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Resolution of an ambiguity in the derivation of the Lorentz-Dirac equation

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An ambiguity in the derivation of the equation of motion of a charged point particle is exhibited. Using momentum conservation in the form $\int T^{\mu\nu} dS_\mu = 0$, one deduces the Lorentz-Dirac equation by integrating over one surface, and a different equation, with variable rest mass, by integrating over another. In the second case, the variability of the rest mass exactly compensates the Larmor radiation, and the Schott term is absent. In both cases, infinite mass renormalizations, of identical form, are required. These renormalizations are both covariant, but since they are the only mathematically insecure parts of the derivations they are obviously responsible for the ambiguity. A distribution theory of the problem is set up to resolve the question. This theory, at its present stage of development, has its own types of arbitrariness, but making very natural choices, a perfectly definite and everywhere finite formulation results, out of which the Lorentz-Dirac equation emerges uniquely. And in this form, as an expression of $\partial_\mu T^{\mu\nu} = 0$, the theory clarifies the energy-momentum exchanges in the problem.

INTRODUCTION

The purpose of this paper is to consider anew the classical, that is, special-relativistic but non-quantum-mechanical, equation of motion of a charged point particle in an external electromagnetic field. The original motivation was to understand precisely how energy-momentum conservation is realized in the Lorentz-Dirac equation[1]

$$m\frac{dv^\mu}{d\tau} = eF^{\mu\nu}_{\text{ext}}v_\nu + \tfrac{2}{3}e^2(\dot{a}^\mu - a^2 v^\mu), \tag{1}$$

where $a^\mu \equiv \dot{v}^\mu$, and the dot means differentiation with respect to the proper time τ. The physical interpretation of the Schott term, $\frac{2}{3}e^2\dot{a}^\mu$, has always been puzzling, and because this term is responsible for the non-Newtonian features of Eq. (1), the microscopic noncausality, preacceleration, and runaway solutions, its elucidation and, indeed, the confirmation of its existence are important.

The Schott-term puzzle, the difficulties associated with it, and a certain intangibility in the problem, have stimulated repeated discussions,[2,5,10,11,16,17] which continue to the present day,[3,6,7] of this classical system. Although these discussions have been theoretically motivated, recent astrophysical research has begun to lend experimental significance to them.[4] This paper is in the first tradition and tries to unite an acceptable (meaningful) physical basis with an acceptable (divergence-free) mathematical formalism.

An understanding of the classical charged-particle problem has ramifications in quantum mechanics as well. Traces of the Schott term, and the term proportional to the Larmor radiation rate ($\frac{2}{3}e^2a^2$), are conspicuous by their absence in quantum mechanics. Indeed, no single-particle relativistic theory exists. It is to be hoped that a clarification of the classical theory will point the way to a quantum-mechanical formulation that can answer these questions.

Dirac's discussion[1] of Eq. (1), when it is looked at with an eye to understanding the physical meaning of the input assumptions, has two problematic features. He integrated the electromagnetic energy tensor, Θ, over a narrow tube surrounding the particle's world line; his electrodynamics follows from equating this integral to a specific function of particle variables—the choice is apparently arbitrary. Also he subtracted an infinite coefficient in the integral from an undetermined constant to produce the rest mass of the particle (the first covariant mass renormalization). This scheme of argument was an open-ended one, which allowed full freedom to creative intuition at its critical stage.

After the event one can see the pattern. Dirac's discussion can be reformulated as an instance of the very general expression of momentum conservation given by

$$\int dS_\mu T^{\mu\nu} = 0. \tag{2}$$

In the present case $T = K + \Theta$, the total energy tensor written as the sum of the separate tensors for the particle and electromagnetic field, and dS is the differential element for the three-surface enclosing a volume of space-time. Such a formulation is given by de Groot and Suttorp, and the necessary simple modification of Dirac argument will be outlined below. The mass renormalization is still required as before.

When the problem is formulated as in Eq. (2), the form of the retarded solutions to Maxwell's equations very strongly suggests applying (2) to a different surface: a tube surrounding the world line, but whose ends are light cones rather than spacelike planes. If this is done, the integral can be calculated exactly, instead of approximately, and much more easily. The equation of motion which emerges, however, is different; not (1), but (see also Refs. 6 and 7)

$$\frac{d}{d\tau}(mv^\mu) = eF^{\mu\nu}_{\text{ext}}v_\nu - \tfrac{2}{3}e^2 a^2 v^\mu, \tag{3}$$

after a mass renormalization of identical form to that needed in the previous treatments. Contracting (3) with v_μ, we find that it is equivalent to

$$m\frac{dv^\mu}{d\tau} = eF^{\mu\nu}_{\text{ext}}v_\nu \tag{4}$$

and

$$\frac{dm}{d\tau} = -\tfrac{2}{3}e^2 a^2. \tag{5}$$

Before attempting to resolve the contradiction between Eqs. (1) and (3), we note that the latter gives a comprehensible description of force and energy within classical electrodynamics. The Schott term is absent, and hence so are the difficult features associated with it. The variability of mass is negligible experimentally for particles of classical size,[6] and for fixed mass elementary particles, one would argue that classical theory was inappropriate anyway, and that only in quantum mechanics could radiation be consistent with unchanging mass. The need for quantum mechanics in a description of the hydrogen atom, precisely because the radiation cannot be neglected, suggests that the boundary between the classical and quantum-mechanical theories is in the region where radiation assumes importance. Finally, Eq. (3), expressed in the instantaneous rest system of the particle [identified by the subscript (0)]

$$\frac{dp^0_{(0)}}{d\tau} = -\tfrac{2}{3}e^2 a^2, \quad \frac{d\vec{p}_{(0)}}{d\tau} = e\vec{E}_{(0)}, \tag{6}$$

contains a simple statement of energy-momentum balance between the elements of the problem we think we know something about: mass, Larmor radiation, and Coulomb force. In particular, the first equation equates the energy loss in Larmor radiation to the decrease in rest mass, the external field doing no work in the instantaneous rest frame. But if this view is correct, there are no energy-momentum changes to take account of, in the "near," as opposed to the asymptotic, electromagnetic field of the particle.

The need for mass renormalization in the application of momentum conservation as expressed in Eq. (2) reminds us forcefully of the singularities in T which prevent pure mathematics from providing protection against inconsistencies. It is especially distressing that it is not sufficient to renormalize covariantly, since this itself can be done in different ways. The most promising means of getting a mathematically consistent scheme is to use distribution theory.[8,9] In place of (2) we shall use

$$\partial_\mu T^{\mu\nu} = 0, \tag{7}$$

to which (2), at least in the differentiable case, is equivalent by Gauss's theorem. The left-hand side of (7) is defined, as a distribution, by

$$(\partial_\mu T^{\mu\nu}, \phi) \equiv -(T^{\mu\nu}, \partial_\mu \phi), \tag{8}$$

in terms of a definition of T. The problem then becomes one of defining T, especially its most singular terms, varying as ρ^{-3} and ρ^{-4} (ρ is the distance, in the rest frame, of the particle from the field point). Adopting what appears to be the most natural definition (but without being able to claim mathematical uniqueness), the whole problem and its solution become well defined, and, for this choice of T, the ambiguity in the equation of motion is settled in favor of (1), the Lorentz-Dirac equation. No mass renormalization of any sort, and certainly not an infinite one, is needed when the problem is formulated in distribution theory terms.

