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Reggeization and the question of higher-loop renormalizability of gravitation

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We study the factorization of gravitational Born amplitudes and the Reggeization of the graviton. Factorization holds at $J = 2$, and the Mandelstam counting argument shows that the graviton must Reggeize, at least to low orders in perturbation theory. The Mandelstam argument also explains the one-loop renormalizability properties of pure gravitation and matter-gravity systems and is used to discuss the higher-loop renormalizability of pure gravitation.

I. INTRODUCTION

In the past twelve years, since it was first posed by Gell-Mann *et al.*,[1] the question whether or not the elementary particles of Lagrangian field theories lie on Regge trajectories has been investigated by a number of authors.[2] It is now believed that the answer is affirmative for renormalizable theories and for those particles which communicate with (two-particle) channels that satisfy the so-called Mandelstam counting criteria.[3,4] These criteria are sufficient conditions which compare the number of arbitrary parameters of, and the number of kinematical constraints on, a unitary analytic scattering amplitude which has an s-channel pole corresponding to the elementary particle.

Necessary conditions can also be formulated, and of these the simplest is the factorization of the Born approximation[1]: For processes involving particles with spin the coefficients of singular terms (in J) of the partial-wave helicity amplitudes $F_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,J)$ form a matrix which factorizes. (We refer the reader to Refs. 1 and 2 for explanation of the nomenclature.) If the Born approximation does not factorize, Reggeization will not take place; if it does, Reggeization *may* take place. We note that the factorization condition is equivalent to the requirement that the rank of the matrix be equal to the rank of the nonsense-nonsense submatrix.[5]

Independent of the question of Reggeization the factorization of the Born approximation is a curious property of some Lagrangian field theories. It is believed on the basis of examples that have been studied that factorization will not take place in a theory which is not renormalizable[6] and that in a renormalizable theory it may be related to the absence of certain counterterms (see Sec. V of Ref. 4). The necessity of renormalizability is seen most strikingly in the examples of massive Yang-Mills theories.[2,7] The requirement of factorization of models involving massive Yang-Mills mesons and scalars forces these models to be of the spontaneously-broken-symmetry type[8] in the same way as requiring tree unitarity does.[9] One might be tempted to conjecture that factorization is a criterion for renormalizability, at least in low (one-loop?) orders of perturbation theory.

We consider in this paper the factorization properties of certain amplitudes involving gravitons. We take as our Lagrangian the sum of the Einstein Lagrangian and those minimally coupled matter Lagrangians whose renormalizability properties have been investigated recently by several authors.[10] These authors have concluded that pure gravitation is one-loop renormalizable whereas gravitation coupled to matter fields is not. We find that the Born amplitudes for graviton-graviton scattering do factorize at $J=2$, albeit in a rather trivial way. With a scalar particle or a photon present factorization holds as well at $J=2$. This is surprising since the theory is not renormalizable. However, we show that these results are consistent with Mandelstam's counting criteria, suitably generalized to amplitudes which exceed the unitarity bounds. These criteria confirm again the striking difference between pure gravity and gravity-matter interactions found in Ref. 10: Whereas at the one-loop level graviton-graviton amplitudes should Reggeize at $J=2$ in pure Einstein theory, they need not when matter is present.

We undertook this work in the hope of shedding some light, from a different angle, on the question of renormalizability of gravitation, but there are additional reasons for looking at the problem. From the work of Scherk and Schwarz[11] we know that the Einstein theory can be obtained (at the tree level) as the zero-slope limit of a certain dual model. Their result suggests Reggeization of the graviton, and it is desirable to check this.

In the next section we exhibit the factorization properties of our amplitudes. In the following section we go through the Mandelstam counting procedure. In the concluding section we discuss our results and speculate on possible implications. The Appendix contains some formulas we have used.

Our conventions are as follows: p (q) stands for the magnitude of the three-momentum of the scalar (graviton) and z is the cosine of the scattering angle in the s channel. Our metric is such that $p_\mu p_\mu = \vec{p}^{\,2} + p_4{}^2 = \vec{p}^{\,2} - p_0{}^2$. The Lagrangian for the coupled graviton-scalar system is

$$\mathcal{L} = g^{1/2}[-2\kappa^{-2}R(g) - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - \tfrac{1}{2}m^2\phi^2], \quad (1)$$

where R is the scalar curvature and ϕ is the scalar field. The metric $g_{\mu\nu}$ is decomposed as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $h_{\mu\nu}$ is the graviton quantum field and $\kappa^2 = 32\pi G$, G being Newton's constant. We refer the reader to Refs. 10, 12, 13, 19, and 20 for an explanation of methods used in graviton quantum processes, and to Ref. 13 in particular for a simplified method for computing gravitational Born amplitudes.

II. FACTORIZATION PROPERTIES

We consider first graviton-graviton scattering. In the Born approximation it is given by s-, t-, and u-channel graviton exchanges and contact terms. For massless gravitons, the helicities are ±2 and the helicity amplitudes are known in the literature,[12,13]

$$F_{2,2;2,2} = \kappa^2\frac{s^4}{4stu}, \quad F_{2,-2;-2,2} = \kappa^2\frac{t^4}{4stu},$$
$$F_{2,-2;2,-2} = \kappa^2\frac{u^4}{4stu}, \quad F_{2,2;2,-2} = F_{2,2;-2,-2} = 0. \quad (2)$$

These amplitudes are defined in terms of the S matrix by

$$\langle\lambda_3,\lambda_4|S-1|\lambda_1,\lambda_2\rangle = i(2\pi)^4\delta^4(p_i - p_f) \times \prod_{i=1}^{4}(2E_i)^{1/2}F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}. \quad (3)$$

