_{\{\lambda_n^*\}}(p_0^*, \eta)$ as a creation operator for a string in a configuration specified by $\{\lambda_n^*\}$, p_0^*, and η. The equal-t commutation rules for the string field follow immediately from this postulate,

$$[\Psi[q,t,x_0^*,x^*],\Psi[q',t,x_0^{*\prime},x^{*\prime}]] = \delta[Z^*-Z^{*\prime}]\,\delta(x_0^*-x_0^{*\prime})\,\Delta_+(q-q')\,. \quad (4.2)$$

When the "times" are unequal we have

$$[\Psi[q,t,x_0^*,x^*],\Psi[q',t',x_0^{*\prime},x^{*\prime}]] = \int dp_0^* \int_0^\infty \frac{d\eta}{(2\eta)^{1/2}} \sum_{\{\lambda_n^*\}} U_{\{\lambda_n^*\}}[Z^*]\,U_{\{\lambda_n^*\}}[Z^{*\prime}] \times (e^{-ip\cdot(x-x')} - e^{ip\cdot(x-x')})\,, \quad (4.3)$$

where p_- is given in Eq. (3.15). We interpret the first term in this expression as the propagator for a positive-energy string in time. We write this as

$$K_+[q-q',t-t',x_0^*-x_0^{*\prime}x^*,x^{*\prime}] = \int_0^\infty \frac{d\eta}{(2\eta)^{1/2}}\, K_\eta^0(q-q',t-t',x_0^*-x_0^{*\prime}) \times K_\eta(x^*,x^{*\prime},t-t')\,, \quad (4.4)$$

where we identify a free particle propagator

$$K_\eta^0 = \int dp_0^* \exp\left(ip_0^*(x_0^*-x_0^{*\prime}) - \eta(q-q') + \frac{p_0^{*2}}{2\eta}(t-t')\right) \quad (4.5)$$

and an internal functional propagator

$$K_\eta[X^*,X^{*\prime},t-t'] = \sum_{\{\lambda_n^*\}} U_{\{\lambda_n^*\}}[Z^*]\,U_{\{\lambda_n^*\}}[Z^{*\prime}] \times \exp\left[-\frac{i}{2\eta}\epsilon_{\{\lambda_n^*\}}(t-t')\right]. \quad (4.6)$$

In terms of free-string functionals we write

$$K_\eta = U_0[Z^*] \sum_{\{\lambda_n^*\}} \prod_n \frac{(\bar{F}_n\cdot\vec{a}^{*\dagger}[Z])^{\lambda_n^*}}{(\lambda_n^*!)^{1/2}} \frac{(F_n\cdot\vec{a}^{*\dagger}[Z'])^{\lambda_n^*}}{(\lambda_n^*!)^{1/2}} \times U_0[Z^{*\prime}]\,, \quad (4.7)$$

where

$$F_n(\sigma,t) = f_n(\sigma)\,e^{-ig_nt/2}$$

With the aid of the multinomial theorem we rewrite Eq. (4.7) as

$$K_\eta = U_0[Z^*]\,e^{\vec{a}^{*\dagger}[Z]\cdot L_{t-t'}\cdot\vec{a}^{*\dagger}[Z']}\,U_0[Z^{*\prime}] \quad (4.8)$$

and we have introduced the kernel

$$L_{t-t'}(\sigma,\sigma') = \sum_{n=1}^\infty e^{-ig_n(t-t')/2} f_n(\sigma) f_n(\sigma')\,. \quad (4.9)$$

To obtain a closed expression we first express the exponent in Eq. (4.8) in terms of functional derivatives only by writing $U_0[Z] = \exp(\frac{1}{2}Z^2)\exp(-Z^2)$ so that

$$K_\eta = e^{(Z^{*2}+Z^{*\prime 2})/2} \exp\left(\frac{1}{2}\frac{\delta}{\delta Z^*}\cdot L\cdot\frac{\delta}{\delta Z^{*\prime}}\right) \times e^{-Z^{*2}-Z^{*\prime 2}}. \quad (4.10)$$

We can linearize the last factors with the functional integral representation

$$e^{-Z^{*2}} = \int DQ^* \exp(-\tfrac{1}{2}Q^{*2} + i\sqrt{2}\,Q^*\cdot Z^*) \quad (4.11)$$

so that evaluating the functional derivatives gives

$$K_\eta = e^{(Z^{*2}+Z^{*\prime 2})/2} \times \int DQ^*\,DQ^{*\prime} \exp(-\tfrac{1}{2}Q^{*2} - \tfrac{1}{2}Q^{*\prime 2} + i\sqrt{2}\,Q^*\cdot Z^* + i\sqrt{2}\,Q^*\cdot Z^{*\prime} - Q^*\cdot L\cdot Q^{*\prime})\,. \quad (4.12)$$

This is a coupled functional Gaussian integral which will be met on several occasions in this paper. Using the results of Appendix A

$$K_\eta = \frac{\exp(-\frac{1}{2}Z^*\cdot L_2\cdot Z^* - \frac{1}{2}Z^{*\prime}\cdot L_2\cdot Z^{*\prime} + 2Z^*\cdot L_1\cdot Z^{*\prime})}{[\det(1-L^2)]^{d^*/2}} \quad (4.13)$$

where

$$L_2(\sigma,\sigma') = \frac{1+L^2}{1-L^2} = -\sum_{n=1}^\infty \coth g_n(\tau-\tau') f_n(\sigma) f_n(\sigma')\,,$$

$$L_1(\sigma,\sigma') = \frac{L}{1-L^2} = -\frac{1}{2}\sum_{n=1}^\infty \operatorname{csch} g_n(\tau-\tau') f_n(\sigma) f_n(\sigma')$$

and the Wick-rotated variable $\tau = it$ has been used. This representation for K_η will be useful frequently in later sections. One can verify that it obeys the necessary reproducing property,

$$\int K_\eta[x^{*\prime\prime}\tau''|x^*\tau]\,K_\eta[x^*\tau|x^{*\prime}\tau']\,DX^* = K_\eta[x^{*\prime\prime}\tau''|x^{*\prime}\tau']\,. \quad (4.14)$$

The factor $\det(1-L^2) = \prod_{n=1}^\infty (1-e^{g_n(\tau-\tau')/2})$ is the correct one to ensure that $U_0[Z^*]$ propagates unchanged in time, since by definition it has a zero-energy eigenvalue. It will be recalled that K_+ propagates string functionals forward in light-cone time t. For relativistic calculations in these coordinates we must sum over all time orderings between interactions. This can usefully be done if one utilizes a different Green's functional which, although employing transverse variables, is similar in structure to a covariant causal propagator. A solution of

$$(\Box + m_0^2 + \eta a^{*\dagger}[Z]\cdot G\cdot a^*[Z])K^F(X,X')$$
$$= \delta(\tau - \tau')\,\delta(q-q')\,\delta[X^* - X^{*\prime}]\,\delta(x_0^* - x_0^{*\prime}) \quad (4.15)$$

is

$$K^F(X,X') = \sum_{\{\lambda_n^*\}} U_{\{\lambda_n^*\}}[Z^*]\, U_{\{\lambda_n^*\}}[Z^{*\prime}] \times \int d^4p \frac{e^{ip\cdot(x_0 - x_0')}}{p^2 - m_0^2 - \eta\epsilon_{\{\lambda_n^*\}}},$$

where the causal p_0 contour is implied. Parametrizing the denominator as

$$\frac{1}{p^2 - m_0^2 - \eta\epsilon_{\{\lambda_n^*\}}} = \int_0^1 dy\, y^{p^2 - m_0^2 - \eta\epsilon\{\lambda_n^*\} - 1} \quad (4.16)$$

and performing the eigenfunctional sum as before yields

$$K^F(X,X') = \int_0^1 \frac{dy}{y} \int d^4p\, e^{ip\cdot(x_0 - x_0')} U_0[Z^*] \times e^{\overleftarrow{a}^\dagger[Z^*]\cdot L(y)\cdot\overrightarrow{a}^\dagger[Z^{*\prime}]} U_0[Z^{*\prime}], \quad (4.17)$$

where

$$L(y,\sigma,\sigma') = \sum_{n=1}^{\infty} y^{\eta g_n} f_n(\sigma) f_n(\sigma').$$

For the evaluation of string vertices and amplitudes it is convenient to work with the Fourier transform of the propagator:

$$\tilde{K}_+[q-q', t-t', p_0^* - p_0^{*\prime}, P^*, P^{*\prime}] = \int \exp(-ip_0^* x_0^* + ip_0^{*\prime} x_0^{*\prime} - iP^*\cdot X^* + iP^{*\prime}\cdot X^{*\prime})$$
$$\times K_+[q-q', t-t', x_0^* - x_0^{*\prime}, X^*, X^{*\prime}]\, dx_0^*\, dx_0^{*\prime}\, DX^*\, DX^{*\prime}$$
$$= \delta(p_0^* - p_0^{*\prime}) \int_0^\infty \frac{d\eta}{(2\eta)^{1/2}} \exp\left[-\eta(q-q') + \frac{ip_0^{*2}}{2\eta}(t-t')\right] \tilde{K}_\eta[P^*, P^{*\prime}, t-t']. \quad (4.18)$$

V. SECOND-QUANTIZED STRING INTERACTIONS

In eonventional field theory, quantized fields interact at points in space. Such local interactions may be interpreted as the fundamental vertices linking the space-time tracks of the point particles created by the fields. In a similar manner, quantum strings are visualized as interacting in light-cone coordinates when they have one or more points in common. This picture is intuitively appealing although it should be stressed that it is unclear whether it generalizes to arbitrary frames of reference in a simple manner. However, in the light-cone representation a second-quantized Hamiltonian in the interaction picture is postulated to describe the general three-string vertex:

$$H_I[\Psi_1, \Psi_2, \Psi_3] = \int dt\, dq\, \Psi^-[q, t, x_{10}^* X_1^*]\Psi^-[q, t, x_{20}^* X_2^*]\; F[x_{10}^*, X_1^*, x_{20}^*, X_2^*, x_{30}^*, X_3^*]\Psi^+[q, t, x_{30}^*\, X_3^*] \prod_{i=1}^3 dx_{i0}^*\, DX_i^*$$
$$= \int dt H_I(t), \quad (5.1)$$

where strings one and two are incoming and string three is outgoing. The precise form of the interaction will depend on the choice of interaction functional F. Since the interaction is local in q we can immediately extract conservation of "longitudinal" momentum η:

$$H_I = \int dt\, \delta(\eta_1 + \eta_2 - \eta_3)\, F[x_{i0}^*, X_i^*]\, \phi_{\eta_1}^-(x_{10}^*, X_1^*, t)\, \phi_{\eta_2}^-(x_{20}^*, X_2^*, t)\, \phi_{\eta_3}^+(x_{30}^*, X_3^*, t) \prod_{i=1}^3 dx_{i0}^*\, DX_i^* \frac{d\eta_i}{(2\eta_i)^{1/2}}, \quad (5.2)$$

where

$$\phi_\eta^\pm(x_0^*, X^*, t) = \int dp_0^* \exp\left(ip_0^* x_0^* - \frac{ip_0^{*2}t}{2\eta}\right) \sum_{\{\lambda_n^*\}} e^{-it\epsilon\{\lambda_n^*\}/2} U_{\{\lambda_n^*\}}[Z^*]\Lambda_{\{\lambda_n^*\}}^\dagger(p_0^*, \eta).$$

In a momentum representation (Appendix B)

$$H_I=\int dt\,\delta(\eta_1+\eta_2-\eta_3)\delta(p_{10}^{*}+p_{20}^{*}-p_{30}^{*})\tilde F[p_{i0}^{*},P_i^{*}]\tilde\phi_{\eta_1}^{-}[p_{10}^{*},P_1^{*},t]\tilde\phi_{\eta_2}^{-}[p_{20}^{*},P_2^{*},t]\tilde\phi_{\eta_3}^{+}[p_{30}^{*},P_3^{*},t]$$
$$\times\exp\left[it\left(\frac{p_{10}^{*2}}{2\eta_1}+\frac{p_{20}^{*2}}{2\eta_2}-\frac{p_{30}^{*2}}{2\eta_3}\right)\right]\prod_{i=1}^{3}dp_{i0}^{*}DP_i^{*}\frac{d\eta_i}{(2\eta_i)^{1/2}}, \tag{5.3}$$

where

$$\tilde\phi_\eta^{\pm}(p_0^{*},P^{*},t)=\sum_{\{\lambda_n^{*}\}}e^{-t\epsilon\{\lambda_n^{*}\}/2}\tilde U_{\{\lambda_n^{*}\}}[P^{*}]\Lambda^{\pm}_{\{\lambda_n^{*}\}}(p_0^{*},\eta),$$

$$\tilde F[\Pi_1,\Pi_2,\Pi_3]=\int F[y_1,y_2,y_3]\times\exp(i\Pi_1\cdot y_1+i\Pi_2\cdot y_2-i\Pi_3\cdot y_3)\times Dy_1Dy_2Dy_3,$$

$$y_i(\sigma)=\theta_i(\sigma)[x_{i0}^{*}+X_i^{*}(\sigma)],$$

$$\Pi_i(\sigma)=\theta_i(\sigma)\left[\frac{p_{i0}^{*}}{\pi\eta_i}+P_i^{*}(\sigma)\right],$$

$$\theta_i(\sigma)=\theta(\sigma)\theta(\pi\eta_i-\sigma)\quad(i=1,3)$$
$$=\theta(\sigma-\pi\eta_1)\theta(\pi\eta_3-\sigma)\quad(i=2).$$

In terms of open-string bases we write

$$P_i^{*}(\sigma)=\sum_{n=1}^{\infty}p_n^{*}f_n^{i}(\sigma), \tag{5.4}$$

where

$$f_n^{i}(\sigma)=\left(\frac{2}{\pi\eta_i}\right)^{1/2}\cos\frac{n\sigma}{\eta_i},\quad i=1,3$$
$$=\left(\frac{2}{\pi\eta_i}\right)^{1/2}\cos\frac{n}{\eta_i}(\pi\eta_1-\sigma),\quad i=2. \tag{5.5}$$