This is not the first time that distribution theory has been applied to the problem of electrodynamics. Some time ago Taylor[10] sketched such a theory for the electromagnetic energy tensor Θ, and showed how Eq. (1) might be derived. His technique was a formal use of Hadamard's method for extracting the finite parts of divergent integrals. Here, precise definitions of the distributions corresponding to functions with singularities ρ^{-3} and ρ^{-4} are based on analogy with the one-dimensional case, for which a quite different characterization reproduces the Hadamard formulas. To the extent that the two theories may be compared, they agree; this itself is interesting because the regularization of divergent integrals is arbitrary up to a distribution with support on the points of singularity. What is still needed, however, is a convincing physical or mathematical principle to cut through this arbitrariness and justify (or not) the choices made here and in Ref. 10.

EQUATIONS OF MOTION

The notational conventions we use are x^k Cartesian spatial coordinates in some inertial frame,

x^0 the time, with $c=1$, $\partial_\mu = \partial/\partial x^\mu$, a spatially favored special-relativistic metric with $g^{ij}=\delta^{ij}$, $g^{00}=-1$, the scalar product of vectors $a\cdot b = a^\mu b_\mu = a^k b^k - a^0 b^0$. In most equations, generality, covariance, and clarity are all served best by omitting the scripts on vectors $(v, a, u, R, z, dS, B, \partial, p, x, j)$ and tensors $(F, \Theta, T, K, A, g, uu, vv, \text{etc.})$. The symbols may then be read as abstract vectors and linear operators (symmetric except for F), or as column (or row) vectors and matrices in a particular coordinate system. Besides the examples above, $F\cdot v$, $\partial\cdot T$, etc., are vectors with components $F^{\mu\nu}v_\nu$, $\partial_\nu T^{\nu\mu}$, etc. Only when a particular frame, e.g. the rest frame, is used are the scripts necessary. Tensors of the form (α,β) and $[\alpha,\beta]$, using anticommutator and commutator brackets, have components $\alpha^\mu\beta^\nu+\beta^\mu\alpha^\nu$ and $\alpha^\mu\beta^\nu-\beta^\mu\alpha^\nu$, respectively.

The four-vector world line of the point particle is denoted by $z(\tau)$, a function of the proper time τ:

$$dz^2 = dz\cdot dz = -d\tau^2,$$

and so the velocity $v \equiv \dot z$ satisfies

$$v^2 = -1,$$

which, by differentiation, gives the relations

$$a\cdot v = 0,$$

$$a^2 + \dot a \cdot v = 0$$

for the acceleration, and $\dot a$. In the instantaneous rest frame, $v^0_{(0)}=1$, $a^0_{(0)}=0$, and $\dot a^0_{(0)} = a^2$.

The various discussions that will be given below, of the equation of motion problem, have in common the Maxwell-Poynting formulation of electromagnetism, with Dirac's point particle source

$$e\,j(x) = e\int d\tau' v(\tau')\,\delta(x - z(\tau')). \tag{9}$$

We use the solution

$$F = F_{\text{ext}} + F_{\text{ret}} \tag{10}$$

of Maxwell's equations, which represents a free external field and the retarded field created by the source, to calculate the electromagnetic energy tensor $\Theta = \Theta_{\text{ext}} + \Theta_{\text{mix}} + \Theta_{\text{ret}}$:

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \tfrac14 g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{11}$$

The notation (except for the sign of Θ which has been changed to make $\Theta^{00}>0$) and the development of the specific forms of F_{ret} and Θ_{ret} needed later, we take as read from Rohrlich's book.[11]

From Maxwell's equations and (11), one can calculate the divergences

$$\partial\cdot\Theta_{\text{ext}} = 0, \tag{12}$$

$$\partial\cdot\Theta_{\text{mix}} = -eF_{\text{ext}}\cdot j. \tag{13}$$

If we could calculate $\partial\cdot\Theta_{\text{ret}}$ there would be no problem; its formal expression contains the factor $\delta(\vec{x})/r^2$, which inhibits all but the boldest.

To interpret Θ, one uses a timelike unit vector t and a spacelike unit vector s, orthogonal to t. Then $-\Theta\cdot t$ is the four-momentum density, and $+\Theta\cdot s$ is the four-momentum flux through a 2-surface with normal s, both these quantities being four-vectors but referring to measurements in a frame with time axis t. The signs are consistent with energy-momentum conservation expressed by (2).

Dirac[1] integrated $\int\Theta\cdot dS$ over the walls (with spacelike normal), but not the ends, of the tube

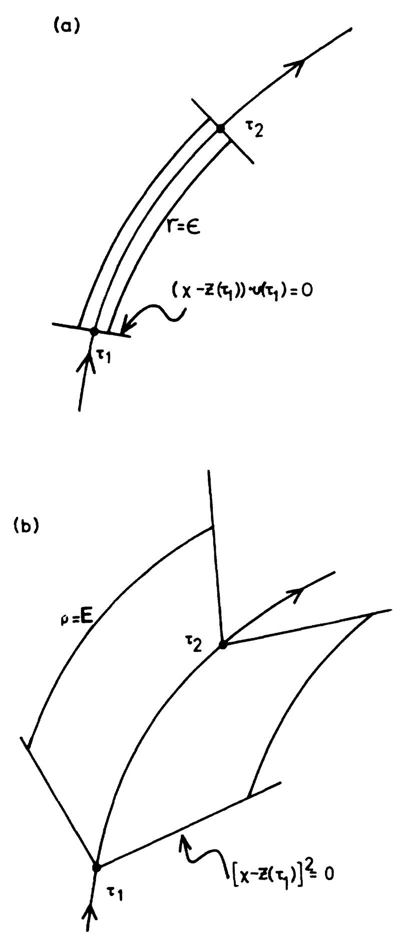

FIG. 1. Two space-time volumes enclosing the portion of the particle's world line between τ_1 and τ_2: (a) the narrow tube with flat ends used by Dirac, (b) finite tube whose ends are light cones.

[see Fig. 1(a)]

$$[x-z(\tau)]^2=\epsilon^2,\qquad [x-z(\tau)]\cdot v(\tau)=0 \tag{14}$$

surrounding the portion of the particle's world line between τ_1 and τ_2. This integral represents the net outflow of electromagnetic momentum from the tube and so should equal the difference in momentum contained within it at the two ends. Dirac did not explicitly use this interpretation, but only the fact that the integral must equal the difference in values of some particle state function at the two ends. Thus he had, for some vector B,

$$\int \Theta\cdot dS=\int_{\tau_1}^{\tau_2} dt\left[-eF_{\text{ext}}\cdot v-\tfrac{2}{3}e^2(\dot a-a^2v)+\frac{e^2a}{2\epsilon}\right]$$

$$=\int_{\tau_1}^{\tau_2} d\tau\,\dot B, \tag{15}$$

in which terms vanishing with ϵ have been omitted. Dirac chose $B=kv$, with a constant k, on the grounds of simplicity. Then to get rid of the divergent ϵ^{-1} term, and to ensure that the equation had the right small velocity form, he set $k=e^2/2\epsilon-m$, and so produced

$$m\dot v=eF_{\text{ext}}\cdot v+\tfrac{2}{3}e^2(\dot a-a^2v). \tag{1}$$

What happens if we try to complete the integral over the surface enclosing Dirac's tube by including the contribution from the ends? The answer is that there is no effective change; the form of the integral remains the same with ϵ replaced by η (say), which also must tend to zero. To see this, we must add to his result

$$\int_{\text{ends}} \Theta\cdot dS=\lim_{\eta\to0}\int_\eta^\epsilon r^2dr\,d\Omega(-v\cdot\Theta)\bigg|_{\tau_1}^{\tau_2}, \tag{16}$$

where r is the radial distance in the three-plane $(x-z)\cdot v=0$, $x=z+r\gamma$, $\gamma^2=1$, and Ω represents the angles defining γ in that plane. But because of Gauss's theorem, and the fact that $\partial\cdot\Theta=0$ off the world line, the integral (15) plus the extra bit (16) for finite η is equal to an integral of the form (15) over a tube of radius η instead of ϵ.

