PHYSICAL REVIEW D VOLUME 12, NUMBER 6 15 SEPTEMBER 1975

Semiclassical scattering of quantized nonlinear waves*

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(Received 1 May 1975)

The relation between semiclassical phase shift and time delay, familiar from quantum mechanics, is extended to quantum field theory. This provides a semiclassical description for the scattering of quantized nonlinear waves. After a study of Levinson's theorem, our methods are applied to the sine-Gordon theory and the nature of the forces arising here is discussed. Finally, a covariant perturbative expansion for the one-particle sector is derived.

I. INTRODUCTION

It is now recognized that solitary wave solutions to nonlinear field equations[1] indicate that in the corresponding quantum theory there exist new states, which resemble heavy elementary particles. The two techniques used so far to exhibit the quantum nature of these states are a field-theoretic WKB method[2] and a systematic coupling-constant expansion.[3,4] However, little is known at present about the quantum scattering of these heavy particles, which we call baryons.

In this paper we show that a consistent semiclassical description, based on the well-known relation between phase shift and time delay in potential theory, can be given for baryon scattering in field theory. When the time delay in a collision is known, as is the case for the sine-Gordon theory,[1] the phase shift can be found very simply. When the time delay is not known, but scattering solutions of the classical theory are available, the phase shift is obtained by solving a first-order differential equation.

As a specific example, the sine-Gordon theory is considered by both methods. Agreement is found, thus erasing any doubt about the equality of the conventional *field-theoretic* time delay and the *particle* time delay. A semiclassical version of Levinson's theorem is derived; it is satisfied in the sine-Gordon theory. However, the full theorem, derived here also since for one-dimensional problems it differs from the familiar result, is violated. The phase shifts are nonvanishing at infinite energy, possibly as a consequence of short-distance singularities in the forces.

The scattering is observed to be essentially relativistic, hence difficult to describe by the existing perturbation theory[3,4] which begins with a static, nonrelativistic approximation. We describe an alternative perturbation theory, which is relativistically covariant.

II. CLASSICAL AND SEMICLASSICAL SCATTERING IN ONE SPACE DIMENSION

A. Classical time delay

Let us consider the classical theory of a particle with energy E, moving in one space dimension in a time-independent, parity-even potential, vanishing at large distances. If at time t_0 the particle is at x_0, then it will arrive at its final position x_f at time t_f given by the formula

$$t_f - t_0 = \int_{x_0}^{x_f} \frac{dx}{v(x,E)}, \tag{2.1a}$$

where $v(x,E)$ is the local velocity. From Hamilton's equation $\dot{x} = \partial H(p,x)/\partial p$, we have $1/v(x,E) = \partial p(x,E)/\partial E$, and (2.1a) is equivalently

$$t_f - t_0 = \frac{\partial}{\partial E}\int_{x_0}^{x_f} dx\, p(x,E)\,. \tag{2.1b}$$

If we use the further relation $p dq = (L+H)dt$, we can write (2.1b) as

$$t_f - t_0 = \frac{\partial}{\partial E}[I + E(t_f - t_0)]\,, \tag{2.1c}$$

where I is the action $I = \int_{t_0}^{t_f} dt\, L(\dot{x},x)$. The arguments of the Lagrangian, $\dot{x}(t,E)$ and $x(t,E)$, satisfy the equations of motion, with boundary conditions $x(t_0,E)=x_0$, $x(t_f,E)=x_f$. The differentiation in (2.1c) is performed at fixed x_f, hence t_f is taken to be a function of E and x_f.

The significance of (2.1c) is that it has been derived without any explicit specification of the dynamics which govern the motion. Moreover, even when L is a field Lagrangian, $L = \int dx\, \mathcal{L}(\dot{\phi}, \phi', \phi)$, the time of flight is still correctly given by (2.1c). Since our subsequent discussion will use that formula for field theory, we demonstrate explicitly its validity in that context:

$$\frac{\partial}{\partial E}\left[\int_{t_0}^{t_f} dt \int dx\, \mathcal{L} + E(t_f - t_0)\right]$$
$$= L_{t=t_f}\frac{\partial t_f}{\partial E} + \int_{t_0}^{t_f} dt \int dx \frac{\partial \mathcal{L}}{\partial E} + E\frac{\partial t_f}{\partial E} + (t_f - t_0)\,. \tag{2.2a}$$

