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Gauge-invariant perturbations of Reissner-Nordström black holes*

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We present the details of previously published results on the stability of the Reissner-Nordström family of black holes and discuss, in some detail, the reduction techniques which were used to simplify the perturbation problem. We develop the Hamiltonian formalism for the perturbations of an arbitrary static solution of the Einstein-Maxwell equations (with vanishing magnetic field) and show explicitly that the perturbed constraints are the generators of the coordinate and electromagnetic gauge transformations of the canonical perturbation variables. We show that the perturbed constraints have vanishing Poisson brackets with one another and use this result as the basis for our reduction technique. The canonical transformations which accomplish the reduction and the perturbation Hamiltonian are given explicitly for the Reissner-Nordström problem. We include a stability result for the (odd- and even-parity) $L = 1$ perturbations which were not previously considered.

I. INTRODUCTION

In this paper we present the details of previously published results[1] on the stability (in linear approximation) of the Reissner-Nordström family of black holes. We also explain in some detail the reduction techniques which were used in simplifying the Reissner-Nordström perturbation problem. The reduction methods are emphasized because they are equally applicable (in a practical way) to a variety of other interesting gravitational perturbation problems.

Our approach is most naturally expressed in terms of a Hamiltonian formulation of the perturbation problem. The reduction technique is essentially an application of the classical method of reducing a Hamiltonian system admitting a commuting set of constants of the motion. For our problem the perturbed constraints are the relevant constants of the motion and the (Abelian) symmetry transformations they generate are just the coordinate and electromagnetic gauge transformations of the canonical perturbation functions. This reduction technique is described in more detail in Sec. II. Section III presents the Hamiltonian formalism for the perturbations of an arbitrary static solution of the Einstein-Maxwell equations (with vanishing magnetic field). In Sec. IV we discuss the commutation properties of the perturbed constraint functions and explain their role as generators of the coordinate and electromagnetic gauge transformations. In Sec. V we specialize to the Reissner-Nordström background, introduce the Regge-Wheeler[2] expansion of the perturbations, and give an explicit canonical transformation which accomplishes the reduction. Sec. VI presents the Hamiltonian for the perturbations expressed in terms of the new canonical variables and recalls the stability results given in Ref. 1. The $L=1$ perturbations, which were not previously considered, are discussed in Sec. VII.

Other studies of the Reissner-Nordström perturbations have recently been made by Zerilli,[3] by Chitre, Price, and Sandberg[4] and by Johnston, Ruffini, and Zerilli.[5] A previous application of the methods used here may be found in Ref. 6.

II. REDUCTION METHODS

The variational integral for the Einstein-Maxwell equations has been put into Hamiltonian form by Arnowitt, Deser, and Misner (ADM).[7] From their expression one may derive another variational integral which determines the equations of perturbation of any particular exact solution. The canonical variables for the linearized ADM equations are just the first-order perturbations of the canonical variables for the original, exact variational problem. The method for deriving the perturbation Hamiltonian has been discussed in Ref. 6 and follows the same pattern developed and applied by Taub[8] to derive the perturbation Lagrangian.

The constraint equations for the perturbation problem are just the linearized versions of the exact constraint equations. The linearized constraints are conserved in time when the dynamical variables are propagated by the linearized evolution equations. Two important properties of the linearized constraint functions are that

(i) they are the generators of the coordinate and electromagnetic gauge transformations of the canonical perturbation functions, and

(ii) they have vanishing Poisson brackets with one another.

In Sec. IV we shall give an explicit proof of these two statements for the restricted problem of perturbing a static solution of the Einstein-Maxwell equations. A proof for the general case may be derived in the same fashion but requires a more lengthy computation. It should be clear, however, that (i) is essentially the standard result that the conserved quantities of a Hamiltonian system are the generators of the symmetry transformations (in this case gauge transformations) of that system. Furthermore (ii) is a necessary consequence of (i) since the constraints themselves must be gauge-invariant.

Our reduction technique is now easy to explain. We seek a transformation to new canonical variables in terms of which the perturbed constraint functions become simply a commuting subset (for example, a subset of the new momentum functions). This is possible in view of (ii) above. The new variables canonically conjugate to the constraints will, in view of (i) above, be gauge-dependent. The remaining canonical pairs, however, will be gauge-invariant since, by construction, they will commute with the perturbed constraints. In terms of the new canonical variables the perturbation problem simplifies considerably. In particular, the evolution equations for the gauge-invariant functions necessarily decouple from all the gauge-dependent functions in the problem (including not only the gauge-dependent canonical variables but also the perturbed lapse and shift functions).