The operators $\Lambda^{\pm}_{\{\lambda_n^{*}\}}(p_0^{*},\eta)$ enable one to construct a Fock space of multistring states from the vacuum. For example, the state

$$|(\{\lambda_n^{*}\}p_0,\eta)^2\rangle=\frac{1}{(2!)^{1/2}}[\Lambda^{+}_{\{\lambda_n^{*}\}}(p_0^{*},\eta)]^2|0\rangle$$

contains two identical strings each in a state specified by the quantum numbers $\{\lambda_n^{*}\}$, p_0^{*}, and η. The second-quantized string-number operator in this space is clearly

$$N=\Lambda^{+}_{\{\lambda_n^{*}\}}(p_0^{*},\eta)\Lambda^{-}_{\{\lambda_n^{*}\}}(p_0^{*},\eta) \tag{5.6}$$

(cf. N_n as the mode number operator in the first-quantized Fock space). This concludes the formal construction of the string field-theory. Light-cone Feynman diagrams follow from the standard Wick procedure applied to matrix elements in the above Fock space of the scattering operator

$$S=T\left(\exp\left[i\int_0^t H_I(t)dt\right]\right). \tag{5.7}$$

The simplest interaction between open strings, first considered by KK, is the one in which two strings fuse at their end points to produce a third. We interpret this description to mean that at some instant $t=t_0$, the parametrization of one string must be indistinguishable from the parametrization of two strings with one parameter point in common. Consequently we write the interaction function in a momentum basis as

$$\tilde F[\Pi_3,\Pi_2,\Pi_1]=\lambda\delta[\Pi_3-\Pi_2-\Pi_1]. \tag{5.8}$$

It should be noted that this functional does not exclude contributions from discontinuous fields $\Pi(\sigma)$. The interaction between arbitrary excited string states is conventionally given in terms of a three-Reggeon vertex containing first-quantized mode operators. In our formalism the matrix element for the string transition

$$\{\lambda_{n_1}^{*}\}+\{\lambda_{n_2}^{*}\}\to\{\lambda_{n_3}^{*}\} \tag{5.9}$$

is

$$T_{t_1t_2t_3}(p_{10}^{*},\eta_1,p_{20}^{*},\eta_2,p_{30}^{*},\eta_3,\{\lambda_{n_1}^{*}\},\{\lambda_{n_2}^{*}\},\{\lambda_{n_3}^{*}\})=\lambda\int\prod_{i=1}^{3}\tilde U_{\{\lambda_n^{*}\}_i}[P_i^{*}]DP_iV[\Pi_3,t_3-t_0,\Pi_2,t_2-t_0,\Pi_1,t_1-t_0], \tag{5.10}$$

where

$$V=\int D\Pi_1'D\Pi_2'D\Pi_3'\tilde K[\Pi_3\tau_3|\Pi_3'\tau_0]\delta[\Pi_3'-\Pi_1'-\Pi_2']\tilde K[\Pi_1'\tau_0|\Pi_1\tau_1]\tilde K[\Pi_2'\tau_0|\Pi_2\tau_2]. \tag{5.11}$$

In terms of the kernels

$$B_i(\tau-\tau',\sigma,\sigma')\equiv G_i^{-1}L_2^i=-\eta_i\sum_{n=1}\frac{1}{n}\coth\frac{n}{\eta_i}(\tau-\tau')f_n^i(\sigma)f_n^i(\sigma'),$$
$$D_i(\tau-\tau',\sigma,\sigma')\equiv G_i^{-1}L_1^i=-\frac{\eta_i}{2}\sum_{n=1}\frac{1}{n}\operatorname{csch}\frac{n}{\eta_i}(\tau-\tau')f_n^i(\sigma)f_n^i(\sigma') \tag{5.12}$$

we can write the propagator

$$\tilde{K}_{\eta_i}[\Pi_i\tau|\Pi_i'\tau'] = \frac{\delta(p_0^* - p_0^{*\prime})e^{-E_i^0(\tau-\tau')}}{[\det(1-L^2)]^{d^*/2}} \exp(-\tfrac{1}{2}P_i^*\cdot B_i\cdot P_i^* - \tfrac{1}{2}P_i^{*\prime}\cdot B_i\cdot P_i^{*\prime} + 2P_i^*\cdot D_i\cdot P_i^{*\prime}), \tag{5.13}$$

where

$$E_i^0 = \frac{1}{2\eta_i}(p_0^{*2} + m_0^2).$$

It will be observed that the interaction functional Eq. (5.8) involves the zero-mode basis functions of the three fields $\Pi_i(\sigma)$ in a nontrivial way. It is useful to exploit the geometric structure of the function space on which the fields are expanded to properly interpret this interaction functional. Each field can be regarded as being expanded in its own (infinite-dimensional) Hilbert space H_i containing one zero-mode vector $(p_0^*/\pi\eta_i)\theta^i(\sigma)$ and a set of internal vectors $P_n^* f_n^i(\theta)\theta^i(\sigma)$. We can decompose this space as

$$H_i = H_i^0 \oplus H_i^+, \tag{5.14}$$

where H_i^0 is the one-dimensional subspace containing $(p_{i0}^*/\pi\eta_i)\theta_i(\sigma)$. When the strings are brought into coincidence the spaces H_1 and H_2 form a direct sum decomposition of H_3:

$$H_3 = H_1 \oplus H_2$$

or (5.15)

$$H_3^0 \oplus H_3^+ = H_1^0 \oplus H_1^+ \oplus H_2^0 \oplus H_2^+.$$

We shall refer to the direct sum of the subspace H_1^+ and H_2^+ as H_3^{++}. This subspace consequently differs from H_3^+ by a linear combination of two zero-mode vectors. It will be important in the following to make precise which subspace is being used to represent the kernels involved in the evaluation of V.

We present three alternative evaluations of Eq. (5.11) using functional methods. *A priori*, the simplest method might appear to be the removal of the δ functional by integration over Π_3, establishing

$$\Pi_3' = \Pi_1' + \Pi_2'. \tag{5.16}$$

Multiplying by $\theta^3(\sigma)$ and integrating gives

$$\begin{aligned}\theta_3\cdot\Pi_3' &= p_{30}^* \\ &= \theta_3\cdot\Pi_1' + \theta_3\cdot\Pi_2' \\ &= p_{10}^{*\prime} + p_{20}^{*\prime}.\end{aligned} \tag{5.17}$$

Using the zero-mode δ function Eq. (5.13) in $\tilde{K}$, the integration over $p_{10}^{*\prime}$ and $p_{20}^{*\prime}$ produces $\delta(p_{10}^* + p_{20}^* - p_{30}^*)$. This leaves an integral over $P_1^{*\prime}$ and $P_2^{*\prime}$ where we eliminate $P_3^{*\prime}$ using Eq. (5.16),

$$P_3^{*\prime} \equiv \Pi_3' - \frac{p_{30}}{\pi\eta_3}\theta_3 = P_1^{*\prime} + P_2^{*\prime} + \tilde{p}^*\theta_-, \tag{5.18}$$

where

$$\begin{aligned}\theta_-(\sigma) &= \frac{1}{\pi\eta_1}\theta_1(\sigma) - \frac{1}{\pi\eta_2}\theta_2(\sigma) \\ &= \left(\frac{2\eta_3}{\pi}\right)^{1/2}\sum_{m=1}^{\infty}\frac{\sin(m\pi\eta_1/\eta_3)}{m}f_m{}^3(\sigma)\end{aligned}$$

and (5.19)

$$\tilde{p}^* = \frac{\eta_2}{\eta_3}p_{10}^* - \frac{\eta_1}{\eta_3}p_{20}^*.$$

Equation (5.18) clearly shows that the difference between the spaces H_3^+ and $H_3^{++} \equiv H_1^+ \oplus H_2^+$ is the vector $\tilde{p}\theta_-(\sigma)$. Inserting this value of $P_3^{*\prime}$ into Eq. (5.11) and defining

$$V = \prod_{i=1}^{3} e^{-iE_i^0(t_i - t_0)}\tilde{V} \tag{5.20}$$

we find,[15] using the formula for coupled Gaussians in Appendix A,

$$\begin{aligned}\tilde{V} = &\{\det(B_1+B_2)(B_2+B_3)[\underline{1}_{(1+)} - (B_1+B_3)^{-1}\cdot B_3\cdot(B_2+B_3)^{-1}\cdot B_3]\}^{-d^*/2} \\ &\times\delta(p_{10}^* + p_{20}^* - p_{30}^*)\{\exp[\tfrac{1}{2}P_{13}^*\cdot B_{13}{}^{-1}\cdot P_{13}^* + \tfrac{1}{2}P_{23}^* B_{23}{}^{-1}\cdot P_{23}^* - P_{13}^*\cdot(B_1+B_3)^{-1}\cdot B_3\cdot B_{23}{}^{-1}\cdot P_{23}^*]\},\end{aligned} \tag{5.21}$$

where

$$B_{13} = B_1 + B_3 - B_3\cdot(B_2+B_3)^{-1}\cdot B_3 \in H_1^+,\quad B_{23} = B_2 + B_3 - B_3\cdot(B_1+B_3)^{-1}\cdot B_3 \in H_2^+,$$

$$P_{j3}^* = 2P_j^*\cdot D_j + 2P_3^*\cdot D_3 - \tilde{p}^*\theta_-\cdot B_3.$$

We observe that since $P_1^{*\prime}$ is defined on the H_1^+ space B_1+B_3 is to be inverted on this space. A more compact expression will result if we use the fact that $B_1+B_2+B_3$ is B_1+B_3 when projected onto the H_1^+

space and B_2+B_3 when projected onto the H_2^+ space since B_i and D_i are only defined in H_i^+. Working in the space H^{++} on which $P_1^{*\prime}+P_2^{*\prime}$ are defined we can consequently rewrite

$$\tilde{V}=\delta(p_{10}^*+p_{20}^*-p_{30}^*)$$
$$\times\int D(P_1^{*\prime},P_2^{*\prime})\exp[-\tfrac{1}{2}(P_1^{*\prime}+P_2^{*\prime})\cdot(B_1+B_2+B_3)\cdot(P_1^{*\prime}+P_2^{*\prime})]$$
$$\times\exp[(P_1^{*\prime}+P_2^{*\prime})\cdot(2D_1\cdot P_1^*+2D_2\cdot P_2^*+2D_3\cdot P_3^*-B_3\cdot\theta_-\tilde{p}^*)] \tag{5.22}$$

as a single Gaussian in $(P_1^{*\prime}+P_2^{*\prime})$. Using Eq. (A7), this gives

$$\tilde{V}=\delta(p_{10}^*+p_{20}^*-p_{30}^*)[\det{}_{++}(B_1+B_2+B_3)]^{-d^*/2}$$
$$\times\exp\left(-\tfrac{1}{2}\sum_{r,s=1}^{3}P_r^*\cdot\Gamma^{rs}\circ P_s^*-\sum_{r=1}^{3}P_r\cdot\Gamma^{r0}\tilde{p}^*-\tfrac{1}{2}\tilde{p}^{*2}\Gamma^{00}\right), \tag{5.23}$$

where

$$\Gamma^{00}=\theta_-\cdot B_3\cdot\theta_--\theta_-\cdot B_3\cdot[B_1+B_2+(B_3)_{++}]^{-1}\cdot B_3\cdot\theta_-,$$
$$\Gamma^{r0}=2\theta_-\cdot B_3\cdot[B_1+B_2+(B_3)_{++}]^{-1}\cdot D_r-2\delta_{r3}\theta_-\cdot B_3,$$
$$\Gamma^{rs}=\delta_{rs}B_r-4D_r\cdot[B_1+B_2+(B_3)_{++}]^{-1}\cdot D_s.$$

In these expressions $(B_3)_{++}$ indicates that B_3 is to be projected into the H^{++} basis (i.e., expanded in terms of f_n^1 and f_n^2). In this space it will be nondiagonal. Its actual form can be obtained from the overlap coefficients detailed in Appendix A. In this method kernels are inverted in the $H^{++}\equiv H_1^+\oplus H_2^+$ subspace.

The second method of evaluating Eq. (5.11) is to break up the interaction functional into its H_1 and H_2 components:

$$\delta[\Pi_1'+\Pi_2'-\Pi_3']=\delta_{(1)}[\Pi_1'-\Pi_3']\,\delta_{(2)}[\Pi_2'-\Pi_3']$$
$$=\delta\left(p_{10}^{*\prime}-\frac{\eta_1}{\eta_3}p_{30}^{*\prime}-\theta_1\cdot P_3^{*\prime}\right)\delta_{(1+)}[P_1^{*\prime}-P_3^{*\prime}]\,\delta\left(p_{20}^{*\prime}-\frac{\eta_2}{\eta_3}p_{20}^{*\prime}-\theta_2\cdot P_3^{*\prime}\right)\delta_{(2+)}[P_2^{*\prime}-P_3^{*\prime}]$$
$$=\delta(p_{10}^{*\prime}+p_{20}^{*\prime}-p_{30}^{*\prime})\,\delta\left(\tilde{p}^*-\frac{\pi\eta_1\eta_2}{\eta_3}\theta_-\cdot P_3^{*\prime}\right)\delta_{(1+)}[P_1^{*\prime}-P_3^{*\prime}]\,\delta_{(2+)}[P_2^{*\prime}-P_3^{*\prime}]. \tag{5.24}$$

In the last equation we have taken sums and differences of the arguments in the zero mode δ functions to isolate $p_0^{*\prime}$ conservation. The three $p_{i0}^{*\prime}$ integrations in Eq. (5.11) now yield an over-all $\delta(p_{10}^*+p_{20}^*-p_{30}^*)$ as before. Furthermore, the $DP_1^{*\prime}DP_2^*$ integrals are trivial, yielding $P_i^{*\prime}=P_3^{*\prime}$ in the appropriate ranges. With the parametrization

$$\delta\left(\tilde{p}^*-\frac{\pi\eta_1\eta_2}{\eta_3}\theta_-\cdot P_3^{*\prime}\right)=\int_{-\infty}^{\infty}du\exp\left[iu\left(\tilde{p}^*-\frac{\pi\eta_1\eta_2}{\eta_3}\theta_-\cdot P_3^{*\prime}\right)\right] \tag{5.25}$$

we can write the $P_3^{*\prime}$ integral as a single functional Gaussian with kernel $B_1+B_2+B_3$ defined in H_3^+. (B_1 and B_2 are nondiagonal in this basis: f_n^3.) The integration yields an ordinary Gaussian in the variable u and this gives

$$\tilde{V}=\{[\det(B_1+B_2+B_3)]\theta_-\cdot(B_1+B_2+B_3)^{-1}\cdot\theta_-\}^{-d^*/2}\delta(p_{10}^*+p_{20}^*-p_{30}^*)$$
$$\times\exp\left(-\tfrac{1}{2}\sum_{r,s}P_r^*\cdot\Gamma^{rs}\cdot P_s^*-\sum_r P_r^*\cdot\Gamma^{r0}\tilde{p}^*-\tfrac{1}{2}\tilde{p}^{*2}\Gamma^{00}\right), \tag{5.26}$$

where

$$\Gamma^{00}=[\theta_-\cdot(B_1+B_2+B_3)^{-1}\cdot\theta_-]^{-1},$$
$$\Gamma^{r0}=\frac{2D_r\cdot(B_1+B_2+B_3)^{-1}\cdot\theta_-}{\theta_-\cdot(B_1+B_2+B_3)^{-1}\cdot\theta_-},$$
$$\Gamma^{rs}=B_s\delta_{rs}-4D_r\cdot(B_1+B_2+B_3)^{-1}\cdot D_s+4D_r\cdot(B_1+B_2+B_3)^{-1}\cdot\theta_-[\theta_-\cdot(B_1+B_2+B_3)^{-1}\cdot\theta_-]^{-1}\theta_-\cdot(B_1+B_2+B_3)^{-1}\cdot D_2.$$

These expressions for Γ^{rs} provide an alternative to the representation in Eq. (5.23). That they are in fact identical expressions follows immediately when we use the block inversion formula

$$(B_1+B_2+B_3)^{-1}=\begin{pmatrix} X & Y^T \\ Y & Z \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} (X-Y^TZ^{-1}Y)^{-1} & -(X-Y^TZ^{-1}Y)^{-1}Y^TZ^{-1} \\ -(Z-YX^{-1}Y^T)YX^{-1} & (Z-YX^{-1}Y^T)^{-1} \end{pmatrix} \tag{5.27}$$

to partition H_3^+ into H_3^{++} and the one-dimensional subspace containing θ_-.