The Lagrangian density depends on E through ϕ which satisfies the Euler-Lagrange equations with boundary conditions $\phi(x, t_f) = \phi_f(x)$ and $\phi(x, t_0) = \phi_0(x)$. This implies

$$\int dx \frac{\partial \mathcal{L}}{\partial E} = \frac{d}{dt}\left(\int dx \frac{\partial \mathcal{L}}{\partial \dot\phi}\frac{\partial \phi}{\partial E}\right).$$

Hence the right-hand side of (2.2a) becomes

$$(t_f - t_0) + (L_{t=t_f} + E)\frac{\partial t_f}{\partial E} + \int dx \frac{\partial \mathcal{L}}{\partial \dot\phi}\frac{\partial \phi}{\partial E}\bigg|_{t=t_0}^{t=t_f}.$$

It is also true that

$$\frac{\partial \phi}{\partial E}\bigg|_{t=t_0}^{t=t_f} = \frac{\partial}{\partial E}\phi_f - \dot\phi_{t=t_f}\frac{\partial t_f}{\partial E} - \frac{\partial}{\partial E}\phi_0\,.$$

Because the initial and final configurations are held constant during the differentiation with respect to E, only $-\dot\phi_{t=t_f}(\partial t_f/\partial E)$ survives on the right-hand side. Therefore, one is left with

$$\frac{\partial}{\partial E}\left[\int_{t_0}^{t_f} dt \int dx\, \mathcal{L} + E(t_f - t_0)\right]$$
$$= (t_f - t_0) + \left(E - \int dx \frac{\partial \mathcal{L}}{\partial \dot\phi}\dot\phi\bigg|_{t=t_f} + L_{t=t_f}\right)\frac{\partial t_f}{\partial E}$$
$$= t_f - t_0\,, \tag{2.2b}$$

which is the desired result.

We also know from (2.1b) that for a free theory

$$t_f - t_0 = \frac{\partial}{\partial E}[p(E)(x_f - x_0)]\,.$$

Thus the total time delay in unbounded motion is

$$\Delta t(E) = \lim \frac{\partial}{\partial E}[I + E(t_f - t_0) - p(E)(x_f - x_0)] \tag{2.3}$$

The limit is taken with t_0 and x_0 tending to $-\infty$, and t_f and x_f tending to $+\infty$, with t_f and x_f properly related by the dynamics.

2. Semiclassical phase shift

Let us next recall the WKB description of potential scattering in quantum mechanics. The Schrödinger equation for a Hamiltonian $H(p, x)$,

$$H\left(\frac{1}{i}\frac{\partial}{\partial x}, x\right)\psi(x) = E\psi(x), \tag{2.4}$$

is solved in the WKB approximation by

$$\psi(x) = \exp[\,i\,u(x)]\,, \tag{2.5a}$$

where for classically unbounded motion $u(x)$ is

$$u(x) = \int_{x_0}^{x} dx'\, p(x', E). \tag{2.5b}$$

Thus only transmission occurs in this approximation. The transmission amplitude, necessarily a pure phase by unitarity, is $T(E) = \exp[\,2i\,\delta(E)]$, and the WKB formula for the phase shift according to (2.5b) is

$$2\delta(E) = \lim \int_{x_0}^{x_f} dx[\,p(x, E) - p(E)]\,. \tag{2.6}$$

Comparing this with the time delay, we arrive at the familiar relation

$$\Delta t(E) = \frac{d}{dE}2\delta(E). \tag{2.7}$$

We propose that the physically sensible result (2.7) is also valid in the semiclassical limit of quantum field theory. Integrating (2.7) gives

$$\delta(E) - \delta(E_{\text{th}}) = \tfrac{1}{2}\int_{E_{\text{th}}}^{E} dE'\,\Delta t(E'), \tag{2.8}$$

where the threshold energy E_{th} is defined by $p(E_{\text{th}}) = 0$. Alternatively from (2.3) it follows that

$$2\delta(E) = \lim[\,I + E(t_f - t_0) - p(E)(x_f - x_0)]\,. \tag{2.9}$$

The constant of integration in (2.9) is adjusted by analogy with potential theory. Also (2.9) has been obtained by Callan and Gross by a superstationary phase approximation to a functional integral.[4] The constant of integration is (2.8) can be evaluated as follows. The semiclassical quantization condition is

