Electromagnetic field of a current loop around a Kerr black hole

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The electromagnetic field due to a current loop placed around a Kerr black hole is investigated within the framework of perturbation theory. The asympotic behavior of the field at large distances from the black hole is studied. The process of charge accretion into the black hole due to the electric field generated by the loop is discussed.

I. INTRODUCTION

Test fields superposed on black-hole backgrounds can be analyzed with the help of perturbation techniques of general relativity. On account of the strong space-time curvatures involved, these fields often exhibit unusual properties that may not be encountered in flat space-times. Exploring new effects thus produced by gravitation is obviously of interest from a theoretical standpoint. The ultimate aim of such investigations, however, is to relate the results to astrophysically realizable situations.

Physical phenomena associated with rotating black holes display, in general, more novel features than those that accompany static ones. For instance, a stationary charge near a Kerr black hole gives rise to a magnetic field as well as the electrostatic field.[1] Analogous effects are to be expected in the case of a current loop placed around a black hole. This problem has been treated recently by Petterson[2] using the Schwarzschild metric. The magnetic field of the loop is modified as compared with that in flat space-time, but the basic structure of the electromagnetic field is unaltered. In the present paper we study the problem with the Kerr black hole as the background. Now, in addition to the expected magnetic field, an electric field is also generated owing to the rotation of the black hole. This electric field in turn leads naturally to the possibility of charge accretion into the black hole. Wald[3] has studied charge accretion into a black hole placed in a uniform magnetic field. Using the same principles as in his procedure we determine the amount of charge that can be attracted by the black hole as a function of the current in the loop. These considerations would be of astrophysical significance, if toroidal currents exist in accreting disks surrounding black holes. Such currents are expected to be created by radiation pressure acting predominantly on electrons.[2] The possible occurrence of this phenomenon in a realistic accretion model calls for a detailed analysis of the relevant physical processes. We are investigating the above aspects of the problem and hope to return to them in the future.

In the next section we present the differential equations governing the electromagnetic field of the current loop placed around a Kerr black hole and obtain the corresponding solutions. In Sec. III, the asymptotic behavior of the field components at large distances from the black hole is examined. The magnetic dipole and the electric quadrupole moments induced by the current loop are computed. The final section is devoted to the process of charge accretion into the black hole.

II. FIELD OF A CURRENT LOOP AROUND A KERR BLACK HOLE

In the Boyer-Lindquist coordinates, the Kerr metric is

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega\, dt)^2 + e^{2\lambda}dr^2 + e^{2\mu}d\theta^2, \tag{1}$$

where

$$\begin{aligned} &e^{2\nu} = \Delta\Sigma/B, \quad e^{2\psi} = B\sin^2\theta/\Sigma, \quad \omega = 2amr/B, \\ &e^{2\lambda} = \Sigma/\Delta, \quad e^{2\mu} = \Sigma, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\Sigma = r^2 + a^2\cos^2\theta, \quad B = (r^2+a^2)^2 - \Delta a^2\sin^2\theta, \\ &\Delta = r^2 - 2mr + a^2. \end{aligned} \tag{3}$$

We use Kinnersley's tetrad the components of which are

$$\begin{aligned} &l^\mu = ((r^2+a^2)/\Delta, 1, 0, a/\Delta), \\ &n^\mu = \frac{1}{2\Sigma}(r^2+a^2, -\Delta, 0, a), \\ &m^\mu = -\frac{\rho^*}{\sqrt{2}}(ia\sin\theta, 0, 1, i/\sin\theta), \end{aligned} \tag{4}$$

where ρ is one of the spin coefficients:

$$\rho \equiv l_{\mu;\nu} m^{\mu} m^{*\nu} = -\frac{1}{r - ia\cos\theta}. \tag{5}$$

The electromagnetic field is characterized by the three complex quantities

$$\phi_0 = F_{\mu\nu} l^{\mu} m^{\nu},$$

$$\phi_1 = \tfrac{1}{2} F_{\mu\nu}(l^{\mu} n^{\nu} + m^{*\mu} m^{\nu}),$$

and

$$\phi_2 = F_{\mu\nu} m^{*\mu} n^{\nu}, \tag{6}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor.

