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Gravitating 't Hooft monopoles*

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The 't Hooft magnetic monopole solution of the SO(3) Higgs model generates, when coupled to gravity, a geometry which for large distances is of the Reissner-Nordström form corresponding to a (magnetic) charge $1/e$. The Higgs fields can contribute a cosmological term. In the absence of scalar fields the corresponding Wu-Yang solution of the SO(3) gauge theory still generates the Reissner-Nordström geometry.

Wu and Yang[1] have found classical solutions of the SO(3) isospin gauge theory. Similar solutions appear also in the Higgs-type model that couples an isotriplet of scalar fields to the isospin gauge fields. There 't Hooft has shown[2] that they correspond to Abelian magnetic monopoles (the corresponding magnetic charge being conserved on topological grounds[3]). The question we want to answer here is what happens when this system is further coupled to gravitation.

For the monopole case, the SO(3) gauge fields approach for large distances precisely one of the Wu-Yang solutions.[2] By a singular gauge transformation[3] this Wu-Yang gauge field can be brought to point in a given, say the third, isospace direction. In this gauge also the Higgs field points in the third isospace direction, so that the theory behaves at large distances (where these considerations apply) as if it were Abelian.[3] As far as the space dependence is concerned, in this "Abelian" gauge, at large distances[2] the surviving Higgs field is constant and the surviving isocomponent of the gauge field is precisely equal to the vector potential corresponding to a usual Abelian Dirac magnetic monopole of magnetic charge $1/e$.[3] If we now switch on gravity, the constant Higgs field will contribute a cosmological term, whereas the magnetic monopole will have an energy-momentum tensor of the same form as for an electric charge with value $1/e$. One therefore must obtain for large distances a Reissner-Nordström geometry corresponding to a (magnetic) charge $1/e$. The energy-momentum tensor and the gravitational field being invariant under local SO(3)-gauge transformations, the same Reissner-Nordström geometry must appear for large distances in every gauge, in particular in the original gauge in which at large distance the gauge fields were of the Wu-Yang form. We shall check this by a direct calculation below. Concerning the cosmological term, it can be removed by hand, by adding an "irrelevant" constant to the original Higgs Lagrangian before coupling it to gravitation.

In the process of treating this problem we shall also find exact (i.e., valid at all not only at large distances), albeit singular, solutions of the coupled gravitational-gauge-scalar system which reduce to the corresponding known exact but singular solutions of the non-Abelian Higgs theory without gravitation. The Reissner-Nordström geometry without cosmological term is valid if one couples just the Wu-Yang solution to gravitation without introducing any scalar Higgs fields.

For a scalar isotriplet of fields ϕ^a $(a=1,2,3)$ coupled to SO(3)-gauge fields and to gravitation, consider the general-relativistic Higgs Lagrangian

$$\mathcal{L}=\sqrt{-g}\left[\frac{R}{16\pi G}-\tfrac{1}{4}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\phi^a\nabla_\nu\phi^a+\tfrac{1}{2}\mu^2\phi^a\phi^a-\frac{\lambda}{4}(\phi^a\phi^a)^2\right]. \qquad (1)$$

Here $\nabla_\mu\phi^a=\partial_\mu\phi^a+ef^a_{bc}A^b_\mu\phi^c$ is the gauge and generally covariant derivative of the Higgs field, while all the other quantities have their usual meanings, and the metric has signature $-+++$.

We are interested in spherically symmetric static solutions for which, in Schwarzschild coordinates, the scalar and vector fields have the form

$$(\phi^1,\phi^2,\phi^3)=S(r)(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),$$
$$A^a_t=A^a_r=0,$$
$$(A_\theta{}^1,A_\theta{}^2,A_\theta{}^3)=Y(r)(\sin\phi,-\cos\phi,0), \qquad (2a)$$
$$(A_\phi{}^1,A_\phi{}^2,A_\phi{}^3)=Y(r)\sin\theta(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta),$$

while the metric corresponds to the general Schwarzschild line element

$$(ds)^2=-A(r)(dt)^2+B(r)(dr)^2+r^2(d\theta)^2+r^2\sin^2\theta\,(d\varphi)^2. \qquad (2b)$$

The scalar and vector field equations take the forms (prime indicates derivative with respect to r)

$$S'' + \left(\frac{1}{2}\frac{A'}{A} - \frac{1}{2}\frac{B'}{B} + \frac{2}{r}\right)S' - \frac{2}{r^2}B(eY-1)^2 S - B(\lambda S^2 - \mu^2)S = 0, \tag{3}$$

$$Y'' + \frac{1}{2}\left(\frac{A'}{A} - \frac{B'}{B}\right)Y' - \frac{1}{r^2}BY(eY-1)(eY-2) - eB(eY-1)S^2 = 0.$$

Independently of the gravitational field [i.e., of $A(r)$ and $B(r)$] they admit for large distances (up to exponentially damped terms) the 't Hooft–Wu-Yang–type solution

$$Y = \frac{1}{e}, \quad S = \mu/\sqrt{\lambda}\ . \tag{4}$$

Equations (4) even provide an exact solution at all distances of the field equations (3). These solutions, however, yield fields singular at the origin which in the absence of gravitation correspond to infinite energy. They are therefore less interesting. All the following considerations are, for that matter, valid at all distances with these qualifications. In addition to solution (4) our considerations also apply to a pure Wu-Yang field without any scalar fields whatsoever.

With the solution (4), the gravitational field equations can be reduced to the form

$$A'/A + B'/B = 0, \qquad \frac{1}{r}\frac{B'}{B} + \frac{B-1}{r^2} = \left(\frac{K}{r^4} - \Lambda\right)B, \tag{5a}$$

where

$$K = G/(e^2/4\pi), \quad \Lambda = -8\pi G V_0, \quad V_0 = -\frac{\mu^2}{4\lambda}. \tag{5b}$$

These equations have the general solution

$$B(r) = \frac{\Lambda}{3}r^2 + 1 - \frac{2GM}{r} + \frac{K}{r^2}, \qquad A(r) = C/B(r). \tag{6a}$$

Here C and $2GM$ are integration constants. C can be determined, in the standard way, by requiring the only departure from asymptotic flatness to originate in the induced cosmological term ($\sim\Lambda$). This yields

$$C = 1. \tag{6b}$$

M is to be equated to the mass of the 't Hooft monopole. We recall that for the physically interesting case the metric takes the form (6) only at large distances. The geometry [(6a) and (6b)] is precisely of the Reissner-Nordström form (with cosmological constant Λ). One can remove the induced cosmological term by adding an explicit cosmological term $-\Lambda\sqrt{-g}$ to the Lagrangian (1). The geometry then becomes asymptotically flat. No induced cosmological term appears in the absence of scalar fields (Wu-Yang case). Topological considerations ensure the quantization of the 't Hooft magnetic charge already at the classical level.[3] In particular for the single monopole case the charge is not arbitrary but $1/e$ even in the presence of gravitation.

To conclude, we have found solutions of non-Abelian gauge theories coupled to gravitation[4] and Higgs fields. The relevant solutions are of the 't Hooft (or Wu-Yang) type for the vector and scalar fields and of the Reissner-Nordström type for the gravitational field. The charge that determines the geometry is the magnetic charge $1/e$, conserved for topological reasons.

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[2]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[3]J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975).

[4]We also note that any solution of the Einstein-Maxwell system is also a solution of the Einstein-Yang-Mills system for any group G in an Abelian gauge.