$$I_T + ET = n\pi. \tag{2.10a}$$

Here I_T is the action for a periodic solution, integrated over a semiperiod T. This is familiar in potential theory; in field theory it represents a simplification (valid for weak coupling) of the WKB condition obtained by Dashen *et al.*[2] The total number of bound states n_B corresponds to the maximum value of n in (2.10a), which occurs for E just below E_{th}. At threshold the semiperiod extends over all time, since the particle ceases to be bound; hence we find

$$\lim(I_T + ET)_{E=E_{\text{th}}} = n_B\pi. \tag{2.10b}$$

Comparing this with (2.9), we get

$$2\delta(E_{\text{th}}) = n_B\pi. \tag{2.11}$$

This provides our principal result, an equation for the semiclassical phase shift:

$$\delta(E)=\tfrac{1}{2}n_B\pi+\tfrac{1}{2}\int_{E_{\rm th}}^{E}dE'\,\Delta t(E'). \tag{2.12}$$

When an explicit expression for $\Delta t(E)$ is not available, but solutions of the classical theory are known so that I in (2.9) can be evaluated, that formula permits an alternate calculation of $\delta(E)$. [This situation may arise if it is not obvious how to extract $\Delta t(E)$ from field-theoretic solutions.] In order to use (2.9), we must exhibit the relationship between x_f and t_f, which asymptotically is

$$x_f=v(E)[t_f-\Delta t(E)-t_0]+x_0,$$

$$x_f-x_0=\frac{dE}{dp}(t_f-t_0)-\frac{d}{dp}2\delta(E),$$

where Hamilton's equation and (2.7) have been used. Hence (2.9) becomes

$$\delta(E)-p\frac{d}{dp}\delta(E)$$
$$=\tfrac{1}{2}\lim\left[I+E(t_f-t_0)-p\frac{dE}{dp}(t_f-t_0)\right]. \tag{2.13}$$

For the field theory with $L=\int dx\,[\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}\phi'^2-U(\phi)]$, $I=\int_{t_0}^{t_f}dt\int dx\,\dot{\phi}^2-E(t_f-t_0)$, and (2.13) can be simplified further:

$$\delta(E)-p\frac{d}{dp}\delta(E)=\int_0^\infty dt\left(\int dx\,\dot{\phi}^2-p\frac{dE}{dp}\right). \tag{2.14}$$

C. Levinson's theorem

Equation (2.11) is the semiclassical Levinson's theorem in one dimension. It should be compared with the exact result

$$2[\delta(E_{\rm th})-\delta(\infty)]=n_B\pi. \tag{2.15}$$

The factor 2 does not occur in three dimensions. We shall give a crude derivation of (2.15) based on a simple state-counting argument[5] for reflectionless potentials. For a particle of momentum p (positive or negative) moving in a reflectionless potential enclosed in a large "box" of length L, periodic boundary conditions require

$$2n\pi=pL+2\delta(p). \tag{2.16a}$$

Thus

$$2\pi\frac{dn}{dp}=L+2\frac{d}{dp}\delta(p). \tag{2.16b}$$

Subtracting the analogous expression for the noninteracting situation leaves

$$2\pi\left[\frac{dn}{dp}-\left(\frac{dn}{dp}\right)_{\rm free}\right]=2\frac{d}{dp}\delta(p),$$
$$2\pi\Delta n_{\rm scatter}=2\int_{-\infty}^{\infty}dp\frac{d\delta(p)}{dp}$$
$$=-4[\delta(E_{\rm th})-\delta(\infty)]. \tag{2.16c}$$

Here $\Delta n_{\rm scatter}$ is the increase in the number of scattering states, after switching on the interactions; since this also equals the negative of the number of bound states, (2.15) is proven. (The validity of (2.15) may be checked in the general class of reflectionless Hamiltonians $p^2-[L(L+1)/\cosh^2 x]$. L is an integer, the phase shift is

$$-\sum_{n=1}^{L}\tan^{-1}\frac{p}{n}+\frac{L\pi}{2}\epsilon(p),$$

and the number of bound states is L.[3])

D. Classical and semiclassical hard-sphere scattering

In the above, we have only considered unbounded motion. However, particularly in the case of collisions between identical particles, there also exists the possibility of an almost perfectly elastic hard-sphere repulsion, which would send the particles back in the directions they came. Since this circumstance is important in the forthcoming discussion on the sine-Gordon equation, we now show that our theory describes it without modification.