In the stationary axisymmetric case, Teukolsky's[4] master perturbation equation for the quantity

$$\Psi = \phi_2/\rho^2$$

takes the form

$$-\Delta \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Psi}{\partial\theta}\right) + (\cot^2\theta + 1)\Psi = 4\pi J, \tag{7}$$

where J contains the source terms. In our case, let $J^{\mu}(r, \theta)$ be the current density for a current loop located in the equatorial plane, $\theta = \pi/2$ placed symmetrically around the black hole. We choose J^{μ} such that the locally nonrotating observer,[5] whose frame of reference is represented by an orthonormal tetrad

$$e_{\hat{t}} = e^{-\nu}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\phi}\right), \quad e_{\hat{\phi}} = e^{-\psi}\frac{\partial}{\partial\phi},$$

$$e_{\hat{r}} = e^{-\lambda}\frac{\partial}{\partial r}, \quad e_{\hat{\theta}} = e^{-\mu}\frac{\partial}{\partial\theta} \tag{8}$$

measures the total current through an $(\hat{r}, \hat{\theta})$ plane as I. In Boyer-Lindquist coordinates, this leads to the current density as

$$J^{\mu} = \left(0, 0, 0, \frac{I}{b^2}\left(\frac{b^2 - 2mb + a^2}{b^2 + 2ma^2/b + a^2}\right)^{1/2} \delta(r - b)\,\delta(\cos\theta)\right). \tag{9}$$

Writing

$$\Psi = \sum_{l=1}^{\infty} R_l(r)\, {}_{-1}S_l^0(\theta)$$

and

$$J = \sum_{l=1}^{\infty} J_l(r)\, {}_{-1}S_l^0(\theta),$$

where ${}_{-1}S_l^0(\theta)$ are spin-weighted spherical harmonics,

$${}_{-1}S_l^0(\theta) = \left[\frac{2l+1}{4\pi l(l+1)}\right]^{1/2} \frac{dP_l(\cos\theta)}{d\theta}, \tag{10}$$

and substituting in Eq. (7), one obtains the radial equation for each l mode:

$$\frac{d^2 R_l}{dr^2} - \frac{l(l+1)}{\Delta} R_l = -4\pi \frac{J_l(r)}{\Delta}, \tag{11}$$

where

$$J_l(r) = 2\pi I \left[\frac{2l+1}{4\pi l(l+1)}\right]^{1/2} \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b}\right)^{1/2} \left[-\frac{i(b^2 + a^2 - 2mb)}{2\sqrt{2}}\delta(r-b)\frac{dP_l(\cos\theta)}{d\theta}\bigg|_{\pi/2} + \frac{al(l+1)}{2\sqrt{2b}}(b^2 - 2mb + a^2)P_l(0)\delta(r-b) - \frac{i(a^2+b^2)}{2\sqrt{2b}}(r^2 - 2mr + a^2)\delta'(r-b)\frac{dP_l(\cos\theta)}{d\theta}\bigg|_{\pi/2}\right] \tag{12}$$

with prime denoting the derivative with respect to r. To solve the inhomogeneous differential equation (11) we use the following linearly independent solutions of the homogeneous part:

$$F_l(r) = -\frac{(2l+1)!}{2^l (l+1)!\, l!\, (m^2 - a^2)^{(l+1)/2}} \Delta Q_l'\left(\frac{r-m}{(m^2-a^2)^{1/2}}\right), \tag{13}$$

$$G_l(r) = \frac{2^l l!\,(l-1)!\,(m^2 - a^2)^{l/2}}{(2l)!} \Delta P_l'\left(\frac{r-m}{(m^2-a^2)^{1/2}}\right). \tag{14}$$

We note that as $r \to \infty$, $F_l(r) \sim 1/r^l$ and $G_l(r) \sim r^{l+1}$, and as $r \to m + (m^2 - a^2)^{1/2}$, $G_l(r) \to 0$ and $F_l(r) \to$ a finite constant. The solution of Eq. (11) is then given by

$$R_l(r) = F_l(r)\left[\int \left(\frac{-4\pi J_l(r)}{\Delta}\right) G_l(r)\, dr\right] \Big/ W, \quad r > b \tag{15}$$

$$R_l(r) = G_l(r)\left[\int\left(\frac{-4\pi J_l(r)}{\Delta}\right)F_l(r)\,dr\right]/W, \quad r<b \tag{16}$$

where W, the Wronskian of $G_l(r)$ and $F_l(r)$, is $-(2l+1)$. Integrating Eq. (15), one finally obtains, for $r>b$,

$$\begin{aligned}\phi_2(r,\theta) &= \pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2} \\ &\quad\times\sum_{l=1}^{\infty}\frac{1}{\sqrt{2}\,l(l+1)}\left[i\left(\frac{a^2+b^2}{b}G_l'(b)-G_l(b)\right)\left(-\frac{2}{\sqrt{\pi}}\sin\frac{\pi l}{2}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\right)\right. \\ &\qquad\left.+\frac{a}{b}l(l+1)G_l(b)\left(\frac{1}{\sqrt{\pi}}\cos\frac{\pi l}{2}\frac{\Gamma((l+1)/2)}{\Gamma((l+2)/2)}\right)\right]\rho^2F_l(r)\frac{dP_l(\cos\theta)}{d\theta} \\ &\equiv\sum_{l=1}^{\infty}H_l^{>}(b)\rho^2F_l(r)\frac{dP_l(\cos\theta)}{d\theta}.\end{aligned} \tag{17}$$