Following Gell-Mann *et al.*[1] we construct parity-conserving amplitudes. We find

$$F^{\pm}_{2,2;2,2} = \kappa^2\frac{s}{1-z^2}, \quad F^{+}_{2,-2;2,-2} = \kappa^2\frac{s}{8(1-z^2)},$$
$$F^{\pm}_{2,2;2,-2} = 0, \quad F^{-}_{2,-2;2,-2} = 0. \quad (4)$$

At $J=2$ the state $|2,2\rangle$ is *sense* whereas the state $|2,-2\rangle$ is *nonsense*. In general the partial-wave amplitudes are expected to contain singular terms in the J plane at $J=J_0$ = an integer of the form

$$F^{+}_{ss}(J) \simeq -b_{ss}(s)\delta_{JJ_0} \quad \text{(sense-sense)},$$
$$F^{+}_{ns}(J) = F^{+}_{sn}(J) \simeq b_{sn}(s)(J-J_0)^{-1/2} \quad \text{(sense-nonsense)},$$
$$F^{+}_{nn}(J) \simeq b_{nn}(s)(J-J_0)^{-1} \quad \text{(nonsense-nonsense)}, \quad (5)$$

the factorization condition being $b_{ss} = b_{sn}b_{nn}{}^{-1}b_{ns}$. We note that for this process the odd-parity amplitudes either are zero or contain no singular terms for nonnegative J.

We obtain the partial-wave amplitudes in the usual way,

$$F^{+}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,J) = \tfrac{1}{2}\int_{-1}^{1}[c^{J+}_{\lambda\mu}(z)F^{+}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,z) + c^{J-}_{\lambda\mu}(z)F^{-}_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,z)]dz. \quad (6)$$

The relevant parts of the c functions we will need are given in Appendix A Since the amplitudes involve massless virtual particles we have defined the partial-wave projections by inserting a small mass into the denominator of the graviton propagator. However, the coefficients of the terms singular in the J plane are continuous and well defined in the limit as this mass goes to zero, as follows from the relation (for $J \simeq J_0$)

$$\frac{1}{2}\int_{-1}^{1}\frac{P_{J-J_0-n}(z)}{z_0-z}dz = Q_{J-J_0-n}(z_0) = P_n(z_0)\ln\frac{1+z_0}{1-z_0} + (J-J_0)^{-1}(\text{terms regular as } z_0 \to \pm 1). \quad (7)$$

(Since we do not change numerators of the propagator, the mass discontinuities of Van Dam and Veltman[14] are absent.)

We write down the matrix b of Eq. (5) at those values of J where singularities may arise, i.e., at $J=0,2$ [at $J=1$ the amplitudes $F^{+}(J)$ are zero because of Bose symmetry.]:

$J=2$.

$$\begin{array}{c|cc} & (2,2) & (2,-2) \\ (2,2) & 0 & 0 \\ (2,-2) & 0 & -12\kappa^2q^2 \end{array}.$$

$J=0$. (8)

$$\begin{array}{c|cc} & (2,2) & (2,-2) \\ (2,2) & 0 & 0 \\ (2,-2) & 0 & -68\kappa^2q^2 \end{array}.$$

Since there are no Kronecker δ terms (corresponding, at $J=2$, to the s-channel graviton pole) in the sense-sense amplitude, and since the sense-nonsense amplitude vanishes, the matrices factorize in a trivial way.

A similar effect occurs in massless Yang-Mills theory at $J=1$. Whereas in the renormalizable massive case[2] the singularity matrix is 4×4 with nonvanishing matrix elements, in the massless case since only the states $|1,1\rangle$ and $|1,-1\rangle$ are physical the matrix is 2×2 and one finds (e.g., setting $m=0$ in Table I of Ref. 2) that again only the nonsense-nonsense entry is nonzero. Although factorization holds one would really like to see it happen in a less trivial way for an amplitude in which the vector meson appears as a $J=1$ Kronecker δ term. This can be achieved by looking at the coupled pion–Yang-Mills system. The helicity matrix becomes 3×3, the sense-sense π-π amplitude receives a contribution from the s-channel vector-meson pole, and factorization holds for this less trivial system (see Table II of Ref. 2). *We note that factorization holds provided the pion is coupled in the usual gauge-invariant renormalizable way.* If one takes for example the gauge-invariant unrenormalizable coupling $\vec{F}_{\mu\nu}\cdot(D_\mu\vec{\pi}\times D_\nu\vec{\pi})$ factorization fails.

We proceed in a similar fashion for gravitation by considering a system of massive scalar particles ϕ, minimally coupled to gravitation. We assume first that there is no scalar self-interaction. From Refs. 12 and 13 we have for scalar-graviton scattering the helicity amplitudes

$$\begin{aligned}F_{2,0;2,0}&=-\kappa^2(m^4-su)^2/[4t(s-m^2)(u-m^2)]\,,\\F_{-2,0;2,0}&=-\kappa^2m^4t^2/[4t(s-m^2)(u-m^2)]\,.\end{aligned}\tag{9}$$

Using crossing symmetry (the helicity crossing matrix is trivial since the graviton is massless and the massive particle ϕ has spin zero) we find the annihilation amplitudes

$$\begin{aligned}F_{2,2;0,0}&=-\kappa^2m^4s^2/[4s(t-m^2)(u-m^2)]\,,\\F_{2,-2;0,0}&=-\kappa^2(m^4-tu)^2/[4s(t-m^2)(u-m^2)]\,.\end{aligned}\tag{10}$$