Consider the motion of a classical particle with the following boundary conditions: At $t\to-\infty$, the particle enters from $x=+\infty$, then it travels towards $x=0$ where it is totally reflected by a perfect barrier so that it returns to $x=+\infty$ at $t\to+\infty$. The time delay due to the presence of forces in the region $x>0$ is

$$\Delta t(E)=2\frac{\partial}{\partial E}\int_0^\infty dx[p(x,E)-p(E)]$$
$$=\lim_{\substack{x_f\to+\infty,\ t_f\to+\infty\\ x_0\to+\infty,\ t_0\to-\infty}}\frac{\partial}{\partial E}[I+E(t_f-t_0)-p(E)(x_f+x_0)]. \tag{2.17}$$

This is exactly the same result as (2.3), with the replacement $x_0\to-x_0$. [If there is a distance of closest approach $\bar{x}$, then $p(x,E)$ vanishes for $\bar{x}\geqslant x\geqslant 0$.]

In the quantum-mechanical treatment of this problem the wave function $\psi(x)$ must vanish at the origin, and its large asymptote defines the phase shift,

$$\psi(x)\underset{x\to\infty}{\sim}\sin(px+\delta).$$

The WKB approximation to the wave function

$$\psi(x) = \sin \int_0^x dx' p(x', E) \tag{2.18}$$

gives the phase shift

$$\delta(E) = \int_0^\infty dx[p(x,E) - p(x)]. \tag{2.19}$$

Comparing with (2.17) we see that (2.7) is regained.

Levinson's theorem continues to hold in the form (2.15). This can be seen as follows. For $x>0$, the number of bound states $(n_B)_{x>0}$ is $(1/\pi)[\delta(E_{th}) - \delta(\infty)]$, a formula expressing the similarity between scattering in a half-line and in three dimensions. But there is an equal number of bound states for $x<0$. Hence the total number of bound states is given by (2.15).

The extension to field theory follows the same pattern as above, with the change that the initial and final separations of the particles have the same sign. In particular, we see that formulas (2.12) or (2.14) may again be used.

III. THE SINE-GORDON THEORY

In the sine-Gordon theory, described by the Lagrangian

$$\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\Phi\right) - 1\right], \tag{3.1}$$

the baryons exhibit the soliton property. The theory possesses soliton-antisoliton and soliton-soliton scattering solutions.[1] In the center-of-mass frame these are

$$\tan\frac{\sqrt{\lambda}}{4m}\phi_{s\bar{s}} = \frac{1}{u}\frac{\sinh m\gamma ut}{\cosh m\gamma x}, \tag{3.2a}$$

$$\tan\frac{\sqrt{\lambda}}{4m}\phi_{ss} = \frac{u\sinh m\gamma x}{\cosh m\gamma ut}, \tag{3.2b}$$

where

$$\gamma = \frac{1}{(1-u^2)^{1/2}}.$$

The asymptotic velocity of each soliton is u. Expressed in terms of the soliton mass, $M = 8m^3/\lambda$, the total energy of each solution is $E = 2M\gamma$ and the relative momentum is $2M\gamma u$.

Examination of the asymptotic form of $\phi_{s\bar{s}}$ as $t\to\pm\infty$ leads to the conclusion that there is only transmission with time delay[1]:

$$\Delta t(u) = \frac{2}{mu\gamma}\ln u. \tag{3.3}$$

Since $\Delta t<0$, it is actually a time advance which indicates attractive forces in the soliton-antisoliton channel, consistent with the existence in this channel of a bound solution, obtained from (3.2a) by taking u to be purely imaginary ($u = ia$, a real):

$$\phi_B = \frac{4m}{\sqrt{\lambda}}\tan^{-1}\left(\frac{1}{a}\frac{\sin m\gamma at}{\cosh m\gamma x}\right), \qquad \gamma = \frac{1}{(1+a^2)^{1/2}}. \tag{3.4a}$$

The energy is $2M\gamma$. Quantization of this solution according to (2.10a) gives the energy spectrum

$$E_n = 2M\sin\frac{m}{2M}n, \tag{3.4b}$$

with $n \le 8\pi m^2/\lambda$, which is the maximum number of bound states. For weak coupling this agrees with the result of Dashen *et al.*[6]