Integrating Maxwell's equations, one obtains[1] $\phi_1(r,\theta)$, for $r>b$:

$$\phi_1(r,\theta)=\sum_{l=1}^{\infty}\sqrt{2}\,H_l^{>}(b)\rho^2\left\{[rF_l'(r)-F_l(r)]P_l(\cos\theta)-\frac{ia}{2l+1}F_l'(r)[lP_{l+1}(\cos\theta)+(l+1)P_{l-1}(\cos\theta)]\right\}. \tag{18}$$

For $r<b$, one finds the analogous expressions for $\phi_2(r,\theta)$ and $\phi_1(r,\theta)$ with F_l and G_l interchanged. Electromagnetic field components can then be computed directly knowing ϕ_2 and ϕ_1 everywhere.

III. THE ASYMPTOTIC FIELD DISTRIBUTION

We will compute the electromagnetic fields at large radii ($r\gg m, b$) for the $l=1$ mode. We substitute the following expressions:

$$G_1(r)=r^2-2mr+a^2, \quad F_1(r)\to\frac{1}{r},$$

and

$$H_1^{>}(b)=-\frac{i\pi I}{2\sqrt{2}}\left(\frac{b^2-2mb+a^2}{b^2+2ma^2/b+a^2}\right)^{1/2}(b^2+a^2-2ma^2/b)$$

in ϕ_2 and ϕ_1 to get

$$\begin{aligned}F_{r\phi} &= -2a\sin^2\theta\,\mathrm{Re}\phi_1-2\sqrt{2}\,\frac{(r^2+a^2)\sin\theta}{\Delta}\,\mathrm{Im}[(r-ia\cos\theta)\phi_2] \\ &\to -\pi I\left(\frac{b^2-2mb+a^2}{b^2+2ma^2/b+a^2}\right)^{1/2}(b^2+a^2-2ma/b)\frac{\sin^2\theta}{r^2},\end{aligned} \tag{19}$$

$$\begin{aligned}F_{\theta\phi} &= 2\sin\theta\,(r^2+a^2)\mathrm{Im}\phi_1-2\sqrt{2}\,a\sin^2\theta\,\mathrm{Re}[(r-ia\cos\theta)\phi_2] \\ &\to \pi I\left(\frac{b^2-2mb+a^2}{b^2+2ma^2/b+a^2}\right)^{1/2}(b^2+a^2-2ma^2/b)\frac{2\sin\theta\cos\theta}{r},\end{aligned} \tag{20}$$

$$\begin{aligned}F_{tr} &= -2\,\mathrm{Re}(\phi_1)-\frac{2\sqrt{2}\,a\sin\theta}{\Delta}\,\mathrm{Im}[(r-ia\cos\theta)\phi_2] \\ &\to \pi I\left(\frac{b^2-2mb+a^2}{b^2+2ma^2/b+a^2}\right)^{1/2}a\,\frac{3(3\cos^2\theta-1)}{2r^4}(b^2+a^2-2m^2/b),\end{aligned} \tag{21}$$

$$\begin{aligned}F_{t\theta} &= 2a\sin\theta\,\mathrm{Im}(\phi_1)-2\sqrt{2}\,\mathrm{Re}[(r-ia\cos\theta)\phi_2] \\ &\to \pi I\left(\frac{b^2-2mb+a^2}{b^2+2ma^2/b+a^2}\right)^{1/2}(b^2+a^2-2ma^2/b)a\,\frac{3\cos\theta\sin\theta}{r^3}.\end{aligned} \tag{22}$$