The $\phi\phi$ elastic amplitude is easily calculated using the Lagrangian of Eq. (1). We find

$$F_{0,0;0,0}=\frac{\kappa^2}{4s}\left[(t-2m^2)(u-2m^2)+2m^4\right]+\text{two crossed terms}\,.\tag{11}$$

The singularity matrix at $J=2$ for the even-parity amplitudes is now

$$\begin{array}{c}\\(0,0)\\(2,2)\\(2,-2)\end{array}\begin{array}{c}\begin{array}{ccc}(0,0)&(2,2)&(2,-2)\end{array}\\\left[\begin{array}{ccc}\dfrac{4\kappa^2p^4}{15s}&0&\dfrac{-i8\kappa^2q^2p^2}{\sqrt5 s}\\0&0&0\\\dfrac{-i8\kappa^2p^2q^2}{\sqrt5 s}&0&\dfrac{-48\kappa^2q^4}{s}\end{array}\right]\end{array},\tag{12}$$

where $s=4q^2=4(p^2+m^2)$. We recall that q (p) denotes the graviton (scalar) 3-momentum. The matrix factorizes and we have thus found a Lagrangian field theory, apparently unrenormalizable,[10] for which factorization holds in a channel containing an elementary particle (the graviton).

At $J=0$ the singularity matrix is

$$\begin{array}{c}\\(0,0)\\(2,2)\\(2,-2)\end{array}\begin{array}{c}\begin{array}{ccc}(0,0)&(2,2)&(2,-2)\end{array}\\\left[\begin{array}{ccc}\dfrac{\kappa^2}{s}(\tfrac{44}{3}p^4+20p^2m^2+5m^4)&0&-i\kappa^2(p^2+q^2)\\0&0&0\\-i\kappa^2(p^2+q^2)&0&-68\kappa^2q^2\end{array}\right]\end{array}.\tag{13}$$

Factorization does not hold. However, since in these channels there is no scalar pole, this matrix tells us nothing about Reggeization of the scalar.

We have considered a number of generalizations. We have allowed for a $\lambda\phi^3$ interaction, in which case the scalar-graviton channel couples to the scalar-scalar channel. The $\phi\phi$ elastic amplitude receives an additional contribution from the scalar pole

$$F_{0,0;0,0}=-\frac{\lambda^2}{s-m^2}+\text{two crossed terms}\,,\tag{14}$$

while the scalar-scalar to scalar-graviton amplitude is

$$F_{2,0;0,0}=-\frac{\kappa\lambda}{2}\frac{(s-m^2)p^2(1-z^2)}{(t-m^2)(u-m^2)}\,.\tag{15}$$

Now the graviton-scalar amplitudes of Eq. (9) also become relevant. At $J=2$ the processes in Eqs. (14) and (15) are sense-sense, but the additional contributions do not give Kronecker δ's so that our previous results at $J=2$ are unchanged. At $J=0$ the singularity matrix becomes

$$\begin{array}{c} \\ (0,0) \\ (2,2) \\ (0,2) \\ (2,-2) \end{array}\begin{array}{c} \begin{array}{cccc} (0,0) & (2,2) & (0,2) & (2,-2) \end{array} \\ \left[\begin{array}{cccc} \frac{2\lambda^2}{s-m^2}+\frac{\kappa^2}{s}(\frac{44}{3}p^4+20p^2m^2+5m^4) & 0 & \frac{-i\kappa\lambda(s-m^2)}{2\sqrt{2}p^2} & -i\kappa^2(p^2+q^2) \\ 0 & 0 & 0 & 0 \\ \frac{-i\kappa\lambda(s-m^2)}{2\sqrt{2}\,p^2} & 0 & \frac{\kappa^2(s^2+m^4)}{2(s-m^2)} & 0 \\ -i\kappa^2(p^2+q^2) & 0 & 0 & -68\kappa^2q^2 \end{array}\right] \end{array} . \tag{16}$$

The nonsense-nonsense matrix has rank two, and it is easy to verify that factorization does not hold for any value of the various parameters. We therefore do not agree with the conclusions of Ref. 19. However, the approach in that paper is somewhat different from ours.

Another generalization we have considered uses the "improved" form of the scalar-graviton Lagrangian.[10,13,15] We have added to the Lagrangian of (1) a term

$$\mathcal{L}' \approx g^{1/2}\phi^2 R(g) . \tag{17}$$

The new term can in principle contribute to the scalar-scalar and scalar-graviton amplitudes, but the contributions to the latter vanish[13] and the contributions to the former do not change our conclusions.

We have also considered the coupled graviton-photon-scalar system.[13] We present the singularity matrix at $J=2$ [we have omitted zero rows and columns corresponding to the states (2,2) and (1,1)]

$$\begin{array}{c} \\ (0,0) \\ (1,-1) \\ (2,-2) \end{array}\begin{array}{c} \begin{array}{ccc} (0,0) & (1,-1) & (2,-2) \end{array} \\ \left[\begin{array}{ccc} \frac{4\kappa^2p^4}{15s} & \frac{\sqrt{6}}{15}\kappa^2p^2 & \frac{-i8\kappa^2p^2q^2}{\sqrt{5}s} \\ \frac{\sqrt{6}}{15}\kappa^2p^2 & \frac{2}{5}\kappa^2q^2 & -i2\left(\frac{6}{5}\right)^{1/2}\kappa^2q^2 \\ \frac{-i8\kappa^2p^2q^2}{\sqrt{5}\,s} & -i\,2\left(\frac{6}{5}\right)^{1/2}\kappa^2q^2 & -\frac{48\kappa^2q^4}{s} \end{array}\right] \end{array} . \tag{18}$$

Factorization still holds at $J=2$. However, it fails at $J=0$.