The asymptote of ϕ_{ss} as $t\to\pm\infty$ lends itself to two interpretations. One may conclude that again there is total transmission with time delay (3.3).[1] Alternatively, since the particles are identical, one may interpret the solution (3.2b) as describing total reflection, with time delay, in the sense of (2.17), again (3.3).[7] However, the fact that $\Delta t<0$ rules out the former, since this would indicate mainly attractive forces in a channel with no bound states. On the other hand, the latter description, based on hard-sphere repulsion, can be reconciled with the time advance, both if the forces are totally repulsive or if they are partially attractive.

If the forces are strongly repulsive and are such that part of the interaction region is excluded from the particles, then one might expect a time advance due to the shorter distance each particle travels. Alternatively, if there is no such excluded region, but there are short-range attractive forces besides the hard-sphere repulsion, then again a time advance would be expected. In this case, of course, the absence of bound states has to be explained, and this can be done if it is recognized that the hard-sphere repulsion at the origin makes the one-dimensional problem analogous to the three-dimensional one. The soliton-soliton solution (3.2b) must, and does, vanish at the origin, and it may be difficult for the field to reach its limiting value of $2\pi m/\sqrt{\lambda}$ within the attractive region. This picture of backward scattering fits nicely with our knowledge that the baryons are probably fermions[3,6,8], and gives a natural place to the exclusion principle in our semiclassical context.

From (2.12) we learn that the phase shifts are

$$\delta_{s\bar{s}}(u) = \frac{4\pi^2 m^2}{\lambda} + \frac{16m^2}{\lambda}\int_0^u dx\,\frac{\ln x}{1-x^2}, \tag{3.5a}$$

$$\delta_{ss}(u) = \frac{16m^2}{\lambda}\int_0^u dx\,\frac{\ln x}{1-x^2}. \tag{3.5b}$$

It perhaps is not obvious that the *particle* time delay is correctly given by the *field-theoretic* time delay (3.3).[9] In order to establish this, as well as the correctness of the integration constants in (3.5), we evaluate the phase shift from the alternate formula (2.14), which becomes with the present kinematics

$$\delta(u)-u(1-u^2)\delta'(u) = \int_0^\infty dt\left(\int dx\,\dot{\phi}^2-\frac{16m^3}{\lambda}u^2\gamma\right). \quad (3.6)$$

The calculation is lengthy and the details are presented in an appendix. The result is

$$\delta_{s\bar{s}}(u)-u(1-u^2)\,\delta'_{s\bar{s}}(u) = \frac{4\pi^2m^2}{\lambda}+\frac{16m^2}{\lambda}\left(\int_0^u dx\,\frac{\ln x}{1-x^2}-u\ln u\right) \quad (3.7a)$$

$$\delta_{ss}(u)-u(1-u^2)\,\delta'_{ss}(u) = \frac{16m^2}{\lambda}\left(\int_0^u dx\,\frac{\ln x}{1-x^2}-u\ln u\right). \quad (3.7b)$$

The solutions of these differential equations are exactly (3.5), except that an arbitrary multiple of $u\gamma$ may be added, which is eliminated by requiring the phase shifts to be finite at infinite energy, $u=1$. Hence (3.3) and (3.5) are verified completely.

The semiclassical Levinson theorem is true, but the complete theorem is not satisfied[10]:

$$\begin{aligned}\delta_{s\bar{s}}(E_{\rm th})-\delta_{s\bar{s}}(\infty)-\frac{n_B\pi}{2} &= -\frac{4\pi^2m^2}{\lambda} -\frac{16m^2}{\lambda}\int_0^1 dx\,\frac{\ln x}{1-x^2}\\ &= -\frac{2\pi^2m^2}{\lambda}\\ &\neq 0\,, \end{aligned} \quad (3.8a)$$

$$\begin{aligned}\delta_{ss}(E_{\rm th})-\delta_{ss}(\infty) &= -\frac{16m^2}{\lambda}\int_0^1 dx\,\frac{\ln x}{1-x^2}\\ &= \frac{2\pi^2m^2}{\lambda}\\ &\neq 0 \end{aligned} \quad (3.8b)$$