The third method of evaluation uses the Fourier representation

$$\delta[\Pi_1'+\Pi_2'-\Pi_3']=\int Dy\, e^{iy\cdot(\Pi_1'+\Pi_2'-\Pi_3')}, \tag{5.28}$$

where $y(\sigma)$ spans the space H_3. By η conservation

$$\theta_1(\sigma)+\theta_2(\sigma)=\theta_3(\sigma), \tag{5.29}$$

so

$$\theta_1=\frac{\eta_1}{\eta_3}\theta_3+\frac{\pi\eta_1\eta_2}{\eta_3}\theta_-, \tag{5.30}$$

$$\theta_2=\frac{\eta_2}{\eta_3}\theta_3-\frac{\pi\eta_1\eta_2}{\eta_3}\theta_-. \tag{5.31}$$

Writing

$$\Pi_1'+\Pi_2'-\Pi_3'=\frac{1}{\pi\eta}\theta_3(p_{10}^{*\prime}+p_{20}^{*\prime}-p_{30}^{*\prime}) +\theta_-\left(\frac{\eta_2}{\eta_3}p_{10}^{*\prime}-\frac{\eta_1}{\eta_3}p_{20}^{*\prime}\right) +P_1^{*\prime}+P_2^{*\prime}-P_3^{*\prime}, \tag{5.32}$$

we can express the exponent in Eq. (5.28) as

$$(x_0^*+X^*)\cdot(\Pi_1'+\Pi_2'-\Pi_3')=x_0^*(p_{10}^{*\prime}+p_{20}^{*\prime}-p_{30}^{*\prime}) +X^*\cdot\theta_-\bar{p}^{*\prime} +X^*\cdot(P_1^{*\prime}+P_2^{*\prime}-P_3^{*\prime}).$$

The x_0^* integration can be performed together with the evaluation of the $p_0^{*\prime}$ integrals in Eq. (5.11) to produce $\delta(p_{10}^*+p_{20}^*-p_{30}^*)$. Integrating the three Gaussians over $P_1^{*\prime}, P_2^{*\prime}, P_3^{*\prime}$ successively produces a Gaussian in X^*. A final integration yields

$$\tilde{V}=\delta(p_{10}^*+p_{20}^*-p_{30}^*)[\det(B_1^{-1}+B_2^{-1}+B_3^{-1})]^{-d^*/2} \times\exp[-\tfrac{1}{2}Q^*\cdot(B_1^{-1}+B_2^{-1}+B_3^{-1})^{-1}\cdot Q^*], \tag{5.33}$$

where

$$Q^*(\sigma)=\bar{p}^*\theta_-+2P_1^*\cdot D_1\cdot B_1+2P_2^*\cdot D_2\cdot B_2^{-1} -2P_3^*\cdot D_3\cdot B_3^{-1}.$$

In this representation all kernels are initially inverted on their own subspace. The final inversion takes place in H_3^+. It is a nontrivial task to show that this form is equivalent to the previous representations, Eqs. (5.26) and (5.23).

The three forms Eqs. (5.33), (5.26), and (5.23), of the three-Reggeon vertex between open strings have been evaluated in terms of certain inverse kernels. In one form or another these kernels appear in Ref. 3. These authors show by a number of indirect methods that for asymptotic times they may be related to the Neumann functions discussed earlier by Mandelstam. They do not, however, make a precise comparison. In the work of Cremmer and Gervais an apparently different first-quantized interaction is used to generate the three-Reggeon vertex. This interaction imposes the condition that there are no discontinuous configurations allowed at the point of interaction. In this case, the authors make a precise comparison with the work of Mandelstam. Since their formalism is different from ours we have investigated the vertex that arises from the interaction:

$$F_1[y_1,y_2,y_3]=\delta[y_1+y_2-y_3]\,\delta(\epsilon_1\cdot y_1-\epsilon_2\cdot y_2), \tag{5.34}$$

where

$$\epsilon_i(\sigma)=\frac{1}{\pi\eta_i}\theta_i(\sigma)+E_i(\sigma) =\delta(\sigma-\pi\eta_1)\theta_i(\sigma).$$

Clearly the last δ function imposes the extra constraint $y_1(\pi\eta_1)=y_2(\pi\eta_1)$ in configuration space.

To compute the Fourier transform

$$\tilde{F}_1[\Pi_1,\Pi_2,\Pi_3]=\int Dy_1Dy_2Dy_3 \times\exp[i(\Pi_1\cdot y_1+\Pi_2\cdot y_2-\Pi_3\cdot y_3)] \times F_1[y_1,y_2,y_3] \tag{5.35}$$

we first integrate over y_3 with the δ functional and then separate out the zero modes in y_1, y_2, Π_1, and Π_2. In terms of the variables

$$u=\tfrac{1}{2}(x_{10}^*+x_{20}^*),$$

$$v=(x_{10}^*-x_{20}^*)$$

we have

$$\tilde{F}_1[\Pi_1,\Pi_2,\Pi_3]=\int_{-\infty}^{\infty}du\,dv\int DX_1^*DX_2^*\delta(v+E_1\cdot X_1^*-E_2\cdot X_2^*)$$
$$\times\exp\left\{iu(p_{10}^*+p_{20}^*-p_{30}^*)+iv\left(\tilde{p}^*-\frac{\pi\eta_1\eta_2}{\eta_3}P_3^*\cdot\theta_-\right)\right.$$
$$\left.+i[P_1^*\cdot X_1^*+P_2^*\cdot X_2^*-P_3^*\cdot(X_1^*+X_2^*)]\right\} \tag{5.36}$$

$$=\delta(p_{10}^*+p_{20}^*-p_{30}^*)\delta_{(1+)}[P_1^*-A^{13}\cdot P_3^*-\tilde{p}^*E_1]\delta_{(2+)}[P_2^*-A^{23}\cdot P_3^*+\tilde{p}^*E_2] \tag{5.37}$$

$$=\delta(p_{10}^*+p_{20}^*-p_{30}^*)\delta_{(3++)}\left[P_1^*+P_2^*-P_3^*+(E_1-E_2)\left(\frac{\pi\eta_1\eta_2}{\eta_3}P_3^*\cdot\theta_--\tilde{p}^*\right)\right], \tag{5.38}$$

where

$$A^{13}(\sigma,\sigma')=\underline{1}_{1+}-\frac{\pi\eta_1\eta_2}{\eta_3}E_1(\sigma)\theta_-(\sigma'),$$

$$A^{23}(\sigma,\sigma')=\underline{1}_{2+}+\frac{\pi\eta_1\eta_2}{\eta_3}E_2(\sigma)\theta_-(\sigma')$$

and we have made explicit the subspace in which each δ functional is defined. Using the form Eq. (5.37) and integrating out the zero modes together with $P_1^{*\prime}$ and $P_2^{*\prime}$ in Eq. (5.11) we obtain

$$\tilde{V}_1=\exp\left[-\tfrac{1}{2}\sum_j P_j^*\cdot B_j\cdot P_j^*-\tfrac{1}{2}\tilde{p}^{*2}(E_1\cdot B_1\cdot E_1+E_2\cdot B_2\cdot E_2)-2\tilde{p}^*P_1^*\cdot D_1\cdot E_1+2\tilde{p}^*P_2^*\cdot D_2\cdot E_2\right]\delta(p_{10}^*+p_{20}^*-p_{30}^*)$$
$$\times\int DP_3^{*\prime}\exp(-\tfrac{1}{2}P_3^{*\prime}\cdot\Delta_1\cdot P_3^{*\prime}+P_3^{*\prime}\cdot R^*) \tag{5.39}$$

where

$$\Delta_1(\sigma,\sigma')=B_3+A^{31}\cdot B_1\cdot A^{13}+A^{32}\cdot B_2\cdot A^{23},$$

$$R^*(\sigma)=2D_3\cdot P_3^*+2A^{31}\cdot D_1\cdot P_1^*+2A^{32}\cdot D_2\cdot P_2^*+\tilde{p}^*A^{31}\cdot B_1\cdot A^{13}-\tilde{p}^*A^{32}\cdot B_2\cdot A^{23}.$$

Integrating over $P_3^{*\prime}$ and simplifying the resulting expression we obtain

$$\tilde{V}_1=\delta(p_{10}^*+p_{20}^*-p_{30}^*)(\det\Delta_1)^{-d^*/2}\exp\left(-\tfrac{1}{2}\sum_{r,s}P_r^*\cdot\Gamma_1^{rs}\cdot P_s^*-\sum_r P_r^*\cdot\Gamma_1^{r0}\tilde{p}^*-\tfrac{1}{2}\tilde{p}^{*2}\Gamma^{00}\right),$$

where (5.40)

$$\Gamma_1^{00}=\theta_-\cdot B_3\cdot\theta_--\theta_-\cdot B_3\cdot\Delta_1^{-1}\cdot B_3\cdot\theta_-,$$
$$\Gamma_1^{r0}=-2D_r\cdot A^{r3}\cdot\Delta_1^{-1}\cdot B_3\cdot\theta_-,$$
$$\Gamma_1^{rs}=B_r\delta_{rs}-4D_r\cdot A^{r3}\cdot\Delta_1^{-1}\cdot A^{3s}\cdot D_s.$$

Comparing Γ_1^{rs} with Γ^{rs} we see that the effect of using the functional Eq. (5.34) is to insert the kernels A^{r3} into the expressions for Γ_1^{rs}. To determine the difference between $\tilde{V}_1$ and $\tilde{V}$ it is worthwhile to follow CG and relate the large-time forms of Γ^{rs} and Γ_1^{rs} to the Neumann coefficients obtained by Mandelstam[5] (see also Ref. 16).

To do this we first extract the leading τ dependence from the kernels in Eqs. (5.23) and (5.40). The use of a Wick-rotated time for the calculation of the Neumann coefficients has been discussed by Mandelstam. In essence, it enables one to map the t,σ mesh on the space-time surface swept out by the interacting strings onto the complex $\tau+i\sigma$ plane. The conformal invariance of the action is then exploited to map the appropriate Neumann function for a simple domain into the function for the desired string domain with a Schwarz-Christoffel transformation (see Appendix C). The large-τ behavior of the Neumann function is then continued to real t. For large $\tau_i-\tau_0$ ($\tau_i>\tau_0$, all $\eta_i>0$) we observe

$$B_i \to G_i^{-1}(1+2L_i^2),$$
$$D_i \to G_i^{-1}L_i \tag{5.41}$$

so that

$$\Gamma^{rs} \to G_r^{-1}\delta_{rs} + L_r \cdot N^{rs} \cdot L_s , \tag{5.42}$$

where

$$\begin{aligned} N^{00} &= \theta_- \cdot G_3^{-1} \cdot \theta_- - \theta_- \cdot G_3^{-1} \cdot (\Delta_{++})^{-1} \cdot G_3^{-1} \cdot \theta_- ,\\ N^{r0} &= 2G_r^{-1} \cdot \theta_- \delta_{r3} - 2G_r^{-1} \cdot (\Delta_{++})^{-1} \cdot \theta_- ,\\ N^{rs} &= 2G_r^{-1}\delta_{rs} - 4G_r^{-1} \cdot (\Delta_{++})^{-1} \cdot G_s^{-1} ,\\ (\Delta_{++})^{-1} &= [G_1^{-1} + G_2^{-1} + (G_{3++})^{-1}]^{-1} \in H^{++} . \end{aligned} \tag{5.43}$$

In terms of

$$\Delta = G_1^{-1} + G_2^{-1} + G_3^{-1} \in H_3^+$$

we note

$$(\Delta_{++})^{-1} = \Delta^{-1} - \frac{(\Delta^{-1} \cdot \theta_-)(\theta_- \cdot \Delta^{-1})}{\theta_- \cdot \Delta^{-1} \cdot \theta_-} . \tag{5.44}$$

Performing the same limits for the kernels in Eq. (5.40) we obtain the same expressions as above for N^{rs} except that $(\Delta_{++})^{-1}$ is replaced by Δ_1^{-1}, where

$$\Delta_1 = A^{31} \cdot G_1^{-1} \cdot A^{13} + A^{32} \cdot G_2^{-1} \cdot A^{23} + G_3^{-1} \in H_3^+ . \tag{5.45}$$

The problem now is to relate Δ_1^{-1} and $(\Delta_{++})^{-1}$ to Mandelstam's Neumann function.