III. MANDELSTAM COUNTING

In this section we check the sufficient conditions for Reggeization, the so-called Mandelstam counting conditions.[3,4] Mandelstam counting is usually carried out on an amplitude which is analytic and unitary. Leaving aside analyticity problems related to infrared behavior, we note that our amplitudes are not unitary since we plan to discuss them in finite orders of perturbation theory. However, the following procedure seems consistent with the Mandelstam approach. To compare the physical and Regge amplitudes start with left-hand and inelastic contributions computed to order n in perturbation theory. (This will play the role of the Mandelstam potential.) Solve now the unitarity and analyticity equations at a given value J_0 for the physical amplitude $F_J(s)$ and at large J for the Regge continuation $F(s,J)$. Continue the latter to $J=J_0$, *then* compare the two amplitudes up to order n in the coupling constant.

We refer the reader to Ref. 4 for a detailed description of the counting procedure, which consists of comparing, at a given J_0, the number of free parameters contained in and the number of kinematical constraints satisfied by both the physical amplitude and the Regge continuation. The amplitude Reggeizes at those values of J where the number of free parameters does not exceed the number of constraints.

The free parameters are Castillejo-Dalitz-Dyson (CDD) pole positions and residues, and subtraction constants. The number of CDD poles equals the number of nonsense states at the value of J_0 under consideration. It must be at least as large as the number of elementary particles of spin J_0 that appear in the s channel of the physical amplitudes. In a (renormalizable) unitary theory with bounded (or logarithmically increasing) partial-wave amplitudes the number of subtraction constants for the case we are considering (equal-mass particles in the initial and final states) equals the number of amplitudes.[4] In an unrenormalizable theory the number is presumably larger. We shall assume that in perturbation theory each amplitude

$F_i(s,J)$ requires $1+\nu_i$ subtractions (where ν_i may increase indefinitely as the order of perturbation theory increases) and that it is correct to thus generalize the Mandelstam procedure. Dicus and Teplitz[6] point out that this does not seem to be the case in an unrenormalizable model of fermions and axial-vector mesons. However, as discussed in Sec. IV C in the second paper of Ref. 2, one should regard their model as a mutilated version of a renormalizable model where the particles acquire a mass by means of a Higgs-Kibble mechanism. There is no reason to expect any generalization of Mandelstam counting to be consistent with the results of an essentially arbitrary tampering with the scattering amplitudes. In our case whether or not the theory is renormalizable and complete we know that a certain number of subtraction constants will be required purely on dimensional grounds, and our generalization of the Mandelstam counting argument should be correct (see also Ref. 4, Sec. V G).

In general the kinematical constraints are of two types: generalized GGMW[4,16] (Goldberger, Grisaru, MacDowell, and Wong) constraints at $s=0$ and threshold constraints. The former are constraints on the helicity amplitudes which lead to constraints on the partial-wave amplitudes. The latter can be obtained either by examining the orbital angular momentum content of each partial-wave amplitude, as done in Ref. 4, or by the following procedure: Each helicity amplitude satisfies a certain number of (pseudo-) threshold constraints which follow from general kinematical considerations and the crossing relations.[17,18] Furthermore, the partial-wave projection of a helicity amplitude which is assumed to satisfy fixed-energy dispersion relations produces additional threshold factors

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,J)\sim(q_iq_f)^{J-\lambda_m}, \tag{19}$$

where $\lambda_m=\max(|\lambda_3-\lambda_4|,|\lambda_1-\lambda_2|)$ and q_i (q_f) is the initial (final) momentum.

In our case, since we are dealing with massless particles there is no distinction between the kinematical constraints at $s=0$ and at threshold. However, from the work of Ader *et al.*[18] we can read off the constraints that amplitudes involving massless particles satisfy at $s=0$; we shall refer to these as kinematical constraints. In addition we shall assume that Eq. (19) still holds although as mentioned earlier we can define partial-wave projections only by some modification of the graviton propagator (or by suitable extraction of a Coulomb-type phase—a procedure we have not attempted; see also Refs. 19 and 20). We shall call the consequences of Eq. (19) threshold constraints.

For graviton-graviton scattering we deal with five amplitudes, $F_{2,2;2,2}$, $F_{2,2;-2,-2}$, $F_{2,2;2,-2}$, $F_{2,-2;2,-2}$, and $F_{2,-2;-2,2}$, the last two being related by Bose symmetry. We have no general argument to suggest that the two amplitudes which vanish in Born approximation do so to all orders. We shall discuss later on how to do Mandelstam counting if helicity conservation is in fact a general property of graviton (and massless scalar) processes and proceed now to consider the case of helicity nonconservation. Under particle interchange Π and the parity operation P we have

$$\Pi|J,\lambda_1,\lambda_2\rangle=(-1)^J|J,\lambda_2,\lambda_1\rangle, \tag{20}$$

$$P|J,\lambda_1,\lambda_2\rangle=(-1)^J|J,-\lambda_1,-\lambda_2\rangle, \tag{21}$$

so that only the following states are allowed:

$$|J,2,2\rangle\pm|J,-2,-2\rangle,\quad J\text{ even, parity}\begin{cases}\text{even}\\\text{odd}\end{cases}, \tag{22}$$

$$|J,2,-2\rangle\pm|J,-2,2\rangle,\quad J\begin{cases}\text{even}\\\text{odd}\end{cases},\quad\text{parity even}. \tag{23}$$

We conclude that if $J\geq 4$, for even J there are three even-parity amplitudes and one odd-parity amplitude, while for odd J there is only one even-parity amplitude. For $J<4$ only the states in (22) are *sense* so that only $J=0$ and $J=2$ amplitudes exist.