The reason for this is that the phase shifts do not vanish at infinite energy in either channel. In ordinary quantum mechanics, it is known from examples such as the inverse-square and δ-function potentials[11] that short-distance singular forces can produce this effect on the phase shift. So if the same mechanism works in field theory, then an explanation of (3.8) can be given by appeal to the short-distance singular forces, needed to produce the exclusion principle dynamically. [Let us observe, however, that a "super" Levinson theorem which counts states in both channels is satisfied:

$$\delta_{s\bar{s}}(E_{\rm th})+\delta_{ss}(E_{\rm th})-\delta_{s\bar{s}}(\infty)-\delta_{ss}(\infty)=\tfrac{1}{2}n_B\pi.]$$

It is clearly important to know to what extent the semiclassical phase shifts are a reliable guide to the exact quantum phase shifts. Although it may be reasonably conjectured[6] that the field-theoretic WKB method gives an exact description of the bound-state spectrum in the sine-Gordon theory, we see no reason for such a claim to be valid for scattering. (In potential theory, even when WKB calculations are exact for discrete states, they remain approximate for the continuum.) We have not used the full apparatus of the field-theoretic WKB method, stability angles, etc.[2,6], but instead a weak-coupling approximation to it has been substituted. Thus our results (3.5) are at best valid for weak coupling. Even here, however, there is doubt at threshold. For (3.5) to be trusted, it must be $O(\lambda^{-1})$ so that it dominates terms $O(\lambda^0)$ which arise as corrections to the weak-coupling limit. Yet when the velocity is small, say $u\sim\lambda/m^2$, $(m^2/\lambda)\int_0^{\lambda/m^2}dx(\ln x)/(1-x^2)$ becomes of the same order as the neglected terms.

Another point worthy of mention is that the phase shifts are essentially relativistic. This is best seen by reintroducing c, the velocity of light, which was set to unity. The logarithmic integrals in (3.5) then become $16(m^2c/\lambda)\int_0^{u/c}dx(\ln x)/(1-x^2)$, which does not tend to a finite limit as $c\to\infty$. This suggests that only a manifestly covariant approach to quantum soliton scattering will be successful.

IV. COVARIANT PERTURBATION THEORY

We have not succeeded in incorporating our semiclassical scattering calculation into the known coupling-constant expansion.[3,4] Part of the difficulty is that the expansion begins with a nonrelativistic static approximation, yet the scattering is essentially relativistic. We describe here an alternate version of the perturbation theory which maintains Lorentz covariance.

Recall that the perturbation theory begins with the observation that, to lowest order, the baryon matrix element of the quantum field Φ depends only on a momentum difference[3]: $\langle p|\Phi(0)|p'\rangle = f(p-p')$. This use of Galilean invariance for the baryons, accurate for weak coupling, gives a simple form for matrix elements of field products, and in conjunction with the operator equations of motion leads to the perturbation series; but manifest Lorentz covariance is lost. To circumvent this shortcoming we make use of Goldstone's observation that if the baryon states are labeled by rapidity instead of momentum, then Lorentz invariance implies that the field matrix element is a function of the rapidity difference, apart from known kinematical factors.[12]

$$\langle p|\Phi(0)|p'\rangle = \frac{2M}{[2E(p)2E(p')]^{1/2}} f(\alpha-\alpha'),$$
$$p = M\sinh\frac{\alpha}{M},\quad E(p) = M\cosh\frac{\alpha}{M}. \tag{4.1}$$

Again matrix elements of products of fields are simple:

$$\begin{aligned}\langle p|\Phi^2(0)|p'\rangle &= \int\frac{dp''}{2\pi}\langle p|\Phi(0)|p''\rangle\langle p''|\Phi(0)|p'\rangle \\ &= \frac{2M}{[2E(p)2E(p')]^{1/2}} \times\int\frac{d\alpha''}{2\pi}f(\alpha-\alpha'')f(\alpha''-\alpha') \\ &= \frac{2M}{[2E(p)2E(p')]^{1/2}} \times\int dx\, e^{i(\alpha-\alpha')x}\phi^2(x).\end{aligned} \tag{4.2}$$