One can see this result happening by writing out the most singular terms in $\Theta\cdot v$ [adapting Dirac's unnumbered equations between his Eq. (60) and his Eq. (61) to the new notation],

$$\Theta\cdot v=\frac{e^2}{4\pi}\left(-\frac{v}{2r^4}+\frac{a\cdot\gamma v}{r^3}\right)+\cdots,$$

at..ng the integrals (16)

$$\int_{\text{ends}}\Theta\cdot dS=\lim_{\eta\to0}\left(-\frac{e^2v}{2r}\bigg|_\eta^\epsilon\right)\bigg|_{\tau_1}^{\tau_2}$$

$$=\lim_{\eta\to0}\int_{\tau_1}^{\tau_2}d\tau\left(-\frac{e^2a}{2r}\bigg|_\eta^\epsilon\right).$$

Adding this to the middle form of (15) just changes ϵ to η.

We now see how Dirac's argument can be put in the form (2), or

$$\int\Theta\cdot dS+\int K\cdot dS=0, \tag{17}$$

in which the integrals are over the closed tube, and K is Minkowski's energy tensor for a particle, but with a "bare" rest mass m_0:

$$K=m_0vv\rho_{(0)}=m_0vj. \tag{18}$$

Here $\rho_{(0)}$ is the proper number density for the particle; in the coordinate system of Eq. (16) it is a δ function at the origin. It therefore contributes to $\int K\cdot dS$ only at the ends of the tube:

$$\int K\cdot dS=\int_0^\epsilon r^2dr\,d\Omega(-v\cdot K)\bigg|_{\tau_1}^{\tau_2}$$

$$=m_0v\bigg|_{\tau_1}^{\tau_2}=\int_{\tau_1}^{\tau_2}d\tau\,m_0a,$$

so the whole of Eq. (17) is

$$\int_{\tau_1}^{\tau_2}d\tau\left[\left(m_0+\frac{e^2}{2\eta}\right)a-eF_{\text{ext}}\cdot v-\tfrac{2}{3}e^2(\dot a-a^2v)\right]=0. \tag{19}$$

Identifying the rest mass $m=m_0+e^2/2\eta$, we get (1), the consistency of the presumption that the rest mass is constant being assured by contracting with v.

Quite apart from its involving an infinite quantity, the mass renormalization process is disturbing because it effects an interchange between the two terms in (17) which otherwise are quite separate and individually interpretable. The philosophy that (17) embodies, imperfectly using renormalization but faithfully using distribution theory, is a view of the world consisting of interacting subsystems, each locally carrying additive and kinematic four-momentum which can be exchan d between subsystems in such a way that the integral condition (2), or the divergence condition (7) is satisfied. That the momentum is kinematic is important because it allows us to take over from the free case the expressions for the individual energy tensors, K in (18), Θ in (11). This formulation began (Pauli[12] has written its early history) with the work of Minkowski and Abraham, where the ideas were conclusions in special cases, and helped along by Einstein's popularization of $\partial \cdot T = 0$ (he, of course, totally opposed the other ideas in it); it has reached acceptance as a fundamental principle,[13] or even dogma.[14] What has prevented complete realization of this has been the dominance of Hamiltonian ideas. In Hamiltonian theory, the subsystems get mixed up, and prominence is given to less physical quantities such as interaction energy and canonical momentum.

The calculations involved in integrating Θ over the tube (14) are complicated because the tube does not "fit" the solution F_{ret}. We can express F_{ret} very simply on a light cone with vertex $z(\tau)$ because all the points on it have the same retarded source point. But to write F_{ret} on a three-plane orthogonal to $v(\tau)$ is difficult and must be done approximately, because all source points earlier than τ are involved, and they have to be related, using Taylor series, with variables at τ. Dirac's initial use of advanced fields, which he added and subtracted, was as a technical aid in these computations.

We shall now work out the consequences of $\int dS\cdot T=0$ by evaluating the integrals in (17) over the invariantly defined tube [see Fig. 1(b)]

$$[x-z(\tau)]^2=0,$$
$$-[x-z(\tau)]\cdot v(\tau)=E \text{ (a constant, big } \epsilon), \tag{20}$$

whose ends are the future light cones with vertices at $z(\tau_1)$ and $z(\tau_2)$. To this end, we define in (21), (26), and (27) the variables whose use simplifies this calculation and those in the next section. After various useful relations are established, the differential elements dS for the integral are computed in (35). The crucial part of the integral is completed in (39), and the variable rest mass equation (3) is the immediate result.

In order to benefit from the close relationship between the surface (20) and the solutions F_{ret}, we introduce light-cone related, intrinsic variables. The notation follows Rohrlich.[11] From any field point x, we "drop a backward light cone" onto the world line, locating a unique point $z(\tau)$, such that

$$x=z(\tau)+R, \tag{21}$$

and R is a future, lightlike vector:

$$R^2=0, \quad R^0=|\vec{R}|. \tag{22}$$

The variables τ and R, three of whose components are independent, can be used in place of x. The Jacobian for the transformation can be easily calculated and one then has the integral formula

$$\int d^4x\, f(x)=\int d\tau \frac{(-v\cdot R)d\vec{R}}{|\vec{R}|} f(z(\tau)+R). \tag{23}$$

At each τ, one can transform $\vec{R}$ to the rest frame, thus converting (23) to

$$\int d^4x\, f=\int d\tau\, d\vec{R}_{(0)} f. \tag{24}$$

This allows us to write

$$\delta(x-z(\tau'))=\delta(\tau-\tau')\delta(\vec{R}_{(0)}),$$

and so, for the current, Eq. (9), we have the alternative expression

$$e\,j(x)=e\,v(\tau)\delta(\vec{R}_{(0)}). \tag{25}$$

The quantity

$$\rho\equiv -v\cdot R \tag{26}$$

is the projection of R on v, and in terms of it we define

$$u\equiv\frac{R}{\rho}-v, \quad R=\rho(u+v) \tag{27}$$

which satisfies

$$u^2=1, \quad u\cdot v=0, \quad \rho=u\cdot R. \tag{28}$$

The interpretation of ρ and u is understood best in the rest system, where

$$\rho=R^0_{(0)}=|\vec{R}_{(0)}|,$$
$$u^0_{(0)}=0, \quad |\vec{u}_{(0)}|=1, \tag{29}$$
$$(R^0_{(0)},\vec{R}_{(0)})=\rho(1,\vec{u}_{(0)}).$$

Therefore, ρ is both the temporal and the spatial displacement of x from $z(\tau)$ in the rest frame at τ, and $\vec{u}_{(0)}$ is a unit vector defining the direction $\vec{x}_{(0)}-\vec{z}_{(0)}$ in the same system. If angles $\Omega_{(0)}$ define $\vec{u}_{(0)}$, we can rewrite the integral formula (24), as

$$\int d^4x\, f=\int d\tau\, \rho^2 d\rho\, d\Omega\, f, \tag{30}$$

in which the subscript is omitted from $d\Omega$ to emphasize its invariant character. It is unnecessary for our purposes to be more specific about the definition of $\Omega_{(0)}$. All that will be needed are the formulas

$$\int d\Omega\, u=\int d\Omega\, uuu=0,$$
$$\frac{1}{4\pi}\int d\Omega\, uu=\tfrac{1}{3}(g+vv), \tag{31}$$

which are the general frame expression of obvious rest frame equations.