The electromagnetic field components in the local Lorentz frame defined in Eq. (8) are then given by

$$
\begin{aligned}
B_{\hat r} &= e^{-\mu} e^{-\psi} F_{\theta\phi} \\
&= \pi \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b} \right)^{1/2} (b^2 + a^2 - 2ma^2/b) I \left(\frac{2\cos\theta}{r^3} \right),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
B_{\hat\theta} &= e^{-\lambda} e^{-\psi} F_{\phi r} \\
&= \pi \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b} \right)^{1/2} (b^2 + a^2 - 2ma^2/b) I \left(\frac{\sin\theta}{r^3} \right),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
E_{\hat r} &= e^{-\nu} e^{-\lambda} F_{rt} + \omega e^{-\nu} e^{-\lambda} F_{r\phi} \\
&\to -\pi \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b} \right)^{1/2} (b^2 + a^2 - 2ma^2/b) a I \, \frac{3(3\cos^2\theta - 1)}{2r^4},
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
E_{\hat\theta} &= e^{-\nu} e^{-\mu} F_{\theta t} + \omega e^{-\nu} e^{-\mu} F_{\theta\phi} \\
&\to -\pi \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b} \right)^{1/2} (b^2 + a^2 - 2ma^2/b) a I \, \frac{3\cos\theta \sin\theta}{r^3},
\end{aligned}
\tag{26}
$$

which are identical to the magnetic field of a dipole with moment

$$
\mu = \pi \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b} \right)^{1/2} (b^2 + a^2 - 2ma^2/b) I,
\tag{27}
$$

and to the electric field of a quadrupole with moment

$$
Q = \mu a.
\tag{28}
$$

We note that setting $a = 0$, one gets the results obtained by Petterson[2] for the current loop around a Schwarzschild black hole.

IV. CHARGE ACCRETION

We will see in this section that the structure of the electric field produced by currents around the Kerr black hole is such that the positively charged particles (from the surrounding interstellar medium) will be pulled into the black hole, while negatively charged particles will be repelled. (The opposite will be true if the current and rotation of the black hole are in the opposite sense.) The magnitude of the accreted charge can be computed by following Carter[6] and Wald,[3] i.e., by calculating the change in the electrostatic energy of a charged particle as it is lowered from infinity to the horizon along the axis of symmetry. Using the same notation as that of Wald,[3] the change in the electrostatic energy of the particle is given by

$$
\begin{aligned}
\epsilon &= E_{\text{final}} - E_{\text{initial}} \\
&= eA_\mu \eta^\mu |_{\text{horizon}} - eA_\mu \eta^\mu |_\infty,
\end{aligned}
\tag{29}
$$

where η^μ in the timelike killing vector $\partial/\partial t$ of the Kerr metric. During accretion, which is treated as a quasistatic process, the vector potential A_μ will consist of two parts, A^I_μ due to the loop and A^Q_μ due to the charge that has already been absorbed by the black hole. We will see that one can sum the contributions of all l modes to A^I_μ exactly. Charge accretion proceeds until the net injection energy ϵ is reduced to zero. By this criterion one can obtain the magnitude and the sign of the total accreted charge Q.

From Eq. (18), on the axis

$$
\begin{aligned}
F_{tr} &= -2\,\mathrm{Re}(\phi_1) \\
&= -2\pi I \left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b} \right)^{1/2} \sum_{l=1}^{\infty} \left[\frac{a}{b} G_l(b) \frac{1}{\sqrt{\pi}} \cos\frac{\pi l}{2} \frac{\Gamma((l+1)/2)}{\Gamma((l+2)/2)} \frac{d}{dr}\left(\frac{r F_l(r)}{r^2 + a^2} \right) \right. \\
&\qquad \left. + \frac{a}{l(l+1)} \left(\frac{a^2 + b^2}{b} G_l'(b) - G_l(b) \right) \frac{2}{\sqrt{\pi}} \sin\frac{\pi l}{2} \frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)} \frac{d}{dr}\left(\frac{F_l(r)}{r^2 + a^2} \right) \right].
\end{aligned}
$$

The vector potential, A^I_μ on the axis will be given by $A^I_\mu = (A_t, 0, 0, 0)$, where

$$A_t = 2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\sum_{l=1}^{\infty}\left[\frac{a}{b}G_l(b)\frac{1}{\sqrt{\pi}}\cos\frac{\pi l}{2}\frac{\Gamma((l+1)/2)}{\Gamma((l+2)/2)}\frac{rF_l(r)}{r^2+a^2}\right.$$
$$\left.+\frac{a}{l(l+1)}\left(\frac{a^2+b^2}{b}G_l'(b)-G_l(b)\right)\frac{2}{\sqrt{\pi}}\sin\frac{\pi l}{2}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\frac{F_l(r)}{r^2+a^2}\right]+K^{>},$$
$$\text{for } r>b \quad (30)$$