We have saturated the intermediate-state sum with a single baryon state; this is a dynamical weak-coupling approximation which does no violence to the relativistic kinematics. The Fourier transform with respect to rapidity of the field form factor $f(\alpha-\alpha')$ has been introduced:

$$\phi(x) = \int\frac{d\alpha}{2\pi}e^{-i(\alpha-\alpha')x}f(\alpha-\alpha'). \tag{4.3}$$

To see how this scheme works, we derive the equation for $\phi(x)$. From the equation of motion

$$\Box\Phi = -U'(\Phi)$$

it follows that

$$2M^2\left(1-\cosh\frac{\alpha-\alpha'}{M}\right)\langle\alpha|\Phi(0)|\alpha'\rangle = \langle\alpha|U'(\Phi)|\alpha'\rangle \tag{4.4a}$$

Upon performing the Fourier transform with respect to $\alpha-\alpha'$, we obtain on the right-hand side simply

$$\frac{2MU'(\phi)}{[2E(p)2E(p')]^{1/2}}.$$

On the left-hand side one encounters integrals of the form $\int(d\alpha/2\pi)e^{-i\alpha(x+i/M)}f(\alpha)$. Assuming convergence, this gives $\phi(x+i/M)$, hence (4.4a) becomes[12]

$$M^2\left[2\phi(x)-\phi\left(x+\frac{i}{M}\right)-\phi\left(x-\frac{i}{M}\right)\right] = U'(\phi). \tag{4.4b}$$

For large M this reduces to the static field equation $\phi''(x)=U'(\phi)$; also one may contemplate solving (4.4b) exactly.

The Galilean perturbation theory[3] can be derived from an effective Hamiltonian by introducing a collective coordinate X, conjugate to momentum, by the transformation $\Phi(x,t)=\phi(x-X(t)) +\chi(x-X(t),t)$.[4] Something similar can be done here; we seek a coordinate conjugate to rapidity. It is convenient to use light-cone variables and quantization[13]:

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0\pm x^1),$$
$$i[\Phi(x^-,x^+),\Phi(y^-,y^+)]|_{x^+=y^+} = \tfrac{1}{4}\epsilon(x^- - y^-).$$

Then if P^+ is conjugate to X^-, $(1/\sqrt{2})\exp[(\alpha/M)X^-]$ is conjugate to α. Hence one is led to

$$\Phi(x^-,x^+) = \phi\left(\frac{1}{\sqrt{2}}e^{\alpha(x^+)/M}[x^- - X^-(x^+)]\right) +\chi\left(\frac{1}{\sqrt{2}}e^{\alpha(x^+)/M}[x^- - X^-(x^+)],x^+\right).$$

This is exactly the transformation suggested by Gervais and Sakita.[4] However, questions of operator ordering, which have been resolved by Tomboulis for the Galilean transformation,[4] appear to be more problematical here.

As yet we have not learned how to apply the above perturbation theory to the scattering problem. We conclude with the optimistic observation that in terms of rapidity, the phase shifts are especially simple:

$$\frac{16m^3}{\lambda}\int_0^u dx\frac{\ln x}{1-x^2} = \int_0^\alpha d\alpha'\ln\tanh\frac{\alpha'}{2M},$$

where $M=8m^3/\lambda$ and $u=\tanh(\alpha/2M)$.

APPENDIX

We give here the lengthy details of the evaluation of (3.5),

$$\frac{m^2}{\lambda}J = \int_0^\infty dt\left(\int_{-\infty}^\infty dx\,\dot\phi^2 - \frac{16m^3}{\lambda}u^2\gamma\right), \tag{A1}$$

$$\gamma = \frac{1}{(1-u^2)^{1/2}}$$

where ϕ is the soliton-antisoliton or the soliton-soliton solution (3.2). The x integration is elementary; a portion of the result cancels against the second term in the bracket of (A1), the rest leads to a convergent integral in t. By changing variables $y=(1+u^2\operatorname{csch}^2 m\gamma ut)^{-1/2}$ in the soliton-antisoliton integral, and $y=(1-u^2\operatorname{sech}^2 m\gamma ut)^{-1/2}$ in the soliton-soliton integral, we are left with

$$J_{s\bar{s}}=8\int_0^1 \frac{dy}{y}[1-(1-u^2)y^2]^{1/2}\ln\frac{1+y}{1-y}$$
$$+16(1-u^2)\int_0^1 dy[1-(1-u^2)y^2]^{-1/2}, \quad \text{(A2a)}$$

$$J_{ss}=-8\int_1^\gamma \frac{dy}{y}[1-(1-u^2)y^2]^{1/2}\ln\frac{y+1}{y-1}$$
$$-16(1-u^2)\int_1^\gamma dy[1-(1-u^2)y^2]^{-1/2}. \quad \text{(A2b)}$$