By differentiating the defining relations (21), (22), and (26) we find the useful gradients

$$\partial\tau=-R/\rho, \tag{32}$$

$$\partial\rho=u+a\cdot uR. \tag{33}$$

Using the new variables, the volume enclosed by the tube (20) is seen to consist of the points

$$\tau_1 \leqslant \tau \leqslant \tau_2,$$
$$0 \leqslant \rho \leqslant E,$$

whose characteristic function (equal to one inside, zero outside) is

$$\psi \equiv \theta(\tau_2 - \tau)\theta(\tau - \tau_1)\theta(E - \rho). \tag{34}$$

We can now easily calculate the differential elements dS for the surface enclosing this volume. Using (30), (32), (33) and (34) we get

$$\begin{aligned}\int dS\cdot T &= \int d^4x\,\psi\partial\cdot T\\ &= -\int d^4x\,\partial\psi\cdot T\\ &= -\int \rho\,d\rho\,d\Omega\,\theta(E-\rho)R\cdot T\Big|_{\tau_1}^{\tau_2}\\ &\quad+\int_{\tau_1}^{\tau_2} d\tau\,E^2 d\Omega(u + a\cdot uR)\cdot T\Big|_{\rho=E},\end{aligned} \tag{35}$$

so $dS = -\rho d\rho d\Omega R$ on the light cone,[13] and $dS = E^2 d\tau d\Omega(u + a\cdot uR)$ on the surface[15,16] $\rho = E$.

Everything is ready for the calculation of (17), whose terms we regroup as

$$\begin{aligned}0 &= \int T\cdot dS\\ &= \int d^4x\,\psi\partial\cdot(K + \Theta_{\text{ext}} + \Theta_{\text{mix}}) + \int dS\cdot\Theta_{\text{ret}}.\end{aligned}$$

Using current conservation, and (25), (26), (32), with (18),

$$\begin{aligned}\partial\cdot K &= \frac{d}{d\tau}(m_0 v)\left(-\frac{R}{\rho}\right)\cdot j\\ &= \frac{d}{d\tau}(m_0 v)\delta(\vec{R}_{(0)});\end{aligned}$$

and since Eqs. (12) and (13) may be written

$$\partial\cdot\Theta_{\text{mix}} = -eF_{\text{ext}}\cdot v\delta(\vec{R}_{(0)}),\quad \partial\cdot\Theta_{\text{ext}} = 0,$$

the stage

$$0 = \int_{\tau_1}^{\tau_2} d\tau\left[\frac{d(m_0 v)}{d\tau} - eF_{\text{ext}}\cdot v\right] + \int dS\cdot\Theta_{\text{ret}} \tag{36}$$

is immediately reached after integrating the δ functions with (24).

We take over from Rohrlich's book the expression

$$F_{\text{ret}} = \frac{e}{\rho^2}[v, u] + \frac{e}{\rho}\left[a - ua\cdot u, \frac{R}{\rho}\right] \tag{37}$$

for the retarded field, and the calculation of Θ_{ret} that follows from it. Setting

$$\begin{aligned}\Theta_{\text{ret}} &= \Theta_{\text{I}} + \Theta_{\text{II}} + \Theta_{\text{III}}\ \text{(different orders of singularity)},\\ \Theta_{\text{I}} &= \frac{e^2}{4\pi}\left(\tfrac{1}{2}g + vv - uu\right)\frac{1}{\rho^4},\\ \Theta_{\text{II}} &= \frac{e^2}{4\pi}\left(a - ua\cdot u, \frac{R}{\rho}\right)\frac{1}{\rho^3},\\ \Theta_{\text{III}} &= \frac{e^2}{4\pi}(a^2 - [a\cdot u]^2)\frac{RR}{\rho^2}\frac{1}{\rho^2}.\end{aligned} \tag{38}$$

The brackets [,] and(,) in these equations are commutators and anticommutators with no factor one-half. If v and a are evaluated at τ, (37) and (38) give the values of F_{ret} and Θ_{ret} at the points $x = x(\tau, \rho, u)$; that is, they are "already" retarded.

The computation of $dS\cdot\Theta_{\text{ret}}$ is simplified by noticing that $\Theta_{\text{III}}\cdot R = \Theta_{\text{II}}\cdot R = 0$. Using the various scalar products (22), (26), and (28) one finds

$$\begin{aligned}\int dS\cdot\Theta_{\text{ret}} &= \int \rho\,d\rho\,d\Omega\,\theta(E-\rho)\frac{e^2}{4\pi}\frac{R}{2\rho^4}\Big|_{\tau_1}^{\tau_2}\\ &\quad+\int_{\tau_1}^{\tau_2} d\tau\,E^2 d\Omega\frac{e^2}{4\pi}\left\{\frac{-u - a\cdot uR}{2\rho^4} + \frac{a - ua\cdot u}{\rho^3}\right.\\ &\qquad\left.+\frac{[a^2 - (a\cdot u)^2]}{\rho^2}\frac{R}{\rho}\right\}_{\rho=E}\end{aligned}$$

Doing the angular integrations with the help of (31),

$$\begin{aligned}\int dS\cdot\Theta_{\text{ret}} &= \left(-\frac{e^2 v}{2\rho}\Big|_{\delta}^{E}\right)_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau\left(\frac{e^2 a}{2E} + \tfrac{2}{3}e^2 a^2 v\right)\\ &= \int_{\tau_1}^{\tau_2} d\tau\left(\frac{e^2 a}{2\delta} + \tfrac{2}{3}e^2 a^2 v\right),\end{aligned} \tag{39}$$

in which the limit, $\delta \to 0$, arising from the integrations over the light cones, is understood.

Combining (36) and (39) we have

$$\begin{aligned}\int dS\cdot T &= \int_{\tau_1}^{\tau_2} d\tau\left[\frac{d}{d\tau}\left(m_0 v + \frac{e^2 v}{2\delta}\right) - eF_{\text{ext}}\cdot v + \tfrac{2}{3}e^2 a^2 v\right]\\ &= 0.\end{aligned}$$

The mass renormalization $m = m_0 + e^2/2\delta$ is identical to that used for (19), and we get, as the differential form of the integral conservation law over the surface (20),

$$\frac{d}{d\tau}(mv) = eF_{\text{ext}}\cdot v - \tfrac{2}{3}e^2 a^2 v, \tag{3}$$

which differs from the Lorentz-Dirac equation (1) in the absence of the Schott term.

Although Eq. (3) provides a not unreasonable electrodynamics, as was outlined in the Introduction, it is to the inconsistency between it and Eq. (1) that we turn. The contradiction between the equations of motion, both supposedly consequences of $\int T\cdot dS=0$, arises from the mass renormalization. This is highlighted by considering the integral $\int T\cdot dS$ between the ends of the tubes (14), and (20) with $E=\epsilon$, at $\tau=\tau_2$ (say). One can check that the "wall" for this volume, consisting of the points $x=z(\tau)+\epsilon u+tv,\ t:0\to\epsilon$, contributes $\frac{2}{3}e^2a$, whereas the ends, from (16) and (39), contribute $\frac{1}{2}e^2v(\delta^{-1}-\eta^{-1})$; the finite Schott term is lost between the canceling infinities. In the next section, distribution theory is used in order to prevent such ambiguous expressions from arising.