We first consider the inversion of Δ_{++}. The essential details of the relations we need may be found in Appendix D. Define on H_3^+ the kernels $\overline{D}^{rs}$ with matrix elements

$$\overline{D}^{rs}_{mn} = -\frac{\alpha_1\alpha_2\alpha_3}{\alpha_r\alpha_s} \frac{m^2n^2}{(\alpha_s m + \alpha_r n)} \bar{a}^{(r)}_m \bar{a}^{(s)}_n , \tag{5.46}$$

and the Mandelstam coefficients with our conventions are given in Eq. (D6). In terms of the matrix δ defined by

$$\delta = \hat{G} - [-1]\overline{D}^{33}[-1] , \tag{5.47}$$

where

$$\hat{G}_{mn} = \eta_r (G_r)_{mn} = m\,\delta_{mn} ,$$
$$[-1]_{mn} = (-1)^m \delta_{mn}$$

we prove that

$$\begin{aligned} \delta W^{31} &= -\left(\frac{\eta_3}{\eta_1}\right)^{1/2}[-1]\overline{D}^{31} ,\\ \delta W^{32} &= -\left(\frac{\eta_3}{\eta_2}\right)^{1/2}[-1]\overline{D}^{32} . \end{aligned} \tag{5.48}$$

In terms of the $f_n^3(\sigma)$ basis in H_3^+ the matrix elements of Δ are

$$\Delta = \eta_1 W^{31}\hat{G}^{-1}W^{13} + \eta_2 W^{32}\hat{G}^{-1}W^{23} + \eta_3\hat{G}^{-1} . \tag{5.49}$$

Thus

$$\begin{aligned} \delta\Delta\delta &= \eta_3[-1]\overline{D}^{31}\hat{G}^{-1}\overline{D}^{13}[-1]\\ &\quad + \eta_3[-1]\overline{D}^{32}\hat{G}^{-1}\overline{D}^{23}[-1]\\ &\quad + \eta_3\delta\hat{G}^{-1}\delta . \end{aligned} \tag{5.50}$$

But

$$\delta\hat{G}^{-1}\delta = \hat{G} - [-1](2\overline{D}^{33} + \overline{D}^{33}\hat{G}^{-1}\overline{D}^{33})[-1] \tag{5.51}$$

using

$$[-1]\hat{G}^{-1}[-1] = \hat{G}^{-1} .$$

With the aid of the further relation[17]

$$\sum_{r=1}^{3} \overline{D}^{3r}\hat{G}^{-1}\overline{D}^{r3} = \hat{G}$$

we have therefore

$$\delta\Delta\delta = 2\eta_3\delta . \tag{5.52}$$

Before solving for Δ^{-1} we observe in Appendix D that

$$\delta\theta_- = 0 . \tag{5.53}$$

The vanishing of this vector means that, as a matrix in H_3^+, δ is bordered by the zeros of elements containing components in the θ_- subspace and is consequently singular. Therefore we may invert δ only on a subspace of H_3^+. Inverting on H_3^{++} Eq. (5.52) gives the relation

$$(\Delta_{++})^{-1} = \frac{1}{2\eta_3}\delta \tag{5.54}$$

and so from Eq. (5.42)

$$N^{33} = \frac{2}{\eta_3} G_3^{-1}[-1]\overline{D}^{33}[-1]G_3^{-1} . \tag{5.55}$$

Returning to the inversion of Δ_1 we observe that

$$\delta\Delta_1\delta = 2\eta_3\delta , \tag{5.56}$$

despite the presence of different overlap coefficients A^{rs} in Δ_1. Consequently we must write

$$(\Delta_{1++})^{-1} = \frac{\delta}{2\eta_3} . \tag{5.57}$$

It is at this point that we depart from the conclusion[17] of Cremmer and Gervais, who assume that δ is nonsingular on H_3^+, and we write $(\Delta_1)^{-1}$ on the right-hand side of Eq. (5.57). Taking into account the difference between Δ_1^{-1} and Δ_{1++}^{-1} we are led to the conclusion that the functional F_1 leads to a different vertex than that based on F, the latter agreeing with Mandelstam's asymptotic form. That Δ_1 and Δ are the same on H_3^{++} follows from the relations

$$A^{3r} = W^{3r} + \frac{\pi\eta_1\eta_2}{\eta_3}(\theta_-)(E_r)(-1)^r , \quad r = 1, 2 \tag{5.58}$$

$$\delta\cdot A^{3r}=\delta\cdot W^{3r},\quad r=1,2. \tag{5.59}$$

To determine the precise difference between $\Delta_1{}^{-1}$ and $(\Delta_{1++})^{-1}\equiv\delta$ we must compute the former inverse using the block inversion formula, Eq. (5.27). Since

$$(X-Y\cdot Z^{-1}\cdot Y)^{-1}=X^{-1}+\frac{(X^{-1}\cdot Y)(Y\cdot X^{-1})}{(Z-Y\cdot X^{-1}\cdot Y)} \tag{5.60}$$

and

$$-Z^{-1}\cdot Y\cdot(X-Y\cdot Z^{-1}\cdot Y)^{-1}=\frac{-Y\cdot X^{-1}}{(Z-Y\cdot X^{-1}\cdot Y)}, \tag{5.61}$$

where Z is a single element, we can write

$$\begin{vmatrix} X & Y^T \\ Y & Z\end{vmatrix}^{-1}=\begin{vmatrix} X^{-1}+\dfrac{(X^{-1}\cdot Y)(Y\cdot X^{-1})}{Z-Y\cdot X^{-1}\cdot Y} & \dfrac{X^{-1}\cdot Y}{Z-Y\cdot X^{-1}\cdot Y} \\ \dfrac{-Y\cdot X^{-1}}{Z-Y\cdot X^{-1}\cdot Y} & \dfrac{1}{Z-Y\cdot X^{-1}\cdot Y}\end{vmatrix}. \tag{5.62}$$

Taking $X=\Delta_{1++}$, $X^{-1}=\delta$, $Y=\underline{1}^{++}\cdot\Delta_1\cdot\theta_-$, and $Z=\theta_-\cdot\Delta_1\cdot\theta_-$ we evaluate first the denominator function

$$(Z-Y\cdot X^{-1}\cdot Y)=\theta_-\cdot(A^{31}\cdot G_1{}^{-1}\cdot A^{13}+A^{32}\cdot G_2{}^{-1}\cdot A^{23}+G_3{}^{-1})\cdot\theta_- -\theta_-\Delta_1\cdot\underline{1}^{++}\cdot\frac{\delta}{2\eta_3}\cdot\underline{1}^{++}\cdot\Delta_1\cdot\theta_-. \tag{5.63}$$

Using the relations

$$\theta_-\cdot A^{3r}=(-1)^r\cdot E_r,\quad \underline{1}^{++}\cdot A^{3r}=\underline{1}^{r+},$$
$$E_rA^{3s}=E_r\delta_{rs},\quad r,s=1,2 \tag{5.64}$$

$$\theta_-\cdot\Delta_1\cdot\underline{1}^{++}=-E_1\cdot G_1{}^{-1}+E_2\cdot G_2{}^{-1}+\theta_-\cdot G_3{}^{-1} \tag{5.65}$$

$$=(-E_1+E_2)(2\eta_3\delta^{-1}-G_{3++}{}^{-1}), \tag{5.66}$$

we can obtain

$$\theta_- -\theta_-\cdot\Delta_1\cdot\delta/2\eta_3=\theta_-+(E_1-E_2)\cdot(\underline{1}^+-G_{3++}{}^{-1}\delta/2\eta_3)+\theta_-\cdot G_3{}^{-1}\cdot\delta/2\eta_3$$
$$=(\epsilon_1-\epsilon_2)\cdot(\underline{1}^+-G_3{}^{-1}\cdot\delta/2\eta_3). \tag{5.67}$$

We have used the result that E_1 and E_2 belong to the subspaces H_1^+ and H_2^+, respectively, and that $\theta_-+E_1-E_2=\epsilon_1-\epsilon_2$. We conclude that the denominator function Eq. (5.63) is

$$Z-Y\cdot X^{-1}\cdot Y=(\theta_-\cdot\Delta_1\cdot\delta/2\eta_3-\theta_-)\cdot\Delta_1\cdot\theta_-$$
$$=(\epsilon_1-\epsilon_2)\cdot(\underline{1}^+-G_3{}^{-1}\cdot\delta/2\eta_3)\cdot\Delta_1\cdot\theta_-. \tag{5.68}$$

Using Eq. (5.67) again gives

$$Z-Y\cdot X^{-1}\cdot Y=(\epsilon_1-\epsilon_2)\cdot(\Delta_1\cdot\theta_- -G_3{}^{-1}\cdot\theta_-)+(\epsilon_1-\epsilon_2)\cdot G_3{}^{-1}\cdot(\underline{1}^+-\delta\cdot G_3{}^{-1}/2\eta_3)\cdot(\epsilon_1-\epsilon_2)$$
$$=(\epsilon_1-\epsilon_2)\cdot(G_3{}^{-1}-G_3{}^{-1}\cdot\delta\cdot G_3{}^{-1}/2\eta_3)\cdot(\epsilon_1-\epsilon_2)$$
$$=\alpha. \tag{5.69}$$

With the aid of projection operators we write Eq. (5.62) as

$$\Delta_1{}^{-1}=\underline{1}^{++}\cdot\left[\delta+(\delta\cdot\Delta\cdot\theta_-)\frac{1}{\alpha}(\theta_-\cdot\Delta\cdot\delta)\right]\underline{1}^{++}-(\delta\cdot\Delta\cdot\theta_-)\frac{1}{\alpha}(\theta_-)-(\theta_-)\frac{1}{\alpha}(\theta_-\cdot\Delta\cdot\delta)+(\theta_-)\frac{1}{\alpha}(\theta_-)$$
$$=\delta+(\theta_- -\delta\cdot\Delta\cdot\theta_-)\frac{1}{\alpha}(\theta_- -\theta_-\cdot\Delta\cdot\delta). \tag{5.70}$$

The required relation between $\Delta_1{}^{-1}$ and δ is then

$$\Delta_1^{-1} = \frac{\delta}{2\eta_3} + \frac{(G_3 - \delta/2\eta_3)\cdot G_3^{-1}\cdot(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_2)\cdot G_3^{-1}\cdot(G_3-\delta/2\eta_3)}{(\epsilon_1-\epsilon_2)\cdot G_3^{-1}\cdot(G_3-\delta/2\eta_3)\cdot G_3^{-1}\cdot(\epsilon_1-\epsilon_2)}\,. \qquad (5.71)$$

We conclude that the Fourier coefficients based on the kernel Δ_1^{-1} differ from those calculated from the kernel $(\Delta_{1++})^{-1}$.

Although we can compare our kernels with the established Neumann coefficients for large times, we do not claim to have closed expressions for the inverses at arbitrary times. These are required in our formalism if we construct the four-Reggeon vertex from a pair of three-Reggeon vertices and the functional propagator Eq. (4.18). The four-Reggeon vertex involves an integration over the interaction time difference. Cremmer and Gervais[4] manage to accommodate this time difference in their formalism by exploiting an over-all time invariance in their operator expressions. The result is that the effective finite-time Neumann coefficients have a trivial time dependence, at least when used to construct on-mass-shell amplitudes. Their procedure is designed to produce the known on-mass-shell dual amplitude. In the formalism discussed in this paper there appears to be no ambiguity in the propagator functionals and both on- and off-mass-shell amplitudes should be considered together in terms of finite-time kernels. Although we do not offer any direct methods for constructing general finite-time inverse kernels (the power series expansions, for example, are not particularly useful), we outline in Appendix C the Fourier expansion of some finite-time Neumann kernels by utilizing the appropriate Schwarz-Christoffel transformation. In this manner we reproduce Mandelstam's coefficients[5] in the large-time limit. If we accept the proof by KK[3] that the kernels in Eq. (5.23) are simply related to the appropriate Neumann function, then the finite-time coefficients follow from the expressions given in Appendix C. We feel, however, that there should be a more direct approach to the explicit calculation of these kernels, particularly if one is to contemplate kernels corresponding to more general string topologies.

Before closing this section we outline how the general excited state amplitude Eq. (5.10) is obtained from the vertex. The threefold functional integral involves Hermite functionals as well as a coupled Gaussian containing the Neumann kernels. The polynomial dependence is handled by writing Eq. (5.10) in the form

$$\begin{aligned} T_{\lambda^*_{n_1}\lambda^*_{n_2}\lambda^*_{n_3}} = {} & \lambda(-i)^{\lambda^*_{n_1}+\lambda^*_{n_2}+\lambda^*_{n_3}}\left\{\prod_{i=1}^{3}\det(1-L_i^{\,2})\right\}^{-d^*/2}\delta(p^*_{10}+p^*_{20}-p^*_{30})e^{-\tilde{p}^{*2}\Gamma^{00}/2} \\ & \times H_{\lambda^*_{n_1}}\left[if^1_{n_1}\cdot G_1^{-1}\cdot\frac{\delta}{\delta j^*_1}\right]H_{\lambda^*_{n_2}}\left[if^2_{n_2}\cdot G_2^{-1}\cdot\frac{\delta}{\delta j^*_2}\right] \\ & \times H_{\lambda^*_{n_3}}\left[if^3_{n_3}\cdot G_3^{-1}\cdot\frac{\delta}{\delta j^*_3}\right]W[j^*_1,j^*_2,j^*_3]\Big|_{j^*_r(\sigma)=\Gamma^{r0}(\sigma)\tilde{p}^*}, \end{aligned} \qquad (5.72)$$

where

$$W[j^*_1,j^*_2,j^*_3] = \int DP^*_1 DP^*_2 DP^*_3 \exp\left[-\tfrac{1}{2}\sum_{r,s}P^*_r\cdot(\Gamma^{rs}+G_r^{-1}\delta_{rs})\cdot P^*_s - \sum_r P^*_r\cdot j^*_r\right].$$

Since W is evaluated in terms of a coupled Gaussian in j^*_r, the results of Appendix B can be used to evaluate $T_{\lambda^*_{n_1}\lambda^*_{n_2}\lambda^*_{n_3}}$.

VI. FOUR-REGGEON VERTICES

Once the three-Reggeon vertex associated with a particular interaction functional has been constructed, one can write down higher-order vertices. As we have already noted, there are now integrations over the various time orderings involved. It is instructive to illustrate our formalism by deriving the form of the kernels that arise in the four-Reggeon vertex containing a direct-channel propagator.