The integrals not involving the logarithm are elementary, giving $+16(1-u^2)^{1/2}\cos^{-1}u$ in (A2a) and $-16(1-u^2)^{1/2}\sin^{-1}u$ in (A2b). The remaining integrals are handled as follows. We consider them functions of $a=(1-u^2)^{1/2}\leq 1$:

$$\tilde{J}_{s\bar{s}}(a)=8\int_0^1 \frac{dy}{y}(1-a^2y^2)^{1/2}\ln\frac{1+y}{1-y}, \quad \text{(A3a)}$$

$$\tilde{J}_{ss}(a)=-8\int_1^{1/a} \frac{dy}{y}(1-a^2y^2)^{1/2}\ln\frac{y+1}{y-1}. \quad \text{(A3b)}$$

The $a=0$ values are readily obtained:

$$\tilde{J}_{s\bar{s}}(0)=+2\pi^2, \quad \text{(A4a)}$$

$$\tilde{J}_{ss}(0)=-2\pi^2. \quad \text{(A4b)}$$

Next the expressions in (A3) are differentiated with respect to a. After an integration by parts, and a change of variable to $z=ya$, the formulas read

$$\tilde{J}'_{s\bar{s}}(a)=-\frac{8}{a}\int_0^a dz[(1-z^2)^{1/2}-(1-a^2)^{1/2}]\left(\frac{1}{a-z}+\frac{1}{a+z}\right), \quad \text{(A5a)}$$

$$\tilde{J}'_{ss}(a)=-\frac{8}{a}(1-a^2)^{1/2}\ln\frac{1-a}{1+a}$$
$$-\frac{8}{a}\int_a^1 dz[(1-z^2)^{1/2}-(1-a^2)^{1/2}]\left(\frac{1}{z-a}-\frac{1}{z+a}\right). \quad \text{(A5b)}$$

The integrals may be evaluated with the help of

$$\int^x \frac{(1-z^2)^{1/2}}{z+\alpha}dz$$
$$=\alpha\sin^{-1}x+(1-x^2)^{1/2}$$
$$+(1-\alpha^2)^{1/2}\ln\left[\frac{\alpha+x-\alpha(1-x^2)^{1/2}-x(1-\alpha^2)^{1/2}}{\alpha+x-\alpha(1-x^2)^{1/2}+x(1-\alpha^2)^{1/2}}\right] \quad \text{(A6)}$$

to obtain the result

$$\tilde{J}'_{s\bar{s}}(a)=-16\sin^{-1}a-16\frac{(1-a^2)^{1/2}}{a}\ln(1-a^2)^{1/2}, \quad \text{(A7a)}$$

$$\tilde{J}'_{ss}(a)=16\cos^{-1}a-16\frac{(1-a^2)^{1/2}}{a}\ln(1-a^2)^{1/2}. \quad \text{(A7b)}$$

Finally integrating with respect to a and using the boundary conditions (A4) gives

$$\tilde{J}_{s\bar{s}}(a)=-16(1-u^2)^{1/2}\cos^{-1}u-16u\ln u$$
$$+16\int_0^u dx\frac{\ln x}{1-x^2}+4\pi^2, \quad \text{(A8a)}$$

$$\tilde{J}_{ss}(a)=16(1-u^2)^{1/2}\sin^{-1}u-16u\ln u$$
$$+16\int_0^u dx\frac{\ln x}{1-x^2}. \quad \text{(A8b)}$$

From (A2) it now follows that

$$J_{s\bar{s}}=16\left(\int_0^u dx\frac{\ln x}{1-x^2}-u\ln u\right)+4\pi^2, \quad \text{(A9a)}$$

$$J_{ss}=16\left(\int_0^u dx\frac{\ln x}{1-x^2}-u\ln u\right), \quad \text{(A9b)}$$

which is equivalent to (3.7).

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*This work is supported in part through funds provided by ERDA under Contract No. AT(11-1)-3069.

†Kennedy Memorial Scholar.

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