From Ref. 18 we find that at $s=0$ an amplitude involving four massless particles whose t and u kinematical singularities have been removed has the behavior

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t)\sim(\sqrt{s}\,)^{|\lambda+\mu|+|\lambda-\mu|+|\Sigma_{i=1}^4\lambda_i|}, \tag{24}$$

and assuming Eq. (19) holds we shall obtain an additional factor $\sim(s)^{J-\lambda_m}$ for the partial-wave amplitudes. In general we shall also find dynamical poles at $s=0$ which modify the actual behavior of the amplitudes. We shall assume that their effect can be separated from the purely kinematic effect described by Eqs. (19) and (24). Therefore, from (24)

$$\begin{aligned}&F^+_{2,2;2,2}+F^-_{2,2;2,2}\sim s^4,\\&F^\pm_{2,-2;2,-2}\sim s^4,\\&F^+_{2,2;2,-2}\sim s^6\end{aligned} \tag{25}$$

and the partial-wave amplitudes pick up additional factors s^J, s^{J-4}, s^{J-4}, respectively. Hence at even $J\geq 4$ there are 4 amplitudes subject to 14 kinematical constraints and $4J-8$ threshold constraints, while for odd $J>4$ there is one amplitude with 4 kinematical constraints and $J-4$ threshold constraints. There are no CDD poles. If each partial-wave amplitude F_i requires $1+\nu_i$ subtractions ($\nu_i = 0$ for a renormalizable theory) we find that at $J \geq 4$ the theory Reggeizes at even J if $\sum_{i=1}^4 \nu_i \leq 4J+2$ and at odd J if $\nu \leq J-1$.

For $J<4$ only $F^{\pm}_{2,2;2,2}$ are physical. At $J=2$ there is one CDD pole in the even-parity amplitude and none in the odd-parity amplitude (the graviton has even parity, and the nonsense state $|2,-2\rangle + |-2,2\rangle$ has even parity), hence two CDD free parameters. We have four kinematical constraints and four threshold constraints on the two amplitudes. We find that the theory should Reggeize at $J=2$ if $\nu_1+\nu_2 \leq 4$. Since an amplitude which is of nth order in κ^2 is proportional to s^n we expect that the physical and Regge amplitudes should agree at $J=2$ up to and including one loop ($n=2$, $\nu_1=\nu_2=2$) but need not agree in higher order.

At $J=0$ the only change is in the number of threshold conditions, which is now zero. The theory will Reggeize if $\nu_1+\nu_2 \leq 0$. Since even in the Born approximation $\nu_1+\nu_2=2$ we do not expect any Reggeization at $J=0$. However, this conclusion will be modified if we take into account helicity conservation.

We consider now the situation in the presence of a scalar particle. For simplicity we ignore scalar self-coupling and assume that the scalar is massless as well, so that we can use Eq. (24). We now have additional amplitudes $F^+_{0,0;0,0}$, $F^+_{0,0;2,2}$, and $F^+_{0,0;2,-2}$ present at even J. From Eq. (24) we find the kinematical constraints

$$F^+_{0,0;0,0} \sim s^0\,,$$
$$F^+_{0,0;2,2} \sim s^2\,, \tag{26}$$
$$F^+_{0,0;2,-2} \sim s^4\,.$$

Hence, at even $J \geq 4$ there are now a total of 20 kinematical constraints, $7J-12$ threshold conditions, no CDD pole, and $7+\sum_{i=1}^7 \nu_i$ subtractions in seven amplitudes. We find that if $\sum_{i=1}^7 \nu_i \leq 7J+1$ the theory will Reggeize.

At $J=2$ only four amplitudes are physical. There is one CDD pole with three parameters, six kinematical constraints, eight threshold conditions, and $4+\sum_{i=1}^4 \nu_i$ subtractions. Hence if $\sum_{i=1}^4 \nu_i \leq 7$ the graviton should Reggeize. In the Born approximation this is the case, but already at the one-loop level since $\sum_{i=1}^4 \nu_i = 8$ there is a difference from the case of pure gravity. At $J=0$ we have three CDD parameters, six kinematical constraints, and no threshold conditions; hence $\sum_{i=1}^4 \nu_i + 1 \leq 0$ for Reggeization. This condition is not satisfied even in the Born approximation.

To summarize, we have found the following: For large J the physical and Regge amplitudes are expected to agree if computed in perturbation theory up to an order in κ^2 roughly equal to J (since nth order in κ^2 requires $n+1$ subtractions in each amplitude). At $J=2$, the two amplitudes should agree up to one loop for pure gravity, but only in the Born approximation if scalars are included. At $J=0$ the two amplitudes are not expected to agree. Allowing the scalars to be massive and self-interacting should not seriously affect our conclusions, although the counting may be modified somewhat.

Finally we consider the situation which obtains if helicity conservation is a property of graviton-graviton scattering beyond the Born approximation. We have in mind doing Mandelstam counting in the case when $F_{2,2;-2,-2}$ and $F_{2,2;2,-2}$ vanish in higher order of perturbation theory. We note that with massless scalars present the amplitude $F_{2,0;-2,0}$ (and hence, by crossing, $F_{0,0;2,2}$) also vanishes in the Born approximation if the scalar-graviton coupling is that of Eq. (1). Although there are other couplings for which this is not the case,[13] we shall discuss below the situation when $F_{0,0;2,2}=0$ in higher order.