We consider the fusing of open strings one and two at time t_a into a third open string. This propagates to time t_b befor it fissions into open strings three and four. For a specific interaction functional $\tilde{F}[\Pi_1,\Pi_2,\Pi_3]$ we define the four-Reggeon vertex (with a propagator from t_a to t_b) to be

$$\begin{aligned} V_{t_a-t_b}[\Pi_1,t_1,\Pi_2,t_2,\Pi_3,t_3,\Pi_4,t_4] = {} & \int\prod_{r=1}^{4}D\Pi'_r D\Pi_a D\Pi_b K_{\eta_1}[\Pi_1,t_1,|\Pi'_1,t_a]K_{\eta_2}[\Pi_2,t_2|\Pi'_2,t_a] \\ & \times\tilde{F}[\Pi'_1,\Pi'_2,\Pi_a]K_{\eta_1+\eta_2}[\Pi_a,t_a|\Pi_b,t_b]\tilde{F}[\Pi_b,\Pi'_3,\Pi'_4] \\ & \times K_{\eta_3}[\Pi'_3,t_b|\Pi_3t_3]K_{\eta_4}[\Pi'_4t_b|\Pi_4t_4]. \end{aligned} \qquad (6.1)$$

We consider first the interaction Eq. (5.8). The zero-mode integrations produce over-all conservation of p_0^*:

$$\int \prod_{r=1}^{4} dp_{r0}^* dp_{a0}^* dp_{b0}^* \delta(p_{r0}^* - p_{r0}^{*\prime})\delta(p_{10}^{*\prime} + p_{20}^{*\prime} - p_{a0}^*)\delta(p_{a0}^* - p_{b0}^*)\delta(p_{b0}^* - p_{30}^{*\prime} - p_{40}^{*\prime}) = \delta(p_{10}^* + p_{20}^* - p_{30}^* - p_{40}^*). \quad (6.2)$$

Performing the integrations over P_a^* and P_b^* with the δ functionals sets

$$P_a^* = P_1^{*\prime} + P_2^{*\prime} + \theta_-^{12}\tilde{p}_{12}^*,$$
$$P_b^* = P_3^{*\prime} + P_4^{*\prime} + \theta_-^{34}\tilde{p}_{34}^*, \quad (6.3)$$

where

$$\theta_-^{rs}(\sigma) = \frac{1}{\pi\eta_r}\theta_r(\sigma) - \frac{1}{\pi\eta_s}\theta_s(\sigma)$$

and

$$\tilde{p}_{rs}^* = \frac{\eta_r}{\eta_r + \eta_s} p_{r0}^* - \frac{\eta_s}{\eta_r + \eta_s} p_{s0}^*.$$

Using the propagators Eq. (4.18) we can perform the remaining integrations over $P_r^{*\prime}$ to obtain

$$V_{t_a - t_b} = \lambda\delta(p_{10}^* + p_{20}^* - p_{30}^*) \prod_{r=1}^{2} [\det(1 - L_{ra}^2)]^{-d^*/2} \prod_{s=3}^{4} [\det(1 - L_{bs}^2)]^{-d^*/2} A_{t_a - t_b}, \quad (6.4)$$

where

$$A_{t_a - t_b} = \{\det(B_1 + B_2 + B_{ab}^{++})(B_3 + B_4 + B_{ab}^{++})[1 - 4(B_1 + B_2 + B_{ab}^{++})^{-1} D_{ab} (B_3 + B_4 + B_{ab}^{++})^{-1} D_{ab}]\}^{-d^*/2}$$
$$\times \exp\left(-\tfrac{1}{2}\sum_{r=1}^{4} P_r^* \cdot B_r \cdot P_r^* - \tfrac{1}{2}\tilde{p}_{12}^{*2}\theta_-^{12} \cdot B_{ab} \cdot \theta_-^{12} - \tfrac{1}{2}\tilde{p}_{34}^{*2}\theta_-^{34} \cdot B_{ab} \cdot \theta_-^{34} + 2\tilde{p}_{12}^*\tilde{p}_{34}^*\theta_-^{12} \cdot B_{ab} \cdot \theta_-^{34} + \tfrac{1}{2}\tilde{P}^* \cdot \Omega^{-1} \cdot \tilde{P}^*\right),$$

$$\tilde{P}^*(\sigma) = \begin{pmatrix} 2\sum_{r=1}^{2} P_r^* \cdot D_r + 2\tilde{p}_{34}^*\theta_-^{34} \cdot D_{ab} - \tilde{p}_{12}^*\theta_-^{12} \cdot B_{ab} \\ 2\sum_{r=3}^{4} P_r^* \cdot D_r + 2\tilde{p}_{12}^*\theta_-^{12} \cdot D_{ab} - \tilde{p}_{34}^*\theta_-^{34} \cdot B_{ab} \end{pmatrix}, \quad (6.5)$$

$$\Omega(\sigma, \sigma') = \begin{pmatrix} B_{12} + B_{ab}^{++} & -2D_{ab} \\ -2D_{ab} & B_{34} + B_{ab}^{++} \end{pmatrix}.$$

In these formulas the time-dependent kernels B_{ab}, D_{ab}, etc. are defined as in Eq. (5.12) in terms of bases on the appropriate string with the corresponding time difference $t_a - t_b$, etc. The general excited state amplitude follows if we integrate $V_{t_a - t_b}$ over $t_a - t_b$ and sew on the external wave function with functional integrals over the variables P_r^*. We refer to the papers by KK[3] and CG[4] for further discussion of the various amplitudes that arise from different time orderings.

If instead of Eq. (5.8) we use the interaction functional given by Eq. (5.34), a similar calculation produces the expression

$$A_{t_a - t_b}^{(1)} = (\det \Omega_{(1)})^{-d^*/2} \exp\Big[-\tfrac{1}{2}\sum_{r=1}^{4} P_r^* \cdot B_r \cdot P_r^* + 2\tilde{p}_{12}^{*2}(E_1 \cdot D_1 \cdot P_1^* - E_2 \cdot D_2 \cdot P_2^*) + 2\tilde{p}_{34}^{*2}(E_3 \cdot D_3 \cdot P_3^* - E_4 \cdot D_4 \cdot P_4^*)$$
$$- \tfrac{1}{2}\tilde{p}_{12}^{*2}(E_1 \cdot B_1 \cdot E_1 + E_2 \cdot B_2 \cdot E_2) - \tfrac{1}{2}\tilde{p}_{34}^{*2}(E_3 \cdot B_3 \cdot E_3 - E_4 \cdot B_4 \cdot E_4) + \tfrac{1}{2}\tilde{P}_{(1)}^* \cdot \Omega_{(1)}^{-1} \cdot \tilde{P}_{(1)}^*\Big], \quad (6.6)$$

where

$$\tilde{P}_{(1)}^*(\sigma) = \begin{pmatrix} 2P_1^* \cdot D_1 \cdot A^{1a} + 2P_2^* \cdot D_2 \cdot A^{2a} - \tilde{p}_{12}^*(E_1 \cdot B_1 \cdot A^{1a} - E_2 \cdot B_2 \cdot A^{2a}) \\ 2P_3 \cdot D_3 \cdot A^{3b} + 2P_4^* \cdot D_4 \cdot A^{4b} - \tilde{p}_{34}^*(E_3 \cdot B_3 \cdot A^{3b} - E_4 \cdot B_4 \cdot A^{4b}) \end{pmatrix}$$

and (6.7)

$$\Omega_{(1)}(\sigma, \sigma') = \begin{pmatrix} \Delta_{12} & 2D_{ab} \\ 2D_{ab} & \Delta_{34} \end{pmatrix}.$$

VII. CONCLUSIONS

In this paper we have developed a formalism to describe the interaction between quantized strings in close analogy with conventional field theory. Little use has been made of the first-quantized mode operators that usually make an appearance in the calculation of dual amplitudes. Whether this is a positive virtue depends on how one handles the problem of finite-times vertices. In our approach there is little alternative to the direct inversion of the kernels that arise during the calculations. However, we feel that apart from this technical obstacle, the formalism is natural and unambiguous. The question does arise as to the nature of the resulting amplitudes and their relation (on and off the mass shell) to the conventional Veneziano model. With the interaction functional $\delta[\Pi_1+\Pi_2-\Pi_3]$ between three open strings, the three-Reggeon vertex of the dual model is obtained provided care is taken when dealing with the zero-mode components of the fields Π_r. We have seen that the coupling of these modes to the internal modes at the time of fusion is responsible for the distinction between this interaction and the one used by Cremmer and Gervais in their formalism.[17]

There are many facets of string field theory that we have not discussed here. Kaku and Kikkawa[3] and Mandelstam[12] have shown that a four-string contact interaction is necessary for consistency with crossing and Lorentz invariance of the dual four-string amplitude. It is clear that this interaction and any others involving the fields at an instant in time can be readily assimilated into our formalism with the correct choice of the functional F. Having established bases for open and closed strings with the same parametric range, it is also straightforward to write down interactions between them. Kaku and Kikkawa[18] have discussed some of these interactions and drawn attention to the extra invariance possessed by the closed string. From our expression for the open- and closed-string propagators one can construct further mixed propagators. For example, the transition propagator from an open to a closed topology is

$$T^{oc}_{\eta,t}[\Pi^o,t_1|\Pi^c,t_2] = \int DX^{*o}DX^{*c}DX^{*\prime o}DX^{*\prime c}\,e^{i(\Pi^o\cdot X^{*o}-\Pi^c\cdot X^{*c})} \times K_\eta[X^{*o},t_1|X^{*\prime o},t]\,\delta[X^{*\prime o}-X^{*\prime c}] \times K_\eta[X^{*\prime c},t|X^{*c},t_2]. \tag{7.1}$$

That different topologies are handled in a unified way is a general feature of the formalism. Once a basis is established for a more complex topology, the kernel $G(\sigma,\sigma')$ can be constructed and the quantum theory of the free system developed from the appropriate Hamiltonian. Quantum transitions between topologies can then be formulated with δ functionals involving fields with different bases as in Eq. (8.1). Goldstone's[19] three-string baryon model is an interesting topology to investigate with these techniques.

We have not attempted to impose Lorentz invariance on the theory in this paper. However, it is well known[2] that the Poincaré algebra of the dual model will only close when d^* and m_0^2 are constrained. It is not difficult to represent the generators of the Poincaré group in terms of free-field functionals. One expects similar constraints in the string field theory.[3] It would, however, be useful to construct the theory in a manifestly covariant fashion. The transition from Eq. (3.12) to a Dirac-Ramond type of equation[20] would then follow naturally. Once intrinsic spin is incorporated into the theory[21] the variety of interactions increases considerably. It is likely that the spin-orbit coupled nonlinear dual models[22] would find a string interpretation. Indeed, if string models are to have any relevance to reality, one is in need of some principle to accommodate the many exotic string topologies (and the transitions between them) that *a priori* should be considered. Despite the complexities the functional calculus is a convenient tool to use in these investigations.

APPENDIX A

Functional integrals are in general notoriously difficult objects to define. We have adopted in this paper the viewpoint that all our manipulations with functional integrals can be independently carried out using instead the limit of an infinitely repeated integral. Once a basis for a field is established:

$$X(\sigma)=\sum_n X_n f_n(\sigma); \tag{A1}$$

we define

$$\int DX\,e^{-X\cdot G\cdot X/2} = \frac{1}{N}\int\cdots\int dx_1\cdots\exp\left(-\sum_{i,j}\tfrac{1}{2}X_i G_{ij}X_j\right), \tag{A2}$$

where in general

$$G(\sigma,\sigma')=\sum_{i,j}G_{ij}f_i(\sigma)f_j(\sigma')$$

and N is a convenient normalization factor. We are aware that there is no well-established way of defining the measure for an arbitrary functional integral. However, with the exception of the δ functional all our integrals contain Gaussian integrands. For the purposes of convergence of these integrals we demand that our fundamental kernels G, G^{-1} be positive-semidefinite so that they contain no negative eigenvalues. In these circumstances we can define an appropriate measure to render our integrals finite. We choose to establish normalizations for the X and P integrals by writing

$$\int DX\, e^{-X\cdot G\cdot X/2}=1, \tag{A3}$$

$$\int DP\, e^{-P\cdot G^{-1}\cdot P/2}=1. \tag{A4}$$

In terms of a Fourier mode representation

$$DX\equiv\prod_n^\infty d\left[\left(\frac{n}{\pi}\right)^{1/2}X_n\right], \tag{A5}$$

$$DP\equiv\prod_n^\infty d\left[\frac{p_n}{(4\pi n)^{1/2}}\right]. \tag{A6}$$

This measure automatically absorbs factors of det$2G$ which would otherwise make an appearance in the formalism. The domain of integration of the functional integrals is over all piecewise continuous functions that can be constructed on the support of the basis functions. With this convention all fields of integration carry their own support.