DISTRIBUTION THEORY

Distribution theory does not, at least in the first instance, provide expressions for integrals, such as (2), of generalized functions. Although the linear functionals (T,ϕ) of the theory can be regarded as extending the definition of the integral

$$\int d^4x\, T\phi \equiv (T,\phi), \tag{40}$$

the test functions ϕ cannot have sharp boundaries, but must be infinitely differentiable, and vanish outside a finite region (these conditions can be weakened later). So in trying to resolve the ambiguities that renormalization has left, we work toward a realization of (7)

$$\partial\cdot T=0, \tag{7}$$

or, more explicitly,

$$\partial\cdot(K+\Theta_{\text{ext}}+\Theta_{\text{mix}}+\Theta_{\text{I}}+\Theta_{\text{II}}+\Theta_{\text{III}})=0. \tag{41}$$

Equation (7) means

$$(\partial\cdot T,\phi)\equiv-(T,\cdot\,\partial\phi)=0, \tag{8}$$

and so the whole problem reduces to finding clear distribution definitions of the terms in T.

In place of (18), we are now able, as it turns out, to use what we would have preferred to use before, Minkowski's expression for the particle's energy tensor with the experimental rest mass

$$K=mvv\rho_{(0)}=pj. \tag{42}$$

We have, using $j=v\delta(\vec{\text{R}}_{(0)})$,

$$(K,\phi)=\int d\tau\, p(\tau)v(\tau)\phi(z(\tau)), \tag{43}$$

which is equivalent to the formal integral

$$\int d^4x\,\phi(x)\int d\tau' pv(\tau')\delta(x-z(\tau')).$$

From (43), (8), and (24),

$$\begin{aligned}(\partial\cdot K,\phi)&=-\int d\tau\, pv\cdot\partial\phi\\ &=\int d\tau\,\dot{p}\phi\\ &=(\dot{p}\delta(\vec{\text{R}}_{(0)}),\phi)\end{aligned} \tag{44}$$

after integrating by parts and using $\phi(\pm\infty)=0$.

If the source of the external field is outside the support of ϕ, then Θ_{ext} will certainly be a locally integrable function, and so the straightforward definition

$$(\Theta_{\text{ext}},\phi)=\int d^4x\Theta_{\text{ext}}\phi \tag{45}$$

can be used. Integrating by parts we find that

$$(\partial\cdot\Theta_{\text{ext}},\phi)=0 \tag{46}$$

because $\partial\cdot\Theta_{\text{ext}}=0$ as a function.

The two pieces Θ_{mix} and Θ_{III} have the same slightly increased degree of difficulty. They are both singular on the world line $\rho=0$, but since the singularities are no worse than ρ^{-2}, they are integrable, and a definition like (45) can still be used. Indeed, taking Θ_{III} (Θ_{mix} is similar) we can write, because the integral is covergent,

$$(\Theta_{\text{III}},\phi)=\int d^4x\,\theta(\rho-\epsilon)\Theta_{\text{III}}\phi, \tag{47}$$

where the limit $\epsilon\to0$ is understood. Then, using the fact[17] that $\partial\cdot\Theta_{\text{III}}=0$ for $\rho\neq0$, and evaluating the angular integrals with (31),

$$\begin{aligned}(\partial\cdot\Theta_{\text{III}},\phi)&=\int d^4x\,\delta(\rho-\epsilon)(u+a\cdot uR)\cdot\Theta_{\text{III}}\phi\\ &=\int d\tau\, d\Omega\frac{e^2}{4\pi}(a^2-[a\cdot u]^2)(u+v)\phi\Big|_{\rho=\epsilon}\\ &=\int d\tau\tfrac{2}{3}e^2a^2v\phi(z(\tau)).\end{aligned}$$

Therefore,

$$\partial\cdot\Theta_{\text{III}}=\tfrac{2}{3}e^2a^2v\delta(\vec{\text{R}}_{(0)}), \tag{48}$$

and similarly

$$\partial\cdot\Theta_{\text{mix}}=-eF_{\text{ext}}\cdot v\delta(\vec{\text{R}}_{(0)}). \tag{49}$$

The real difficulty in this treatment is the problem of defining Θ_{I} and Θ_{II}. They have singularities, ρ^{-4} and ρ^{-3}, that are not integrable. From now on we shall take them together, as a sum, $\Theta_{\text{I+II}}$. This is suggested not only by the difficulties they present, but because[17]

$$\partial \cdot \Theta_{\mathrm{I+II}} = 0, \quad \rho \neq 0. \tag{50}$$

Since $\Theta_{\mathrm{I+II}}$ is well defined for $\rho \neq 0$, we need a distribution definition which reduces to

$$(\Theta_{\mathrm{I+II}}, \phi) = \int d^4x\, \Theta_{\mathrm{I+II}}\phi \tag{51}$$

when $\phi = 0$ in a neighborhood of $\rho = 0$; that is, we need a regularization of the integral for general ϕ. About the problems that regularization poses, Gel'fand and Shilov[9] remark that only partial answers exist. However, they give enough clues in discussing simple problems to make one particular regularization in our case seem highly attractive.

The regularization of (51), whose consequences will be followed up, is what appears to be the simplest in the natural τ, ρ, u coordinate system. It is based on analogy with more clear cut cases. The remarks that follow, up to Eq. (61), are meant only to motivate the analogy. As it has been defined so far, the regularization problem has no unique solution because to any regularization satisfying (51), one can add a distribution concentrated on the world line to provide another. But the presumption here is that a mathematical characterization will eventually single out the solutions chosen below.

Discussing functions of one variable, of the form $x^\lambda \theta(\pm x) \equiv x_\pm^\lambda$ (λ complex) and x^{-n} (n integral), Gel'fand and Shilov find a particular regularization which has such attractive properties (relations with derivatives, consistency under multiplication by differentiable functions) that they call it canonical. In the two cases of most interest to us this regularization may be written

$$(x^{-1}, \phi) = \int dx \frac{\phi(x) - \theta(1 - |x|)\phi(0)}{x}, \tag{52}$$

$$(x^{-2}, \phi) = \int dx \frac{\phi(x) - \phi(0) - \theta(1 - |x|)x\phi'(0)}{x^2}. \tag{53}$$

These are also the pseudofunctions of Schwartz,[8] and the finite parts of Hadamard. Despite their rather odd looking forms, these definitions are distinguished uniquely by satisfying

$$\left(\frac{d}{dx}\ln|x|, \phi\right) = (x^{-1}, \phi), \quad \left(\frac{d}{dx}x^{-1}, \phi\right) = (-x^{-2}, \phi). \tag{54}$$

The effect of the subtraction terms in $\phi(0)$ and $\phi'(0)$ is to make the integrals convergent near $x = 0$, but the subtractions need not be regarded as having been made *ad hoc* for this purpose as the relations (54) require them. If $\phi = 0$ near $x = 0$, the definitions (52) and (53) return to the form $\int dx\, x^{-n}\phi$. The cutoff $\theta(1 - |x|)$ ensures that each integral exists as $x \to \pm\infty$; the fact that it is at $|x| = 1$ is because $\ln x = 0$ there. The θ factor is not necessary and is not present in the similar formulas for the analytic (as functions of λ) $x_\pm^\lambda$; but as a consequence, these distributions have poles at the negative integers. They cancel in the combinations used to form x^{-n}.