The vanishing of $F_{2,2;-2,-2}$ gives $F^+_{2,2;2,2} = F^-_{2,2;2,2}$, so that the graviton will not make its presence felt as a Kronecker δ term (since it must be absent in the negative-parity amplitude). We can now drop $F^-_{2,2;2,2}$ from our discussion. The vanishing of $F_{2,2;2,-2}$ means that there is no coupling between sense and nonsense states for pure gravity. Furthermore, the amplitudes $F^+_{2,2;2,2}$ and $F^+_{2,-2;2,-2}$ are completely decoupled in the unitarity equations and will be discussed separately.

The amplitude $F^+_{2,2;2,2}$ is subject to $J+4$ constraints. There are no CDD poles; hence $F^+_{2,2;2,2}$ Reggeizes if $1+\nu \leq J+4$. The amplitudes $F^{\pm}_{2,-2;2,-2}$ are subject to J constraints and are physical for $J \geq 4$ only. As there are again no CDD poles, they Reggeize when $1+\nu \leq J$.

If massless scalars are present, we have additional amplitudes $F^+_{0,0;0,0}$ and $F^+_{0,0;2,-2}$ at even J coupled to $F^+_{2,-2;2,-2}$. The amplitudes $F^+_{2,2;2,2}$ and $F^-_{2,-2;2,-2}$ remain uncoupled and counting is unchanged for them. For $J \geq 4$, the coupled system $F^+_{0,0;0,0}$, $F^+_{0,0;2,-2}$, and $F^+_{2,-2;2,-2}$ is subject to $3J$ constraints. The system Reggeizes, therefore, if $\nu_1+\nu_2+\nu_3 \leq 3J-3$. At $J=2$, the graviton can appear as a Kronecker δ singularity in $F_{0,0;0,0}$, and $F_{0,0;2,-2}$ provides the coupling to the necessary nonsense state. At $J=2$ there are two constraints and two CDD parameters and the Reggeization condition reads $\nu \leq -1$. Nonetheless, as we have seen in Sec. II, this system factorizes.

IV. DISCUSSION AND CONCLUSIONS

Let us first summarize our results. We have found that the Born amplitudes for gravitons and minimally coupled (massive) scalars and photons factorize at $J=2$. At $J=0$ graviton-graviton amplitudes factorize trivially but factorization does not hold when scalars are included. These results are not affected by modifications of the original model such as allowing for scalar self-coupling or using the "improved" scalar-graviton theory.

These factorization results will be compared below with the Mandelstam counting results that we have examined in the previous section. The counting, for a theory which may not have an S matrix with the usual properties, is on rather shaky grounds. Because of the long-range nature of the forces, Coulomb-type phases have to be removed before a partial-wave projection can be carried out. We have not exhibited well-defined partial-wave amplitudes, but have assumed that they exist. Presumably, properly defined partial-wave amplitudes will satisfy the threshold behavior assumed in Eq. (19), as is the case for Coulomb scattering.

We have also assumed that the Mandelstam counting procedure may be applied to amplitudes which do not satisfy unitarity bounds, by simply counting the number of additional subtraction constants required in partial-wave dispersion relations. At fixed angle and to order n in κ^2 amplitudes are expected to increase like s^n on dimensional grounds and therefore $n+1$ subtractions will be required.

We conclude that the equality between physical and Regge amplitudes at a given J will hold only up to a certain order n in κ^2. We have seen that $n \simeq J$ for large J. We have also seen that we obtain different results depending on whether we do or do not include scalars and whether helicity is conserved or not. We summarize the results of Mandelstam counting in tables I and II, where we show, for various J, to what order n of perturbation theory (in κ^2) the amplitudes are expected to Reggeize. In the notation of the previous section we have assumed that the number of subtractions in a given amplitude F_i is $1+\nu_i = 1+n$. We have indicated separately the situation at $J>4$ and $J=4$, where all amplitudes are physical, and at $J<4$, where some of the amplitudes are unphysical.

We note that there are some significant differences between the cases of helicity conservation and nonconservation. They come mainly from the fact that in the first case $F_{2,2;2,2}$, the amplitude subject to the strongest kinematical constraints, decouples from the others. In spite of these differences there is one common result that we wish to emphasize: At $J=2$ the physical and Regge amplitudes should agree up to and including one loop for pure gravity ($n=2$) but not when scalars are included. For pure gravity the Born approximation (which certainly conserves helicity) should

TABLE I. Reggeization conditions for pure gravitation. The theory Reggeizes at J up to and including order n in κ^2, i.e., $(n-1)$ loops.

	Helicity nonconservation	Helicity conservation
$J>4$ even	$\sum_{i=1}^{4} \nu_i \le 4J+2,\ n \le J$	$F^{+}_{2,2;2,2}$: $\nu \le J+3,\ n \le J+3$ $F^{-}_{2,-2;2,-2}$: $\nu \le J-1,\ n \le J-1$
$J>4$ odd	$\nu \le J-1,\ n \le J-1$	$F^{-}_{2,-2;2,-2}$: $\nu \le J-1,\ n \le J-1$
$J=4$	$\sum_{i=1}^{4} \nu_i \le 18,\ n \le 4$	$F^{+}_{2,2;2,2}$: $\nu \le 7,\ n \le 7$ $F^{+}_{2,-2;2,-2}$: $\nu \le 3,\ n \le 3$
$J=2$	$\nu_1+\nu_2 \le 4,\ n \le 2$	$F^{+}_{2,2;2,2}$: $\nu \le 5,\ n \le 5$
$J=0$	$\nu_1+\nu_2 \le 2,\ n = 0$	$F^{+}_{2,2;2,2}$: $\nu \le 3,\ n \le 3$

Reggeize (i.e., factorize) at both $J=0$ and $J=2$, and indeed it does factorize. For gravity with scalars the Born approximation should factorize at $J=2$ if helicity is not conserved, but need not if helicity is conserved. In fact it does factorize, and we might try to understand this result as follows: As we have already mentioned, it may be necessary to insert a small mass in the denominator of the graviton propagator in order to define partial-wave projections and obtain sensible threshold constraints. This mass will very likely introduce a certain amount of helicity nonconservation (and, one hopes, nothing more serious) even in the Born approximation, so that our helicity-nonconservation results may hold before the mass

TABLE II. Reggeization conditions for gravitons coupled to massless scalars. The theory Reggeizes at J up to and including order n in κ^2, i.e., $(n-1)$ loops. The amplitudes denoted by "others" are $F^{+}_{2,-2;2,-2}$, $F^{+}_{0,0;2,-2}$, and $F^{+}_{0,0;0,0}$.