The functional integrals that occur in the text can all be generated from a single simple Gaussian integral

$$\int DQ\, e^{-Q\cdot A\cdot Q/2+iQ\cdot B}=\frac{e^{-B\cdot A^{-1}\cdot B/2}}{\{\det A\}^{1/2}}, \tag{A7}$$

where

$$DQ=\prod_n^\infty d\left[\frac{Q_n}{(2\pi)^{1/2}}\right].$$

Repeated application of this formula produces the important integral

$$\begin{aligned}\int DQDQ'\exp(-\tfrac{1}{2}Q\cdot A\cdot Q-\tfrac{1}{2}Q'\cdot A'\cdot Q'-Q\cdot B\cdot Q'+iQ\cdot\alpha+iQ'\cdot\alpha')\\ =[\det A\det A'(1-A'^{-1}\cdot B^T\cdot A^{-1}\cdot B)]^{-1/2}\\ \times\exp[-\tfrac{1}{2}\alpha\cdot(A-B\cdot A'^{-1}\cdot B^T)^{-1}\cdot\alpha-\tfrac{1}{2}\alpha'\cdot(A'-B^T\cdot A^{-1}\cdot B)^{-1}\cdot\alpha'+\alpha\cdot A^{-1}\cdot B\cdot(A'-B^T\cdot A^{-1}\cdot B)^{-1}\cdot\alpha'].\end{aligned} \tag{A8}$$

This formula is valid even when $Q(\sigma)$ and $Q'(\sigma)$ have different supports. In this case the last term may also be written as $\alpha\cdot(A-B\cdot A'^{-1}\cdot B^T)^{-1}\cdot B\cdot A'^{-1}\cdot\alpha'$. But if the fields have the same support this can be written $\alpha\cdot(A'\cdot B^{-1}\cdot A-B^T)^{-1}\cdot\alpha'$. Of frequent occurrence are the overlap coefficients W^{rs}_{mn} between two fields defined on the support of strings r and s,

$$\Pi_r\cdot\Pi_s=\sum_{m,n}\Pi^r_m\Pi^s_n W^{rs}_{mn}, \tag{A9}$$

where

$$W^{rs}_{mn}=f^r_m\cdot f^s_n.$$

In terms of our open-string bases, Eq. (5.5), with

$$f^r_0(\sigma)=\frac{\theta^r(\sigma)}{(\pi\eta_r)^{1/2}},$$

the three-string overlaps are

$$\begin{aligned}W^{31}_{00}&=\left(\frac{\eta_1}{\eta_3}\right)^{1/2},\\ W^{32}_{00}&=\left(\frac{\eta_2}{\eta_3}\right)^{1/2},\\ W^{31}_{m0}&=\left(\frac{2\eta_3}{\eta_1}\right)^{1/2}\frac{\sin(m\pi\eta_1/\eta_3)}{m\pi}\\ &=-W^{32}_{m0}\left(\frac{\eta_2}{\eta_1}\right)^{1/2},\\ W^{31}_{mj}&=\frac{2m}{\pi}(-1)^j\frac{\eta_3}{\eta_1}\frac{\sin(m\pi\eta_1/\eta_3)}{(m^2-j^2\eta_3^2/\eta_1^2)}\\ &=(-1)^{j+1}\left(\frac{\eta_2}{\eta_1}\right)^{1/2}W^{32}_{mj}\\ W^{31}_{0j}&=W^{32}_{0j}=0,\end{aligned} \tag{A10}$$

$m, j \geq 1$.

With these coefficients the Fourier decomposition of any kernel is easily rendered, e.g.,

$$(A^r \cdot B^s \cdot B^t \cdot C^w)_{mn} = \sum_{k,l,j} A^r_m W^{rs}_{mk} B^s_k W^{st}_{kl} B^t_{lj} W^{tw}_{jn} C^w_n \tag{A11}$$

is the m, n component in the r, w bases, respectively. In this example A^r, B^s, C^w are taken diagonal in the r, s, w bases, respectively.

APPENDIX B

We record here some useful representations for the Hermite functionals that feature in the free-string eigenfunctionals. It should be stressed that with the appropriate identification of the eigenvalues and eigenvectors of the fundamental kernel G this section is generally applicable to more than one string topology.

We write the general eigenfunctional equation in the form

$$(\delta \cdot \delta + 2\bar{\epsilon} - X \cdot \Gamma \cdot X) U[X] = 0, \tag{B1}$$

where $\delta(\sigma) = \delta/\delta X(\sigma)$ and we suppress Lorentz structure in this appendix for ease of presentation. We observe that $U_0 = \exp(-\frac{1}{2} X \cdot G \cdot X)$ is a solution provided

$$\begin{aligned} \bar{\epsilon} &= \tfrac{1}{2}\operatorname{Tr} G \\ G^2 \cdot X &= \Gamma \cdot X. \end{aligned} \tag{B2}$$

If Γ is a positive-semidefinite kernel there exists a positive-semidefinite square-root kernel G of Γ.

Writing $\bar{\epsilon} = \epsilon + \frac{1}{2}\operatorname{Tr} G$ we therefore seek solutions of

$$[(\delta - X \cdot G) \cdot (\delta + G \cdot X) + 2\epsilon] U[X] = 0 \tag{B3}$$

or, with

$$U[X] = e^{-X \cdot G \cdot X/2} \Omega[X], \tag{B4}$$

$$(\delta \cdot \delta - 2X \cdot \delta + 2\epsilon) \Omega[X] = 0. \tag{B5}$$

If we try the solution $\Omega[X] = \exp(\delta \cdot B \cdot \delta) V[X]$, we find

$$B = \tfrac{1}{4} G^{-1} \tag{B6}$$

(at least in the subspace spanned by eigenvectors of G with positive eigenvalues), where V must satisfy

$$(X \cdot G \cdot \delta - \epsilon) V[X] = 0. \tag{B7}$$

Let $f_n(\sigma)$ be the eigenvectors of $G(\sigma, \sigma')$ with eigenvalues $g_n > 0$

$$G \cdot f_n = g_n f_n. \tag{B8}$$

But

$$\begin{aligned} (X \cdot G \cdot \delta)(f_n \cdot X)^{\lambda_n} &= \lambda_n (X \cdot G \cdot f_n)(f_n \cdot X)^{\lambda_n - 1} \\ &= \lambda_n g_n (X \cdot f_n)^{\lambda_n}. \end{aligned} \tag{B9}$$

So for any collection of integers $\{\lambda_n \geq 0\}$ we have solutions

$$V_{\{\lambda_n\}}[X] = \prod_n (f_n \cdot X)^{\lambda_n}, \tag{B10}$$

$$\epsilon_{\{\lambda_n\}} = \sum_n \lambda_n g_n, \tag{B11}$$

or

$$\Omega_{\{\lambda_n\}}[X] = \exp(-\tfrac{1}{4}\delta \cdot G^{-1} \cdot \delta) \prod_n (f_n \cdot X)^{\lambda_n} \tag{B12}$$

$$= \prod_n [f_n \cdot (X - \tfrac{1}{2} G^{-1} \cdot \delta)]^{\lambda_n} \exp(-\tfrac{1}{4}\delta \cdot G^{-1} \cdot \delta) 1 \tag{B13}$$

$$= \exp(\tfrac{1}{2} X \cdot G \cdot X) \prod_n [\tfrac{1}{2} f_n \cdot (X - G^{-1} \cdot \delta)]^{\lambda_n} \exp(-\tfrac{1}{2} X \cdot G \cdot X) \tag{B14}$$

$$= \exp(X \cdot G \cdot X) \prod_n (-\tfrac{1}{2} f_n \cdot G^{-1} \cdot \delta)^{\lambda_n} \exp(-X \cdot G \cdot X). \tag{B15}$$

Each of these forms generates a functional polynomial in X. Consider now the action of $\frac{1}{2} f_n \cdot (X - G^{-1} \cdot \delta)$ on the functional

$$\Omega_{\{\lambda_n\}}[i G^{-1} \cdot \delta] \exp(-\tfrac{1}{2} X \cdot G \cdot X).$$

Since

$$[\tfrac{1}{2} f_k \cdot X, \Omega_{\{\lambda_n\}}[i G^{-1} \cdot \delta]] = -\tfrac{1}{2} f_k \cdot G \cdot \delta \; \Omega_{\{\lambda_n\}}[i G^{-1} \cdot \delta] \tag{B16}$$

we find

$$\tfrac{1}{2} f_k \cdot (X - G^{-1} \cdot \delta) \Omega_{\{\lambda_n\}}[i G^{-1} \cdot \delta] e^{-X \cdot G \cdot X/2} = i \left[f_k \cdot \left(i G^{-1} \cdot \delta - \frac{1}{2i} G \cdot \delta \right) \right] \Omega_{\{\lambda_n\}}[i G^{-1} \cdot \delta] e^{-X \cdot G \cdot X/2} \tag{B17}$$

$$= i \Omega_{\{\lambda_n + \delta_{nk}\}}[i G^{-1} \cdot \delta] e^{-X \cdot G \cdot X/2}. \tag{B18}$$

Hence from Eq. (B14) we have the simple result

$$U_{\{\lambda_n\}}[X] = i^{\Sigma\lambda_n}\Omega_{\{\lambda_n\}}[iG^{-1}\cdot\delta]e^{-X\cdot G\cdot X/2}. \quad \text{(B19)}$$

Multiplying by the normalization factor $N_{\{\lambda_n\}} = \prod_n (g_n/2\lambda_n!)^{1/2}$, to ensure the normalization Eq. (3.10) in the text, we can write the normalized eigenfunctional as

$$U_{\{\lambda_n\}}[X] = \prod_n i^{\lambda_n} H_{\lambda_n}(if_n\cdot G^{-1}\cdot\delta)e^{-X\cdot G\cdot X/2}, \quad \text{(B20)}$$

where $H_\lambda(y)$ is the classical Hermite polynomial.

To illustrate the utility of this last representation we calculate the wave functional on the momentum representation,

$$\begin{aligned}\tilde{U}_{\{\lambda_n\}}[P] &= \int DX\, e^{-P\cdot X}U_{\{\lambda_n\}}[X] \\ &= N_{\{\lambda_n\}}\int DX\, e^{-iP\cdot X}\Omega_{\{\lambda_n\}}[X]e^{-X\cdot G\cdot X/2} \\ &= N_{\{\lambda_n\}}\Omega_{\{\lambda_n\}}\left[i\frac{\delta}{\delta P}\right]\int DX\, e^{-X\cdot G\cdot X/2-iP\cdot X} \\ &= N_{\{\lambda_n\}}\Omega_{\{\lambda_n\}}\left[i\frac{\delta}{\delta P}\right]e^{-P\cdot G^{-1}\cdot P/2} \\ &= i^{-\Sigma\lambda_n}U_{\{\lambda_n\}}[G^{-1}\cdot P] \\ &= \prod_n i^{\lambda_n}H_{\lambda_n}(f_n\cdot G^{-1}\cdot P)e^{-P\cdot G^{-1}\cdot P/2}.\end{aligned} \quad \text{(B21)}$$

For the open string, the kernel G and the basis functions are defined in Eqs. (3.3) and (3.4). For the closed string

$$f_{2n}(\sigma) = \left(\frac{2}{\pi\eta}\right)^{1/2}\cos\frac{2n\sigma}{\eta}, \qquad f_{2n-1}(\sigma) = \left(\frac{2}{\pi\eta}\right)^{1/2}\sin\frac{2n\sigma}{\eta} \quad \text{(B22)}$$

$$g^d_{2n} = \frac{2n}{\eta}, \quad g^d_{2n-1} = \frac{2n}{\eta}. \quad \text{(B23)}$$

In both cases $0 \le \sigma \le \pi\eta$. In this notation both topologies have the internal propagator

$$\begin{aligned}K_\eta[P_1, P_2, t_1 - t_2] = [\det(1 - L^2)]^{-d^*/2} \times\exp(&-\tfrac{1}{2}P_1\cdot G^{-1}\cdot L_2\cdot P_1 \\ &-\tfrac{1}{2}P_2\cdot G^{-1}\cdot L_2\cdot P_2 \\ &+ 2P_1\cdot G^{-1}\cdot L_1P_2),\end{aligned} \quad \text{(B24)}$$

where

$$L(\sigma, \sigma', t_1 - t_2) = \sum_{n\ge 1} f_n(\sigma)e^{ig_n(t_1-t_2)}f_n(\sigma')$$

and L_1, L_2 are given by Eq. (4.13) in the text.

APPENDIX C

In this appendix we consider the evaluation of finite-time Neumann coefficients for interacting strings. In particular we shall show how the time dependence of these coefficients differs from the infinite-time case as well as illustrating some of the technical problems involved in dealing with finite-time Neumann functions.

The equations defining the Neumann function are[5]

$$\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right)N(\rho, \rho') = 2\pi\delta^2(\rho - \rho'), \quad \text{(C1)}$$

$$\frac{\partial}{\partial n_\rho}N(\rho, \rho') = f(\rho), \quad \text{(C2)}$$

where $\rho = \tau + i\sigma$, $\partial/\partial n_\rho$ is the normal derivative at the boundary of the area traced out in the ρ plane by the interacting strings, and $f(\rho)$ is an arbitrary function of ρ only. As pointed out by Mandelstam, the Neumann function defined in this way is conformally invariant. Since the Neumann function for the upper-half z plane is

$$N(z, z') = \ln|z - z'| + \ln|z - \bar{z}'| \quad \text{(C3)}$$

we can, in principle, find it for any string configuration once a transformation $z = z(\rho)$ mapping the area traced out in the ρ plane by the strings into the upper-half z plane is known. In practice it is easier to find the inverse of this mapping. If we limit ourselves to considering strings which trace out polygons in the ρ plane, then the mapping which takes these into the upper-half z plane is provided by the Schwarz-Christoffel transformation

$$\frac{d\rho}{dz} = A\prod_{r=1}^{n}(z - z_r)^{\alpha_r/\pi - 1}, \quad \text{(C4)}$$

where z_r is the point on the real z axis to which the vertex r is mapped and α_r is the internal angle of the polygon at that vertex (see Fig. 1). A is a constant determining the over-all size of the polygon.

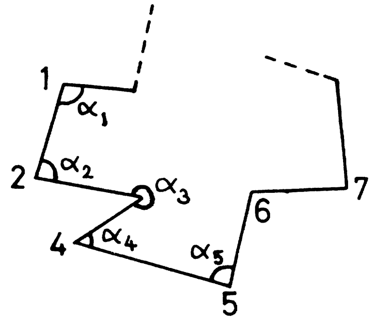

FIG. 1. A polygon in the ρ plane.

As particular examples we may consider the free-string and the vertex function diagrams shown in Figs. 2 and 3. For these the Schwarz-Christoffel transformations are

$$\rho(z)=A\int\frac{dz}{[(z-z_1)(z-z_2)(z-z_3)(z-z_4)]^{1/2}} \tag{C5}$$

and

$$\rho(z)=A\int dz[(z-z_1)(z-z_2)(z-z_3)(z-z_4)\times(z-z_6)(z-z_7)]^{-1/2}(z-z_5)\,. \tag{C6}$$

Allowing one of the arms to go to infinity corresponds to allowing an adjacent pair of z_r to coincide. If, for example, we take $z_1=z_2$, $z_3=z_4$, $z_6=z_7$, for the vertex we obtain

$$\rho(z)=A\int\frac{dz(z-z_5)}{(z-z_1)(z-z_3)(z-z_6)} =\alpha_1\ln(z-z_1)+\alpha_2\ln(z-z_2)+\alpha_3\ln(z-z_3)\,, \tag{C7}$$

where

$$\alpha_1=A\frac{(z_1-z_5)}{(z_1-z_3)(z_1-z_6)}\,,\quad \alpha_2=A\frac{(z_3-z_5)}{(z_3-z_1)(z_3-z_6)}\,,\quad \alpha_3=A\frac{(z_6-z_5)}{(z_6-z_1)(z_6-z_3)}\,. \tag{C8}$$

Note that

$$\alpha_1+\alpha_2+\alpha_3=0\,. \tag{C9}$$

Equation (C7) is the transformation for the string diagrams shown in Fig. 4. This is the transformation used by Mandelstam[5] for the infinite-time vertex.