For the case of several variables (we are interested primarily in three, to regularize functions with singularities ρ^{-n} on surfaces with τ constant), the canonical regularization is apparently impossible. a result quoted by Gel'fand and Shilov. Nonetheless. the regularizations

$$(r^{-3}, \phi) = 4\pi \int r^2 dr \frac{\overline{\phi}(r) - \theta(1 - r)\overline{\phi}(0)}{r^3}, \tag{55}$$

$$(r^{-4}, \phi) = 4\pi \int r^2 dr \frac{\overline{\phi}(r) - \overline{\phi}(0) - r\theta(1 - r)\overline{\phi}'(0)}{r^4}, \tag{56}$$

$$\overline{\phi}(r) \equiv \int \frac{d\Omega}{4\pi}\phi, \tag{57}$$

based on (52) and (53), suggest themselves immediately. These definitions, in a slightly different form, are also given by Schwartz. For our infinitely differentiable ϕ, of course $\overline{\phi}(0) = \phi(0)$, $\overline{\phi}'(0) = 0$, $\overline{\phi}'''(0) = 0$, etc., but the forms given above are suggestive of the further slight generalization we shall need. The regularizations of r^{-3} and r^{-4} have been based on x^{-n} rather than x_+^λ as the range of integration might suggest. The latter have poles at the negative integers, although for even negative integers the residues vanish [$\overline{\phi}'(0) = 0$, etc.].

Setting

$$\Theta_{\mathrm{I+II}} - A\rho^{-4}, \tag{58}$$

where, from (38),

$$A = \frac{e^2}{4\pi}[(\tfrac{1}{2}g + vv - uu) + \rho(a - ua \cdot u, u + v)], \tag{59}$$

we see that A is not continuous at $\rho = 0$ because of the lack of definition of the radial vector u at those points. This is why it is necessary to introduce, corresponding to (57),

$$\overline{\psi}(\rho, \tau) \equiv \int \frac{d\Omega}{4\pi} A\phi. \tag{60}$$

We now adopt, as the regularization of (51),

$$(\Theta_{\mathrm{I+II}}, \phi) = 4\pi \int d\tau \int \rho^2 d\rho \frac{\overline{\psi}(\rho) - \overline{\psi}(0) - \rho\theta(1 - \rho)\overline{\psi}'(0)}{\rho^4}, \tag{61}$$

which is based on (56), and in which the τ dependence of the integrand is suppressed. If we insert a $\theta(\rho-\epsilon)$ in the integral, as we may, since it converges, we can deal more freely with the separate pieces

$$(\Theta_{\mathrm{I+II}},\phi) = 4\pi\int d\tau\,\rho^2 d\rho\,\theta(\rho-\epsilon)\frac{\bar\psi(\rho)-\bar\psi(0)-\rho\theta(1-\rho)\bar\psi'(0)}{\rho^4}. \qquad (62)$$

The limit $\epsilon\to 0$ is again understood.

To get the subtraction terms, we calculate the first two terms in a series expansion of $\bar\psi(\rho)$. Since τ is fixed and we are therefore expanding on the light cone, we can use

$$\phi(x)=\phi(z(\tau)+R) = \phi(z(\tau))+R\cdot\partial\phi(z(\tau))+\tfrac{1}{2}(R\cdot\partial)^2\phi(z(\tau))+\cdots, \qquad (63)$$

where, as before, $R=\rho(u+v)$. Putting this expression for ϕ, and that for A given by (59), in the definition (60), we get

$$\bar\psi=\frac{e^2}{4\pi}[(\tfrac{1}{6}g+\tfrac{2}{3}vv)\phi(z(\tau)) +\rho(\tfrac{1}{6}g+\tfrac{2}{3}vv)v\cdot\partial\phi(z(\tau))+\rho\tfrac{2}{3}(a,v)\phi(z(t))+\cdots], \qquad (64)$$

with the help of the integral formulas (31).

The terms in (64) which are linear in ρ enter (62) as $\rho\bar\psi'(0)$. The first of them, containing $v\cdot\partial\phi$ as a factor, can be integrated by parts, and what remains cancels the other. Thus (62) has, effectively, only the $\bar\psi(0)$ subtraction, and it may be written

$$(\Theta_{\mathrm{I+II}},\phi)=\int d^4x\,\theta(\rho-\epsilon)\Theta_{\mathrm{I+II}}\phi -\frac{e^2}{\epsilon}\int d\tau(\tfrac{1}{6}g+\tfrac{2}{3}vv)\phi(z(\tau)), \qquad (65)$$

which is a convenient expression for the regularization of (51). This contains an explicit form, for general ϕ, for the divergent part of a symbolic equation for a special class of ϕ given by Taylor.[10]

The calculation of $\partial\cdot\Theta_{\mathrm{I+II}}$ is now straightforward. Using (50) and (33), with (65) and (8),

$$(\partial\cdot\Theta_{\mathrm{I+II}},\phi)=\int d\tau\,\rho^2 d\rho\,d\Omega\,\delta(\rho-\epsilon)(u+a\cdot uR)\cdot\Theta_{\mathrm{I+II}}\phi +\frac{e^2}{\epsilon}\int d\tau(\tfrac{1}{6}g+\tfrac{2}{3}vv)\cdot\partial\phi(z(\tau)). \qquad (66)$$

In the first integral, the scalar products are done as usual and ϕ may be expanded as in (63); omitting terms which vanish with ϵ, this integral is

$$e^2\int d\tau\frac{d\Omega}{4\pi}\left\{-\frac{u}{2\epsilon^2}[\phi+\epsilon(u+v)\cdot\partial\phi+\tfrac{1}{2}\epsilon^2(u+v)_\mu(u+v)_\nu\partial^\mu\partial^\nu\phi] + \frac{1}{\epsilon}[a-\tfrac{3}{2}ua\cdot u-\tfrac{1}{2}va\cdot u][\phi+\epsilon(u+v)\cdot\partial\phi]\right\},$$

in which ϕ is evaluated at $z(\tau)$. The angular integrals are evaluated with (31)

$$e^2\int d\tau\left[-\frac{1}{6\epsilon}(\partial+vv\cdot\partial)_\mu\phi-\tfrac{1}{4}\times 2\times\tfrac{1}{3}(g+vv)_{\mu\nu}v_\rho\partial^\nu\partial^\rho\phi +\frac{1}{2\epsilon}a_\mu\phi-\tfrac{1}{6}v_\mu\cdot\partial\phi+\frac{a_\mu}{2}v\cdot\partial\phi\right],$$

and after integrations by parts, we get

$$\frac{e^2}{\epsilon}\int d\tau\left(-\frac{\partial\phi}{6}-\tfrac{2}{3}vv\cdot\partial\phi\right)-e^2\int d\tau\tfrac{2}{3}\dot a\phi(z(\tau)).$$

Equation (66) therefore reads

$$(\partial\cdot\Theta_{\mathrm{I+II}},\phi)=-\tfrac{2}{3}e^2\int d\tau\,\dot a\phi(z(\tau)) =(-\tfrac{2}{3}e^2\dot a\delta(\vec{\mathrm{R}}_{(0)}),\phi)$$

or (67)

$$\partial\cdot\Theta_{\mathrm{I+II}}=-\tfrac{2}{3}e^2\dot a\delta(\vec{\mathrm{R}}_{(0)}).$$

Teitelboim[18] gives a similar divergence relation, but with an additional infinite renormalization term which, besides being purely formal, is a reminder of its derivation—an integral over the surface (14)—and the susceptibility of it to ambiguity, by comparison with integral (39) over (20).