$$A_t = 2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\sum_{l=1}^{\infty}\left[\frac{a}{b}F_l(b)\frac{1}{\sqrt{\pi}}\cos\frac{\pi l}{2}\frac{\Gamma((l+1)/2)}{\Gamma((l+2)/2)}\frac{rG_l(r)}{r^2+a^2}\right.$$
$$\left.+\frac{a}{l(l+1)}\left(\frac{a^2+b^2}{b}F_l'(b)-F_l(b)\right)\frac{2}{\sqrt{\pi}}\sin\frac{\pi l}{2}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\frac{G_l(r)}{r^2+a^2}\right]+K^{<},$$
$$\text{for } r<b. \quad (31)$$

The $K^{>}$ and $K^{<}$ are some constants. The continuity of A_t at $r=b$ gives the condition

$$K^{<}-K^{>} = 2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\sum_{l=1}^{\infty}\frac{2}{\sqrt{\pi}}\sin\frac{\pi l}{2}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\frac{a}{l(l+1)}\frac{1}{b}\left[-F_l'(b)G_l(b)+G_l'(b)F_l(b)\right]$$
$$= 2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\frac{a}{b}\sum_{l=1}^{\infty}\frac{2\sin(\pi l/2)}{\sqrt{\pi}}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\frac{(2l+1)}{l(l+1)}$$
$$= 2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\frac{a}{b}, \quad (32)$$

since

$$\sum_{l=1}^{\infty}\frac{2}{\sqrt{\pi}}\sin\frac{\pi l}{2}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\frac{(2l+1)}{l(l+1)}=1. \quad (33)$$

One way to prove the above equality is to convert the series into a difference of two convergent hypergeometric functions[7]:

$$\sum_{l=1}^{\infty}\frac{2}{\sqrt{\pi}}\sin\frac{\pi l}{2}\frac{\Gamma((l+2)/2)}{\Gamma((l+1)/2)}\frac{(2l+1)}{l(l+1)} = \sum_{n=0}^{\infty}\frac{2}{\sqrt{\pi}}(-1)^n\frac{[\Gamma(n+\frac{3}{2})](4n+3)}{[\Gamma(n+1)](2n+1)(2n+2)}$$
$$= \sum_{n=0}^{\infty}\frac{2}{\sqrt{\pi}}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}-\frac{1}{4}\sum_{n=0}^{\infty}\frac{2}{\sqrt{\pi}}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+2)}$$
$$=2F(1,\tfrac{1}{2},1;-1)-\tfrac{1}{2}F(1,\tfrac{1}{2},2;-1)$$
$$=2\frac{1}{\sqrt{2}}F\left(1,0,1;\frac{1-\sqrt{2}}{2}\right)-\tfrac{1}{2}F\left(2,1,2;\frac{1-\sqrt{2}}{2}\right) \text{(see Ref. 7 for linear transformations)}$$
$$=2\left(\frac{1}{\sqrt{2}}\right)-\tfrac{1}{2}(-2+2\sqrt{2})$$
$$=1.$$

Noting the behavior of $F_l(r)$ and $G_l(r)$ at the horizon and at infinity, one finds that

$$e(A^I_\mu\eta^\mu|_{\text{horizon}}-A^I_\mu\eta^\mu|_\infty)$$
$$\equiv e(A_t|_{\text{horizon}}-A_t|_\infty)$$
$$=e(K^{<}-K^{>})$$
$$=e\left[2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\frac{a}{b}\right]. \quad (34)$$

The vector potential due to charge Q on the black hole is given by $A^Q_\mu = -Q/2m\,\eta_\mu$. Thus, the net electrostatic injection energy is given by

$$\epsilon = e\left[2\pi I\left(\frac{b^2-2mb+a^2}{b^2+a^2+2ma^2/b}\right)^{1/2}\frac{a}{b}-\frac{Q}{2m}\right]. \quad (35)$$

We have derived the above expression only for injection along the symmetry axis, but it is valid for any general injection of particles since, as Carter[6] has shown, the electrostatic injection energy is constant over the black hole. The final value of accreted charge is determined by setting ϵ to zero.[8] Hence a rotating black hole with a current loop around it will charge up to a value

$$Q = 4\pi I \frac{ma}{b}\left(\frac{b^2 - 2mb + a^2}{b^2 + a^2 + 2ma^2/b}\right)^{1/2}. \tag{36}$$

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[8]We would like to point out that we have not necessarily calculated the complete minimum energy field configuration, and that it could correspond to the current loop carrying a charge opposite to that of the black hole.