Any three of the z_r may be chosen arbitrarily. It is often convenient to take one of the z_r, say z_j, to be at infinity. This may be effected by omitting the term $(z-z_j)$ from Eq. (C4). The transformation for the diagram of Fig. 2 may be written in terms of Jacobi elliptic functions, whereas that for Fig. 3 is not expressible in terms of standard function. As more manageable examples we shall consider the string diagrams shown in Figs. 5 and 6.

For Fig. 5 the Schwarz-Christoffel transformation

$$\rho(z)=A\int\frac{dz}{[(z-1)(z+1)]^{1/2}} =A\cosh^{-1}z+\text{const} \tag{C10}$$

which can be obtained from Eq. (C5) by putting $z_2=z_3$ to infinity and $z_1=-z_4=1$ thus using the available freedom in the choice of the z_r. We can rewrite Eq. (C10) as

$$z=\cosh\frac{(\rho-\tau_e)}{\eta}\,, \tag{C11}$$

where the arbitrary constants have been chosen so that the width of the string in the ρ plane is $\pi\eta$ and its end is at time $\tau=\tau_e$.

We can now write the Neumann function as

$$N(\rho,\rho')=\ln\left|\cosh\frac{(\rho-\tau_e)}{\eta}-\cosh\frac{(\rho'-\tau_e)}{\eta}\right| +\ln\left|\cosh\frac{(\rho-\tau_e)}{\eta}-\cosh\frac{(\bar\rho'-\tau_e)}{\eta}\right|\,. \tag{C12}$$

Expressed as a Fourier series, this becomes

$$N(\rho,\rho')=\frac{2}{\eta}(\tau-\tau_e)-2\ln2 -4\sum_{n=1}^{\infty}\frac{e^{-(n/\eta)(\tau-\tau_e)}}{n}\cosh\frac{n}{\eta}(\tau'-\tau_e) \times\cos\frac{n\sigma}{\eta}\cos\frac{n\sigma'}{\eta}\,. \tag{C13}$$

We note that if we take $\tau_e\to-\infty$,

$$\lim_{\tau_e\to-\infty}N(\rho,\rho')=\frac{2\tau}{\eta}-2\ln2 -2\sum_{n=1}^{\infty}\frac{e^{-(n/\eta)(\tau-\tau')}}{n}\cos\frac{n\sigma}{\eta}\cos\frac{n\sigma'}{\eta} \tag{C14}$$

up to an infinite constant. This is the infinite-strip Neumann function obtained by Mandelstam.[5]

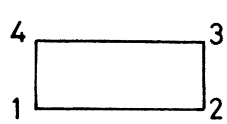

FIG. 2. ρ-plane diagram for the propagation of a free string between finite times.

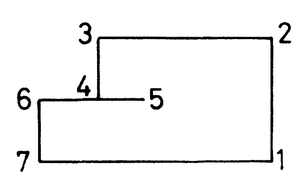

FIG. 3. ρ-plane diagram for the interaction between three strings propagating between finite times.

The Neumann coefficients for the infinite and semi-infinite strings differ primarily in their dependence on τ' (the time nearest the end of the strip). The simple exponential dependence on τ' for the infinite strip becomes a hyperbolic cosine dependence when the strip does not extend to infinity in both directions. It is interesting to note that if τ' is chosen so that it is always on the end of the semi-infinite strip (i.e., $\tau' = \tau_e$) the simple exponential dependence on τ' is regained. (However, the Neumann coefficients do not equal the infinite-strip coefficients.)

Unfortunately, the method of obtaining the Neumann coefficients used for the semi-infinite strip is difficult to apply to more general string configurations due to problems in inverting the Schwarz-Christoffel transformation. In order to study the vertex shown in Fig. 6 we proceed in a different manner. The Schwarz-Christoffel transformation mapping Fig. 6 into the upper-half z plane is given by

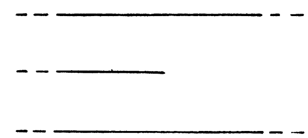

FIG. 4. ρ-plane diagram for the interaction between three strings propagating between asymptotic times.

$$\rho(z) = A \int \frac{z - z_0}{z[(z-1)(z-z_1)]^{1/2}}\, dz\,, \tag{C15}$$

where the correspondence between ρ and z plane points is shown in Fig. 7. Full use of the freedom in choosing the z_r has been made in order to obtain the simplest possible form for $\rho(z)$. Evaluating the integral we obtain

$$\begin{aligned}\rho(z) = {}& A \ln\left(2\exp\{\tfrac{1}{2}[\ln(1-z) + \ln(z_1 - z)]\} - 2z + 1 + z_1\right) \\ & - A\frac{z_0}{\sqrt{z_1}} \ln\left(2z_1 - (1+z_1)z + 2\sqrt{z_1}\exp\{\tfrac{1}{2}[\ln(1-z) + \ln(z_1 - z)]\}\right) + A\frac{z_0}{\sqrt{z_1}}\ln z\,,\end{aligned} \tag{C16}$$

with the usual convention that $\ln z$ is made single-valued by cutting the z plane from zero to minus infinity. A line drawn along the real-z axis going above all branch points maps into the boundary of the vertex in the ρ plane. In the limit as z_1 tends to unity we obtain the transformation for the vertex with all arms going to infinity:

$$\lim_{z_1 \to 1} \rho(z) = \eta_1 \ln z + \eta_3 \ln 4(1-z)\,, \tag{C17}$$

where

$$\eta_1 = A\frac{z_0}{\sqrt{z_1}}\,,$$

$$\eta_3 = A\left(1 - \frac{z_0}{\sqrt{z_1}}\right).$$

The transformation Eq. (C16) has a logarithmic branch point at $z=0$ and square-root branch points at 1 and z_1. With η_1 and η_3 defined as in Eq. (C17) the widths of arms one and three in Fig. 6 are $\pi\eta_1$ and $\pi\eta_3$, respectively. The Neumann coefficients where τ is taken across arm one and τ' across arm three are then given by

$$N_{nm} = \frac{4}{\pi^2\eta_1\eta_3}\int_0^{\pi\eta_3} d\sigma' \cos\frac{n\sigma'}{\eta_3} \times \int_0^{\pi\eta_1} d\sigma \cos\frac{m\sigma}{\eta_1} N(\rho, \rho')\,, \tag{C18}$$

$n, m \neq 0$. This can be rewritten as a contour integral[17]

$$N_{nm} = \frac{2}{\pi^2\eta_1\eta_3}\oint_{c_2} d\sigma' \cos\frac{n\sigma'}{\eta_3}\oint_{c_1} d\sigma \cos\frac{m\sigma}{\eta_1}\ln(z - z')\,, \tag{C19}$$

where the contours in the z and z' planes are the curves mapping to the dashed lines shown in Fig. 6 together with the reflections of these curves in

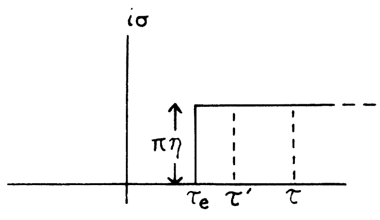

FIG. 5. The free string with one end at finite time.

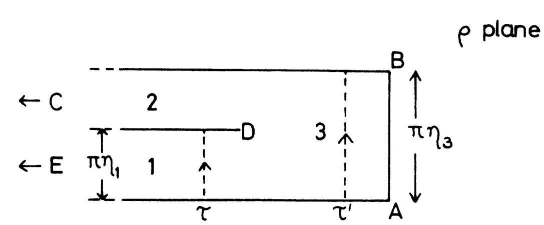

FIG. 6. The three-string vertex with one end at finite time.

the real axis (see Fig. 7). At first sight it seems likely that for n-string vertex diagrams the discontinuities in the integrands arising from the cuts along the real axis in $\rho(z)$ will lead to the contours being disjoint. However, these discontinuities vanish at the points where the contours meet the real axis, allowing the contours to be closed. When n and m are nonzero we can integrate by parts to obtain

$$N_{nm}=\frac{2}{\pi^2 nm}\oint_{c_2}dz\oint_{c_1}dz'\sin\frac{n\sigma'}{\eta_3}\sin\frac{m\sigma'}{\eta_1}\frac{1}{(z-z')^2}. \tag{C20}$$

Considering first the z integration we use Eq. (C16) to substitute for σ and then write $\sin(m\sigma/\eta_1)$ as a sum of exponentials. Only the $e^{-in\sigma/\eta_1}$ term has singularities inside c_1. (The pole at $z=z'$ lies outside c_1.) Thus

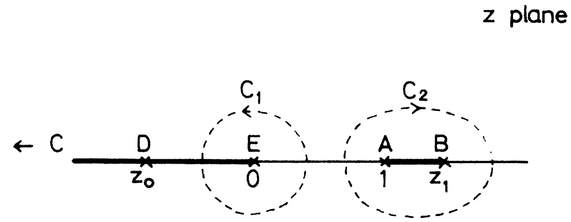

FIG. 7. Diagram illustrating the correspondence between z- and ρ-plane points for the vertex of Fig. 6 together with the contour used in evaluating its finite-time Neumann coefficients.

$$\begin{aligned}\oint_{c_1}dz\sin\frac{m\sigma}{\eta_1}\frac{1}{(z-z')^2}&=-\oint_{c_1}\frac{dz}{2i}\frac{z^{-m}}{(z-z')^2}e^{m\tau/\eta_1}\left\{2z_1-(1+z_1)z+2\sqrt{z_1}\exp\left[\frac{\ln(1-z)+\ln(z_1-z)}{2}\right]\right\}^m\\&\quad\times\left\{2\exp\left[\frac{\ln(1-z)+\ln(z_1-z)}{2}\right]-2z+1+z_1\right\}^{-m\sqrt{z_1}/z_0}\\&=-\pi e^{m\tau/\eta_1}\sum_{r=0}^{m-1}\frac{(r+1)}{(m-1-r)!\,z'^{2+r}}T^m_{m-1-r},\end{aligned} \tag{C21}$$

where

$$\begin{aligned}T^m_{m-1-r}(z_1,\eta_1,\eta_3)&=\partial^{m-1-r}_{z=0}\left\{2z_1-(1+z_1)z+2\sqrt{z_1}\exp\left[\frac{\ln(1-z)+\ln(z_1-z)}{2}\right]\right\}^m\\&\quad\times\left\{2\exp\left[\frac{\ln(1-z)+\ln(z_1-z)}{2}\right]-2z+1+z_1\right\}^{-m\sqrt{z_1}/z_0}.\end{aligned} \tag{C22}$$

The z' integration is more difficult as $\sin(n\sigma'/\eta_3)$ is found to have a cut from $z=1$ to $z=z_1$. Therefore instead of just picking up the residue of a pole the z' integration reduces to an integral from $z=1$ to z_1 of the discontinuity of the integrand. We obtain

$$N_{nm}=\frac{2}{\pi mn}e^{m\tau/\eta_1}[e^{-n\tau'/\eta_3}(z_1-1)^n+e^{n\tau'/\eta_3}(z_1-1)^{-n}]\sum_{r=0}^{m-1}\frac{(r+1)}{(m-1-r)!}T^m_{m-1-r}Q^n_r,\quad n,m\neq 0 \tag{C23}$$

where

$$Q^n_r(z_1,\eta_1,\eta_3)=\int_1^{z_1}\frac{dz'}{z'^{2+r}}\sin\left\{\frac{nA}{\eta_3}\sin^{-1}\frac{2[(z'-1)(z_1-z')]^{1/2}}{z_1-1}-\frac{nAz_0}{\eta_3\sqrt{z_1}}\sin^{-1}\frac{2[z_1(z'-1)(z_1-z')]^{1/2}}{z'(z_1-1)}\right\} \tag{C24}$$

and the positive square root is to be taken. By putting $z=1$ in Eq. (C16) we may obtain the position of the end of arm three (at $\tau=\tau_e$, say)

$$\tau_e=\eta_3\ln(z_1-1). \tag{C25}$$

Using this we can rewrite

$$N_{nm}=\frac{4}{\pi mn}e^{m\tau/\eta_1}\cosh\frac{n}{\eta_3}(\tau'-\tau_e)\times\sum_{r=0}^{m-1}\frac{(r+1)}{(m-1-r)!}T^m_{m-1-r}Q^n_r,\quad n,m\neq 0. \tag{C26}$$

Since the coefficients T^m_{m-1-r} and Q^n_r contain no τ' or τ dependence, we see that the effect of keeping arm three of the vertex finite is exactly the same as for the semi-infinite strip of Fig. 5, i.e., the exponential dependence on τ' becomes a hyperbolic cosine dependence. In this case we cannot determine whether a simple exponential dependence is obtained if τ' is chosen to be on the end of arm three ($\tau'\equiv\tau_e$) because of the complicated forms of the coefficients in Eq. (C26) which depend on z_1 (τ_e).

In the limit, as z_1 tends to unity we can show that Mandelstam's coefficients for the vertex with all arms infinite are obtained. In this limit

$$T^m_{m-1-r}=4^{-m\eta_3/\eta_1}(-1)^{m-1-r}\times\left(-\frac{m\eta_3}{\eta_1}\right)\left(-\frac{m\eta_3}{\eta_1}-1\right)\cdots\times\left(-\frac{m\eta_3}{\eta_1}-m+r+2\right), \tag{C27}$$

and by going back to the z' contour integral

$$\lim_{z_1\to 1}(z_1-1)^{-n}Q^n_r(z_1)=-\frac{\pi(-1)^n}{4^n(n-1)!}\left(-\frac{\eta_1}{\eta_3}n-2-r\right)\times\left(-\frac{\eta_1}{\eta_3}n-3-r\right)\cdots\times\left(-\frac{\eta_1}{\eta_3}n-n-r\right), \tag{C28}$$

giving

$$\lim_{z_1\to 1}N_{nm}=2nm\exp[m(\tau-\eta_3\ln 4)/\eta_1+n(\tau'-\eta_3\ln 4)/\eta_3]\times(-1)^{m+n}\frac{(\eta_1+\eta_3)}{m\eta_3+n\eta_1}a_m\left(-\frac{\eta_3}{\eta_1}\right)a_n\left(-\frac{\eta_1}{\eta_3}\right),$$

$$n,m\neq 0 \tag{C29}$$

where

$$a_n(\gamma)=\frac{(n\gamma-1)(n\gamma-2)\cdots(n\gamma-n+1)}{n!}.$$

[The factor $(-1)^{m+n}$ which does not occur in Ref. 5 arises because we use a different convention in our transformation Eq. (C17). In his paper Mandelstam takes σ' increasing from top to bottom in arm three in Fig. 6, whereas we take both σ and σ' as increasing from bottom to top of arms one and three, respectively.]