Bringing together (44), (46), (48), (49), and (67), the explicit form of $\partial\cdot T=0$, Eq. (41), becomes

$$(\dot p-eF_{\mathrm{ext}}\cdot v-\tfrac{2}{3}e^2\dot a+\tfrac{2}{3}e^2a^2v)\delta(\vec{\mathrm{R}}_{(0)})=0, \qquad (68)$$

from which the Lorentz-Dirac equation is recovered. Equation (68), with the terms derived from Θ grouped together, expresses the equality of the three-dimensional densities for the rate of gain of particle momentum and the rate of loss of electromagnetic momentum. But having split the field into F_{ext} and F_{ret}, we change our language and describe $eF_{\mathrm{ext}}\cdot v\delta(\vec{\mathrm{R}}_{(0)})$, which arises from Θ_{mix}, as a force density; the other terms,

$$\partial\cdot\Theta_{\mathrm{ret}}=\tfrac{2}{3}e^2(a^2v-\dot a)\delta(\vec{\mathrm{R}}_{(0)}), \qquad (69)$$

which would be present even if the force were not electromagnetic, are the density for the rate of gain of electromagnetic momentum carried by the retarded field created by the particle.

The first term in (69) supplies the Larmor radiation at the invariant rate $\tfrac{2}{3}e^2a^2v$, and the second governs changes, at the rate $-\tfrac{2}{3}e^2\dot a$, of momentum, which, because it does not reach the radiation zone, may be described as bound.[17]

An integral which expresses the decomposition just mentioned is the distribution theory version of $\int dS\cdot\Theta_{\rm ret}$ over the surface (20), illustrated in Fig. 1(b). It was in the calculation, (39), of this integral using renormalization theory, that the Schott term was lost. Here we shall see how that term reappears, and of course there will be no divergences. To set the integral up, we use

$$\begin{aligned}(\partial\cdot\Theta_{\rm ret},\phi)&=\int d\tau\tfrac{2}{3}e^2(a^2v-\dot a)\phi(z(\tau))\\&=-(\Theta_{\rm ret},\cdot\,\partial\phi)\\&\equiv\mathrm{Reg}\int d^4x\,\Theta_{\rm ret}\cdot(-\partial\phi),\end{aligned}\tag{70}$$

where "Reg" means the regularization of the integral; so far as $\Theta_{\rm III}$ is concerned, Reg has no effect, but for $\Theta_{\rm I+II}$, the limiting process (65) is meant. The test functions ϕ are supposed to have compact support and to be infinitely differentiable. So we use a sequence of smoothed forms of the characteristic function (34),

$$\phi=\mathrm{smooth}[\theta(\tau_2-\tau)\theta(\tau-\tau_1)\theta(E-\rho)],\tag{71}$$

and look at the limit of (70) as the smoothing is reduced. This provides the distribution theory expression of the integral:

$$\begin{aligned}\int dS\cdot\Theta_{\rm ret}&\equiv\lim\mathrm{Reg}\int d^4x\,\Theta_{\rm ret}\cdot(-\partial\phi)\\&=\int_{\tau_1}^{\tau_2}d\tau\tfrac{2}{3}e^2(a^2v-\dot a).\end{aligned}\tag{72}$$

Before seeing exactly how (72) is realized, it will be convenient to have available a slightly different form of the regularization (65), namely,

$$\begin{aligned}(\Theta_{\rm I+II},\phi)&\equiv\mathrm{Reg}\int d^4x\,\Theta_{\rm I+II}\phi\\&=\int d^4x\,\theta(r-\epsilon)\Theta_{\rm I+II}\phi\\&\quad-\frac{e^2}{\epsilon}\int d\tau(\tfrac{1}{6}g+\tfrac{2}{3}vv)\phi(z(\tau)),\end{aligned}\tag{73}$$

in which the region excluded from the first integral is the interior of the tube (14), and, as before, r is the radial distance in the plane $[x-z(\tau)]\cdot v(\tau)=0$; that is,

$$x=z(\tau)+r\gamma,\quad\gamma^2=1,\quad\gamma\cdot v(\tau)=0,$$

for points on the simultaneity plane at τ. To show that the two regularizations are the same, we must show that

$$\int d^4x[\theta(\rho-\epsilon)-\theta(r-\epsilon)]\Theta_{\rm I+II}\phi=0$$

in the (understood) $\epsilon\to0$ limit. This condition may be written

$$\int d\tau(1+ra\cdot\gamma)r^2dr\,d\Omega(\gamma)[\theta(\rho-\epsilon)-\theta(r-\epsilon)]\Theta_{\rm I+II}\phi=0,\tag{74}$$

using the four-dimensional differential[16] in terms of τ, r, and γ.

To check the condition (74), we must rewrite the most singular parts of $\Theta_{\rm I+II}$ in terms of the planar variables τ, r, and γ. The geometry is in Fig. 2, and the condition

$$\begin{aligned}r\gamma&=x-z(\tau)\\&=z(\tau-t)+\rho(u+v(\tau-t))-z(\tau)\end{aligned}$$

leads, using $\gamma\cdot v(\tau)=0$, $\gamma^2=1$, $u\cdot v(\tau-t)=0$, to

$$\begin{aligned}t&=\rho-\rho^2a\cdot u+\cdots,\\r&=\rho-\tfrac{1}{2}\rho^2a\cdot u+\cdots,\\\rho&=r+\tfrac{1}{2}r^2a\cdot u+\cdots\\&=r+\tfrac{1}{2}r^2a\cdot\gamma+\cdots,\\u&=\gamma-\tfrac{1}{2}ra\cdot\gamma\gamma-ra\cdot\gamma v+\tfrac{1}{2}ra+\cdots,\\v(\tau-t)&=v-ra+\cdots,\end{aligned}\tag{75}$$

in which v, a are the vectors at τ. Substituting in (38), and keeping only the r^{-4} and r^{-3} terms,

$$\Theta_{\rm I+II}=\frac{e^2}{4\pi}\left[\frac{\tfrac{1}{2}g+vv-\gamma\gamma}{r^4}+\frac{\tfrac{1}{2}(a,\gamma)+a\cdot\gamma(-g+\gamma\gamma-2vv)}{r^3}+\cdots\right],\tag{76}$$

wherein again, $v=v(\tau)$, $a=a(\tau)$, the simultaneous values.

Putting (76) in the left-hand side of (74), which may now be written

$$\int d\tau\int d\Omega\int_\epsilon^{\epsilon-\epsilon^2a\cdot\gamma/2+\cdots}r^2dr\,(1+ra\cdot\gamma)\Theta_{\rm I+II}\phi,$$

one finds that the integral vanishes as $\epsilon\to0$, and

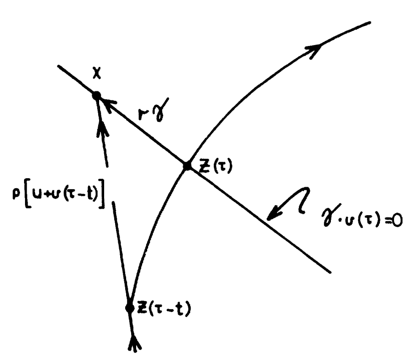

FIG. 2. The connection between planar variables and light-cone variables, Eq. (75).

so the alternative regularization is verified.

To return to the dissection of (72). There is no difficulty in taking the required limit in (72), except where $\Theta_{\rm ret}$ is singular, at the vertices of the light cones $\tau=\tau_1$ and $\tau=\tau_2$. For the part of the integral over $\rho=E$, one gets, by the same method used for (39), that is, with $-\partial\phi=\delta(\rho-E)(u+a\cdot uR)$,

$$\lim \mathrm{Reg}\int_{(\rho=E)} d^4x\,\Theta_{\rm ret}\cdot(-\partial\phi)=\int_{\tau_1}^{\tau_2}d\tau\left[\frac{e^2a}{2E}+\tfrac{2}{3}e^2a^2v\right], \tag{77}$$

which, in the radiation zone limit $E\to\infty$, gives the integral over the Larmor radiation rate, the contribution of $\Theta_{\rm III}$. Nor is there any difficulty with the $\Theta_{\rm III}$ parts of the light-cone integrals. Because these parts are convergent, and $\Theta_{\rm III}\cdot(-\partial\phi)=0$ since $\partial\tau=-R/\rho$, there is no contribution.