	Helicity nonconservation	Helicity conservation
$J \le 4$ even	$\sum_{i=1}^{7} \nu_i \le 7J+1,\ n \le J$	$F^{+}_{2,2;2,2}$: $\nu \le J+3,\ n \le J+3$ others: $\sum_{i=1}^{3} \nu_i \le 3J-3,\ n \le J-1$
$J>4$ odd	$\nu \le J-1,\ n \le J-1$	$F^{-}_{2,-2;2,-2}$: $\nu \le J-1,\ n \le J-1$
$J=4$	$\sum_{i=1}^{7} \nu_i \le 29,\ n \le 4$	$F^{+}_{2,2;2,2}$: $\nu \le 7,\ n \le 7$ others: $\sum_{i=1}^{3} \nu_i \le 9,\ n \le 3$
$J=2$	$\sum_{i=1}^{4} \nu_i \le 7,\ n \le 1$	$F^{+}_{2,2;2,2}$: $\nu \le 5,\ n \le 5$ $F^{+}_{0,0;0,0}$: $\nu \le -1,\ n = 0$
$J=0$	$\sum_{i=1}^{4} \nu_i \le -1,\ n = 0$	$F^{+}_{2,2;2,2}$: $\nu \le 3,\ n \le 3$ $F^{+}_{0,0;0,0}$: $\nu \le -3,\ n = 0$

is allowed to tend to zero. One might hope that those helicity-nonconservation results which are stronger than the helicity-conservation ones will still hold in the zero-mass limit. If this rather optimistic view is correct our factorization results at $J=2$ will be explained.

As mentioned in the Introduction, one of the main reasons we undertook this work was to shed some light on the question of renormalizability of gravitation since it is believed that there is a relation between renormalizability and Reggeization.[2,4,6] Although our understanding is still quite minimal we offer the following possible interpretations of our results:

(a) The factorization of the Born approximation, perhaps a property of renormalizable theories, suggests that there exists a theory of massless spin-two particles and matter which has the same tree structure as the Einstein theory but is renormalizable. As a (somewhat imperfect) analogy we offer the case of spin-$\frac{1}{2}$ massive Yang-Mills scattering considered by Abers, Keller, and Teplitz.[21] The theory of this system, in the absence of a Higgs mechanism, is unrenormalizable, but the particular amplitudes factorize. It so happens that the renormalizable theory has the same tree structure for the process they considered since the Higgs scalar does not contribute in Born approximation.[2]

In our case we know in fact that pure gravitation is one-loop renormalizable. Gravitation with scalars is not, but perhaps by introducing additional particles, which do not contribute to scalar-graviton scattering in the Born approximation, one could make it renormalizable. The main difficulty and main difference from the example of Abers *et al.* is the fact that this theory has a dimensional coupling constant so that if it is renormalizable it is so in a manner different from what we are accustomed to. Conceivably a different curved-space quantization scheme might be needed. Insofar as the Born approximation reflects the classical aspects of the theory, a very optimistic attitude would be that our factorization results suggest the existence of such a scheme in which matter-gravity might be renormalizable.

(b) Pure gravitation is one-loop renormalizable and this is consistent with the (trivial) factorization of the graviton-graviton amplitudes. Gravitation with scalars is not renormalizable and the factorization of Eq. (12) is an accident. After all, whether or not the graviton Reggeizes should have little to do with the presence of scalars. The graviton Regge pole can be thought of as being extracted out of a sum of ladder diagrams with gravitons in the internal lines which should dominate diagrams with internal scalars. In graviton-scalar scattering the relevant term would come from the coupling of external scalars to such ladders. All that is required is that the coupling be right, and it happens to be so.

In following up this possibility we conjectured that the coupling happens to be right because it is generally covariant. We then investigated the somewhat similar, gauge invariant but unrenormalizable pion-Yang-Mills coupling $\vec{F}_{\mu\nu}\cdot(D_\mu\vec{\pi}x D_\nu\vec{\pi})$ only to discover that it destroyed the factorization properties of this system. We conclude that the scalar-graviton coupling is "right" for reasons other than general covariance.

(c) The view that we are inclined to espouse is the following: Pure gravitation is one-loop renormalizable, gravitation with scalars is not, and this can be understood on the basis of Mandelstam counting.

Let us consider a diagram or set of diagrams of a given order, describing a scattering process. We assume that all subdiagrams have been made finite. Infinite four-point counterterms may be required to make the corresponding amplitude finite, and they can be accompanied in general by arbitrary finite parts. We note that the counterterms give only low angular momentum contributions and therefore can introduce arbitrary parameters in the low partial waves. By the same token the amplitude at large J is finite and presumably can be continued to complex J. With the usual assumptions we can continue it now to low J and if we have an argument to suggest that the continuation must equal the physical amplitude we must conclude that the counterterms cannot introduce arbitrary parameters in the amplitudes. (Our concern, of course, is not with counterterms which only renormalize already present interactions.)