When m or n is zero we have

$$N_{0m}=-\frac{1}{\pi^2 m\eta_3}\oint d\sigma'\oint \sin\frac{m\sigma}{\eta_1}\frac{1}{(z-z')}dz$$

$$=-\frac{2A}{\pi m\eta_3}e^{m\tau/\eta_1}\sum_{r=0}^{m-1}\frac{T^m_{m-1-r}}{(m-1-r)!}\times\int_1^{z_1}\frac{dz'(z'-z_0)}{z'^{2+r}[(z'-1)(z_1-z')]^{1/2}} \tag{C30}$$

and

$$N_{n0}=\frac{1}{\pi^2 n\eta_1}\oint dz'\oint d\sigma\sin\frac{n\sigma'}{\eta_3}\frac{1}{(z-z')}$$

$$=\frac{4}{\pi n}\cosh\frac{n}{\eta_3}(\tau'-\tau_e)Q^n_{-1}. \tag{C31}$$

Again we find that in the limit $z_1\to 1$ Mandelstam's infinite vertex coefficients are obtained

$$\lim_{z_1\to 1}N_{0m}=-2e^{(m/\eta_1)(\tau-\eta_3\ln 4)}(-1)^{m-1}a_m\left(-\frac{\eta_3}{\eta_1}\right), \tag{C32}$$

$$\lim_{z_1\to 1}N_{n0}=2e^{(\eta/\eta_3)(\tau-\eta_3\ln 4)}(-1)^{n-1}a_n\left(-\frac{\eta_1}{\eta_3}\right). \tag{C33}$$

APPENDIX D

In this appendix we outline the relevant steps used to derive Eqs. (5.48) and (5.53) used in the text. We follow fairly closely the proof given by Cremmer and Gervais,[17] with some significant differences, however. We consider the inverse mapping $z=z(\rho)$ that relates the infinite three-string domain in the ρ plane to the z plane (see Appendix C). We introduce the functions $h^i(\rho)$, which equal $\ln z(\rho)$ on the ith string and have sections of the line $\rho=\tau_0+ia$ in common. In terms of parameters α_i the interaction time τ_0 is given by

$$\tau_0=(\alpha_1)^{\alpha_1}(\alpha_2)^{\alpha_2}(-\alpha_3)^{\alpha_3}, \tag{D1}$$

with Mandelstam's convention

$$\alpha_1+\alpha_2+\alpha_3=0,\quad \alpha_3<0. \tag{D2}$$

In terms of the parametrization used in this paper

$$\alpha_1=\eta_1,\quad \alpha_2=\eta_2,\quad \alpha_3=-\eta_3. \tag{D3}$$

The inverse mappings are explicitly

$$h^{(1)}(\rho)=\sum_{n>0}a^{(1)}_n e^{n\rho/\alpha_1}, \tag{D4}$$

$$h^{(2)}(\rho)=\frac{\rho-i\pi\alpha_1}{\alpha_2}-\frac{\alpha_1}{\alpha_2}\sum_{n>0}a^{(2)}_n e^{n(\rho-i\pi\alpha_1)/\alpha_2}, \tag{D5}$$

$$h^{(3)}(\rho)=-\frac{\rho}{\alpha_3}+\frac{\alpha_1}{\alpha_3}\sum_{n>0}a^{(3)}_n(-1)^n e^{n\rho/\alpha_3}, \tag{D6}$$

where

$$a^{(r)}_n=a_n\left(-\frac{\alpha_{r+1}}{\alpha_r}\right)$$

and

$$a_n(\gamma)=\frac{1}{n\gamma}\binom{n\gamma}{n}=(-1)^n a_n(1-\gamma).$$

We introduce the operators

$$\Gamma_q(\rho_0)=e^{-q\rho/\alpha_3}\int_{\rho_0}^{\rho}d\rho\, e^{q\rho/\alpha_3}, \tag{D7}$$

$$\Delta_q=e^{-q\rho/\alpha_3}\frac{\partial}{\partial\rho}e^{q\rho/\alpha_3}, \tag{D8}$$

and calculate

$$\Gamma_q \Delta_0^2 h^{(1)}(\rho) = \sum_{n>0} a_n^{(1)} \frac{n^2}{\alpha_1^{\,2}} \left(\frac{q}{\alpha_3} + \frac{n}{\alpha_1}\right)^{-1} \left(e^{n\rho/\alpha_1} - e^{n\tau_0/\alpha_1 - q(\rho-\tau_0)/\alpha_3}\right), \tag{D9}$$

$$\Gamma_q \Delta_0^{\,2} h^{(2)}(\rho) = -\frac{\alpha_1\alpha_3}{\alpha_2^{\,2}} \sum_{n>0} a_n^{(2)} \frac{n^2}{\alpha_2^{\,2}} e^{-q\rho/\alpha_3} \left(\frac{n}{\alpha_2} + \frac{q}{\alpha_3}\right)^{-1} \left(e^{n(\rho - i\pi\alpha_1)/\alpha_2 + q\rho/\alpha_3} - e^{n(\rho_3 - i\pi\alpha_1)/\alpha_2 + q\tau_0/\alpha_3}\right), \tag{D10}$$

$$\Gamma_q \Delta_0^{\,2} h^{(3)}(\rho) = \frac{\alpha_1}{\alpha_3} \sum_{n>0} a_n^{(3)} (-1)^n \frac{n^2}{\alpha_3(q+n)} \left(e^{n\rho/\alpha_3} - e^{n(\tau_0 - i\pi\alpha_3)/\alpha_3 - q(\rho-\tau_0-i\pi\alpha_3)/\alpha_3}\right). \tag{D11}$$

Since

$$\operatorname{Re}\Gamma_q \Delta_0^{\,2} h^{(r)}(\tau_0 + i\sigma) = \operatorname{Re}\Gamma_q \Delta_0^{\,2} h^{(3)}(\tau_0 + i\sigma), \quad r = 1, 2 \tag{D12}$$

we obtain

$$\sum_{n>0} \frac{(-1)^n}{(n+q)} n^2 \bar{a}_n^{(3)} \tilde{W}_{nm}^{31} = \left(\frac{\alpha_3}{\alpha_1}\right)^2 \frac{\alpha_3 m^2 \bar{a}_m^{(1)}}{(\alpha_1 q + \alpha_3 m)} - \tilde{W}_{qm}^{31} \sum_{n>0} n^2 \left[\left(\frac{\alpha_3}{\alpha_1}\right)^2 \frac{\alpha_3 \bar{a}_n^{(1)}}{(\alpha_1 q + \alpha_3 n)} - (-1)^n \frac{\bar{a}_n^{(3)}}{(q+n)}\right], \tag{D13}$$

$$\sum_{n>0} \frac{(-1)^n}{(n+q)} n^2 \bar{a}_n^{(3)} \tilde{W}_{nm}^{32} = -\left(\frac{\alpha_3}{\alpha_2}\right)^2 \frac{\alpha_3 m^2 \bar{a}_m^{(2)}}{(\alpha_2 q + \alpha_3 m)} + (-1)^q \tilde{W}_{qm}^{32} \sum_{n>0} (-1)^n n^2 \left[\left(\frac{\alpha_3}{\alpha_2}\right)^2 \frac{\alpha_3 \bar{a}_n^{(2)}}{(a_1 q + \alpha_3 n)} - (-1)^n \frac{\bar{a}_n^{(3)}}{(q+n)}\right] \tag{D14}$$

by integrating Eq. (D12) with $\cos n\sigma/\alpha_r$. In the formula we have defined

$$\bar{a}_n^{(r)} = a_n^{(r)} e^{n\tau_0/\alpha_r}, \qquad \tilde{W}_{nm}^{3s} = \left(\frac{\alpha_1}{-\alpha_3}\right)^{1/2} W_{nm}^{3s}. \tag{D15}$$

In comparing these formulas with similar ones in Ref. 17 we stress that Eqs. (D13) and (D14) involve different overlap coefficients. One can generate similar formulas containing the A^{rs} by explicitly using the fact that the functions $h^{(1)}(\rho)$ and $h^{(2)}(\rho)$ are equal at the interaction point $\rho = \tau_0 + i\pi\alpha_1$.

We also require identities among the $\bar{D}^{rs}$ matrices defined in the text. Since these relations are independent of arm overlap coefficients we have[17]

$$\sum_{p>0} \left[\bar{D}_{np}^{33}(-1)^p - \frac{\alpha_3}{\alpha_1} \bar{D}_{np}^{31}\right] = n(-1)^n, \tag{D16}$$

$$\sum_{r=1}^{3} \bar{D}^{3r} \hat{G}^{-1} \bar{D}^{r3} = \hat{G}. \tag{D17}$$

In terms of the Mandelstam coefficients Eq. (D16) becomes

$$n^2 \bar{a}_n^{(3)} \sum_{p>0} \left[\left(\frac{\alpha_3}{\alpha_1}\right) \frac{\alpha_3}{\alpha_1 n + \alpha_3 p} \bar{a}_p^{(1)} - \frac{(-1)^p}{n+p} \bar{a}_p^{(3)}\right] = \frac{\alpha_3^{\,2}}{\alpha_1\alpha_2} n(-1)^n. \tag{D18}$$

Multiplying Eq. (D13) by $q^2 \bar{a}_q^{(3)}$ we can use Eq. (D18) immediately to obtain

$$\sum_{n>0} [\hat{G}_{qn} - (-1)^q \bar{D}_{qn}^{33} (-1)] W_{nm}^{31} = -\left(-\frac{\alpha_3}{\alpha_2}\right)^{1/2} (-1)^q \bar{D}_{qm}^{31}, \tag{D19}$$

which is the same as Eq. (5.48) in the text. $\delta \cdot W^{32} = -(\eta_1/\eta_1)^{1/2}[-1]\bar{D}^{32}$ follows in a similar manner, using Eq. (D14).

To prove $\theta_- \cdot \delta = 0$ we first integrate Eq. (D12) for $r = 1$ directly to obtain

$$-\frac{\alpha_3}{\alpha_1} \sum_{n>0} n^2 \frac{\bar{a}_n^{(1)}}{\alpha_1 q + \alpha_3 n} W_{q0}^{31} = \frac{\alpha_1}{\alpha_3} \sum_{n>0} \frac{\bar{a}_n^{(3)}(-1)^n}{\alpha_3(q+n)} n^2 (W_{n0}^{31} - W_{q0}^{31}). \tag{D20}$$

Multiplying by $(-1)^q q^2 \bar{a}_q^{(3)}$ and using Eq. (D18) then gives

$$\sum_{n>0} (-1)^q \bar{D}_{qn}^{33} (-1)^n W_{n0}^{31} = q W_{q0}^{31} \tag{D21}$$

or

$$(\delta)_{mn} W_{n0}^{31} = 0.$$

But since

$$\delta \cdot \theta_- = (\delta)_{mn} \left[\frac{W_{n0}^{31}}{(\pi\alpha_1)^{1/2}} - \frac{W_{n0}^{32}}{(\pi\alpha_2)^{1/2}}\right] = -\frac{-\alpha_3}{\alpha_2(\pi\alpha_1)^{1/2}} \delta \cdot W^{31} \tag{D22}$$

we have the desired result.

[1]J. H. Schwarz, Phys. Rep. 8C, 269 (1973).
[2]P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56, 109 (1973).
[3]M. Kaku and K. Kikkawa, Phys. Rev. D 10, 1110 (1974).
[4]E. Cremmer and J. L. Gervais, Nucl. Phys. B90, 410 (1975).
[5]S. Mandelstam, Nucl. Phys. B64, 205 (1973).
[6]P. Ramond, Yale Report No. C00/3075/63 (unpublished); C. Marshall and P. Ramond, Nucl. Phys. B85, 375 (1975).
[7]P. A. M. Dirac, Can. J. Math. 2, 129 (1950). See also J. L. Gervais and B. Sakita, Phys. Rev. Lett. 30, 716 (1973).
[8]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 2.2 and Eq. 4.59.
[9]For a clear exposition on this point see J. Scherk, Rev. Mod. Phys. 47, 123 (1975), and references therein.
[10]Lorentz indices will be supressed and transverse vectors with d^* components distinguished by an asterisk. Scalar products are to be understood where necessary.
[11]This is precisely the term discussed in the calculation of the critical dimension by L. Brink and H. B. Nielsen, Phys. Lett. 45B, 332 (1973).
[12]S. Mandelstam, in *Proceedings of the XVII International Conference on High Energy Physics, London, 1974*, edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974), p. I-260.
[13]It should be noted that by expanding the operators a, $a^\dagger$ in terms of the open-string basis, Eq. (3.4), we may recover the operator formalism commutation relations $[a^i_m, a^{j\dagger}_n]=\delta_{mn}\delta^{ij}$.
[14]J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).
[15]Strictly, the determinant in this and similar expressions should be interpreted as a product of determinants. Each one is to be restricted to the appropriate subspace which is determined by reference to the preceding functional integral.
[16]M. Ademollo, E. Del Giudice, P. Di Vecchia, and S. Fubini, Nuovo Cimento 19A, 181 (1974).
[17]E. Cremmer and J. L. Gervais, Nucl. Phys. B76, 209 (1974).
[18]M. Kaku and K. Kikkawa, Phys. Rev. D 10, 1823 (1974).
[19]J. Goldstone (unpublished).
[20]P. Ramond, Phys. Rev. D 3, 2415 (1971).
[21]S. Mandelstam, Nucl. Phys. B69, 77 (1974).
[22]K. Bardakci and M. B. Halpern, Phys. Rev. D 3, 2493 (1971).