But because $\partial\tau=-R/\rho=-(u+v)$ is undefined at $\rho=0$, where $\Theta_{\rm I+II}$ is singular and not integrable in the ordinary sense, the $\Theta_{\rm I+II}$ part of the integral over the light cones in (72) must be evaluated with care. Instead of trying to approach the light cone $\tau=\tau_2$ (and similarly for $\tau=\tau_1$) directly as the smoothing is removed, we approach the light cone truncated (see Fig. 3) by the plane $[x-z(\tau_2+\delta)]\cdot v(\tau_2+\delta)=0$, and then take the limit as $\delta\to0$. The characteristic function for the volume bounded by the truncated light cone has a well-defined gradient, $v(\tau_2+\delta)\delta(v(\tau_2+\delta)\cdot[x-z(\tau_2+\delta)])$, where the world line crosses the boundary, and the processes of removal of smoothing and of regularization may be interchanged.

The points on the truncating plane may be written

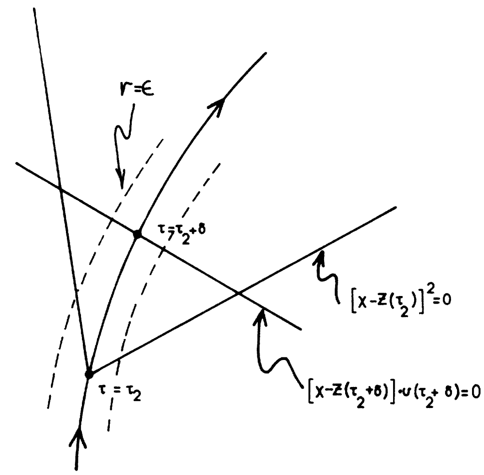

FIG. 3. Before the double limit [regularization ($\epsilon\to0$) and then removal of smoothing ($\delta\to0$)] is taken in Eqs. (78) and (80).

$$x=z(\tau_2+\delta)+r\gamma,\quad \gamma^2=1,\quad \gamma\cdot v(\tau_2+\delta)=0.$$

Where the plane cuts the light cone $\tau=\tau_2$,

$$\rho=\delta+\delta^2a\cdot u+\cdots,$$

$$r=\delta+\tfrac{1}{2}\delta^2a\cdot\gamma+\cdots,$$

with identical expressions at the earlier end ($\tau=\tau_1$). The remaining terms of (72) may now be written as a sum of integrals over the portion of the light cone with $\rho\geq\delta+\delta^2a\cdot u+\cdots$, and the region of the truncating plane with $r\leq\delta+\frac{1}{2}\delta^2a\cdot\gamma+\cdots$:

$$\lim_{\delta\to0}\left[\int d\Omega(u)\int_{\delta+\delta^2a\cdot u+\cdots}^{E}\rho^2d\rho\,\Theta_{\rm I+II}\cdot(-R/\rho)+\mathrm{Reg}\int d\Omega(\gamma)\int_0^{\delta+\frac{1}{2}\delta^2a\cdot\gamma+\cdots}(1+ra\cdot\gamma)r^2dr\,\Theta_{\rm I+II}\cdot(-v(\tau+\delta))\right]_{\tau_1}^{\tau_2}. \tag{78}$$

The first integral in (78) gives

$$\left[-\frac{e^2v(\tau)}{2E}+\frac{e^2v(\tau)}{2\delta}-\frac{e^2a(\tau)}{6}\right]_{\tau_1}^{\tau_2}, \tag{79}$$

with the omission of terms which vanish with δ. To calculate the second integral in (78), we use the form of regularization given in (73):

$$\left[\int d\Omega(\gamma)\int_\epsilon^{\delta+\delta^2a\cdot\gamma/2+\cdots}r^2dr\,\Theta_{\rm I+II}\cdot(-v(\tau+\delta))(1+ra\cdot\gamma)-\frac{e^2}{\epsilon}(\tfrac{1}{8}g+\tfrac{2}{3}vv(\tau+\delta))\cdot(-v(\tau+\delta))\right]_{\tau_1}^{\tau_2}$$

$$=\left[-\frac{e^2v(\tau+\delta)}{2\delta}\right]_{\tau_1}^{\tau_2}=\left[-\frac{e^2v(\tau)}{2\delta}-\frac{e^2a(\tau)}{2}\right]_{\tau_1}^{\tau_2}. \tag{80}$$

The integration has been done using (76), and again terms vanishing with δ have been omitted, as have those vanishing with ϵ.

Combining (79) and (80), and letting $\delta \to 0$, we have

$$\lim \operatorname{Reg} \int_{(\text{light cones})} d^4x \Theta_{\text{ret}} \ (-\partial\phi) = \left(-\frac{e^2 v}{2E} - \tfrac{2}{3}e^2 a\right)_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} d\tau \left(-\frac{e^2 a}{2E} - \tfrac{2}{3}e^2 \dot{a}\right). \quad (81)$$

The whole contribution here comes from $\Theta_{\text{I+II}}$. The term involving E vanishes in the limit $E \to \infty$, leaving an integral over the Schott term. But for finite E, the E-dependent terms cancel in the sum of (77) and (81), which constitutes a verification of (72). The integral of Θ_{ret} over the light cone, a sum of $\rho^2 d\rho d\Omega \Theta_{\text{ret}} \cdot (-R/\rho)$ is a sum of the momentum $\rho^2 d\rho d\Omega \Theta_{\text{ret}} \cdot (-v)$, less the radial flux $\rho^2 d\Omega d\rho \Theta_{\text{ret}} \cdot u$, a process which removes the contribution of all free radiation.

Although the decomposition, in (77) and (81), of an integral form of the divergence relation (69) for $\partial \cdot \Theta_{\text{ret}}$ involves a light-cone integration whose physical interpretation is unfamiliar, the divergence relation itself already shows that any confusion about the term radiation reaction can only be semantic. If we use it, as perhaps we should, for the reaction due to the whole of the particle's retarded field, then its density is the whole of the right-hand side of (69), $\frac{2}{3}e^2(a^2 v - \dot{a})\delta(\vec{R}_{(0)})$. If we use it in a narrower sense, referring only to radiation that reaches infinity, just the Larmor part applies, and the near field must be dealt with separately. There is certainly no need to introduce an acceleration energy concept, intrinsic to the particle, if by that is meant an energy without alternative physical explanation.

Since $\dot{a}^0_{(0)} = a^2$, the equations of motion in the instantaneous rest system, replacing (6), are

$$\frac{dp^0_{(0)}}{d\tau} = 0, \quad \frac{d\vec{p}_{(0)}}{d\tau} = e\vec{E}_{(0)} + \tfrac{2}{3}e^2 \dot{\vec{a}}_{(0)}, \quad (82)$$

where, here, the interpretation of the second term on the right-hand side of the second equation is the negative of the rate of change of the particle's retarded field momentum, to which only the near field contributes in the rest system. The first equation embodies unchanging rest mass, and in the rest system, no net change of energy in the retarded field, the near and asymptotic changes canceling.

If electrodynamics is formulated using distributions, there is no need for a Coulomb self-energy concept; there are no divergent integrals to which the term could be applied, and the idea is not needed in the physical interpretation; there is no need to account for an infinite difference, as between charged and uncharged particles, in "bare" masses.

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