Therefore, if Mandelstam counting indicates that the Regge and physical amplitudes should agree down to a given J we must conclude that the only possible (four-point) counterterms have angular momentum content which is less than J. For example, in graviton-graviton scattering we must conclude that the only possible counterterms to the one-loop amplitude can contribute only at $J=0$. On the other hand, on dimensional grounds they must contain four derivatives of the gravitational field. The reader may convince himself that the only possible such terms which contribute only to $J=0$ and not to $J=2$ vanish on shell. Hence the one-loop amplitude requires no renormalization.

On the other hand, in the presence of scalars Mandelstam counting indicates that there may be counterterms which contribute at $J=2$, e.g.,

$\kappa^4[(h_{\alpha\beta,\lambda})^2]^2$ (or, covariantly, $g^{1/2}R^2$) since the two amplitudes need not agree there. Thus the theory need not be one-loop renormalizable.

Obviously our arguments lack rigor. We can at best talk about four-point amplitudes in low orders of perturbation theory. We also do not really know if our Mandelstam counting procedure is correct. There are other holes in our arguments that the reader will have no difficulty finding. Nonetheless, if we are correct we might speculate about what will happen for pure gravitation beyond the one-loop approximation. We consider first the case of helicity nonconservation.

We note that in order $(\kappa^2)^n$ counterterms will contain $2n$ derivatives of the graviton field, and will contribute a contact term with $2n$ momentum factors to the $(n-1)$-loop amplitude. If we assume that in this order each $\nu_i = n$, we have obtained the result that the physical and Regge amplitudes must agree at even J if $J \geq n$ and at odd J if $J \geq n+1$. We can now attempt to generalize the argument at $J=2$ by asking if it is possible to reconcile $2n$ momentum factors with the absence of contributions at J.

For $n=3$ the two-loop amplitude can have a counterterm with 6 momentum factors and hence a possible contribution at $J=3$. However, on shell the physical amplitudes vanish at $J=3$, so that unless the 6 momentum factors can be arranged to give nonzero contributions only to $J \leq 2$ the counterterms must be absent and the two-loop amplitude will be renormalizable on shell.

For $n=4$ the physical and Regge amplitudes must agree at $J=4$. If it is impossible for 8 momentum factors to give vanishing contributions at $J=4$, but nonvanishing contributions for $J \leq 2$, the three-loop amplitude will be renormalizable on shell.

For $n=5$ our argument would break down since the condition is $J \geq n+1$ at odd J. The two amplitudes need not agree at $J=5$ and the 4-loop amplitude need not be renormalizable. If the condition were $J \geq n$ at all J we might be able to continue the argument and perhaps prove that pure gravitation is in fact renormalizable in arbitrary (finite) orders of perturbation theory.

However, we can already stop at the two-loop level since in fact it may be possible to construct a counterterm with 6 momentum factors which does not contribute at $J=3$, for instance $g^{1/2}(R_{\mu\nu\alpha\beta})^2R^2$. Indeed the amplitude $F_{2,2;-2,-2}$ can receive a contribution from such a counterterm having the unique form (see Ref. 13; the methods explained there readily give this result) $\kappa^6(s^3+t^3+u^3)$. Whether such a counterterm is actually present is an open question.

The situation is very different if helicity conservation is true in higher orders of perturbation theory. Not only must $F_{2,2;-2,-2}$ vanish, but, as exhibited in Table I, the theory must Reggeize at $J=2$ up to and including four-loop diagrams. We are then back to the case discussed above which suggests that two-loop amplitudes should be renormalizable on shell.

Obviously it would be interesting, but extremely difficult, to investigate the question of helicity conservation and to check our conjectures by a direct calculation of the $n=2,3,\ldots$-loop amplitudes.

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APPENDIX

The relevant parts of the c functions in the partial-wave expansion Eq. (6) are given here. For non-negative λ and μ, $c^J_{\lambda\mu}$ can be written as linear combinations of Legendre functions $P_{J-\lambda_m}, \ldots, P_{J+\lambda_m}$ with constant coefficients, where $\lambda_m = \max(|\lambda|, |\mu|)$. For sense-sense transitions, $J-\lambda_m$ is non-negative. For sense-nonsense and nonsense-nonsense transitions, $J-\lambda_m$ is negative and these Legendre functions of negative degree generate poles in J in the partial-wave amplitudes; therefore, we only keep the terms having Legendre functions of negative degree.

The c's can be generated by using the recursion relations (A13) and (A14) of Ref. 1 [note the misprint in Eq. (A13): $(L+1-2\lambda)$ should be replaced by $(L+1+2\lambda)$]. The relevant parts of the c's in our calculations are as follows.

$J=0$:

$$c^0_{00} \simeq P_0\,,$$

$$c^0_{04} \simeq i\sqrt{J}\left(\tfrac{4}{5}P_{J-4}+\tfrac{16}{5}P_{J-2}\right),$$

$$c^0_{44} \simeq -\tfrac{672}{5}P_{J-2}-\tfrac{8}{5}P_{J-4}\;.$$

$J=2$:

$$c^2_{00} \simeq P_2\,,$$

$$c^2_{04} \simeq -i\frac{4}{\sqrt{5}}(J-2)^{1/2}P_{J-4}\,,$$

$$c^2_{44} \simeq -24P_{J-4}\,,$$

$$c^2_{22} \simeq \tfrac{4}{5}P_0+\tfrac{8}{7}P_2+\tfrac{2}{35}P_4\,,$$

$$c^2_{24} \simeq -i4\left(\tfrac{6}{5}\right)^{1/2}(J-2)^{1/2}P_{J-4}\;.$$

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