The usefulness of this technique depends, of course, on the tractability of performing the required canonical transformation. When the background space-time has considerable symmetry the transformation problem may be greatly simplified by expanding the perturbations in a suitable set of tensor harmonics such as the Regge-Wheeler[2] harmonics for spherically symmetric backgrounds or the Bonanos[9] or Regge-Hu[10] harmonics for spatially homogeneous backgrounds. Regarding the general problem, it has been possible to define a complete set of gauge-invariant functions of the (purely gravitational) perturbations of an arbitrary vacuum space-time with compact Cauchy surfaces.[11] However, this construction is highly nonlocal as it requires the solution of certain (well-posed) elliptic systems of equations. This method is probably extendable to the noncompact case provided suitable asymptotic conditions are imposed. The nonlocality of this approach, however, may render it impractical as a tool for standard perturbation problems. An open question is the extent to which our reduction program is applicable (in a practical sense) to the perturbations of space-times of moderate symmetry.

III. VARIATIONAL INTEGRAL FOR THE PERTURBATIONS

The ADM form of the variational integral for the Einstein-Maxwell equations is given by

$$16\pi I=\int_\Omega d^4x\,(\pi^{ij}g_{ij,t}+A_i\mathcal{E}^i{}_{,t}-N\mathcal{H}-N_i\mathcal{H}^i-A_0\mathcal{E}^i{}_{,i})\,, \tag{3.1}$$

in which

$$\begin{aligned}\mathcal{H}&=g^{-1/2}(\pi^{ij}\pi_{ij}-\tfrac{1}{2}\pi^2)-g^{1/2}R\\&\quad+\tfrac{1}{2}g^{-1/2}g_{ij}(\mathcal{E}^i\mathcal{E}^j+\mathcal{B}^i\mathcal{B}^j)\,,\\ \mathcal{H}^i&=-2\pi^{ij}{}_{|j}-g^{ik}\epsilon_{klm}\mathcal{E}^l\mathcal{B}^m\,,\end{aligned} \tag{3.2}$$

with $\mathcal{B}^i=\frac{1}{2}\epsilon^{ijk}(A_{k,j}-A_{j,k})$ and $\pi=g_{ij}\pi^{ij}$. Our notation is the same as that of Ref. 7 except that we have absorbed a factor of 2 into the definitions of both $\mathcal{E}^i$ and A_μ and we use ϵ_{ijk} and ϵ^{ijk} (with $\epsilon_{123}=\epsilon^{123}=1$) to represent the totally antisymmetric tensor densities. Independent variation of the lapse function N, shift vector field N_i, and scalar potential A_0 gives the equations of constraint

$$\mathcal{H}=\mathcal{H}^i=\mathcal{E}^i{}_{,i}=0\,, \tag{3.3}$$

while independent variation of the canonical pairs $(g_{ij},\ \pi^{ij})$ and $(\mathcal{E}^i,\ A_i)$ gives the evolution equations of the Einstein-Maxwell system.

A static solution with vanishing magnetic field ($\mathcal{B}^i=0$) has $\pi^{ij}=N_i=\mathcal{B}^i=0$ and obeys

$$\begin{aligned}&\mathcal{E}^i{}_{,i}=0,\quad -g^{1/2}R+\tfrac{1}{2}g^{-1/2}g_{ij}\mathcal{E}^i\mathcal{E}^j=0\,,\\&\mathcal{B}^i{}_{,t}=\tfrac{1}{2}\epsilon^{ijk}[(Ng^{-1/2}\mathcal{E}_j)_{,k}-(Ng^{-1/2}\mathcal{E}_k)_{,j}]\\&\quad=0\,,\\&\pi^{ij}{}_{,t}=-Ng^{1/2}(R^{ij}-\tfrac{1}{2}g^{ij}R)+g^{1/2}(N^{|ij}-g^{ij}N_{|k}{}^{|k})\\&\quad+Ng^{-1/2}(\tfrac{1}{4}g^{ij}g_{kl}\mathcal{E}^k\mathcal{E}^l-\tfrac{1}{2}\mathcal{E}^i\mathcal{E}^j)\\&\quad=0\,.\end{aligned} \tag{3.4}$$

The remaining static field equations, $g_{ij,t}=\mathcal{E}^i{}_{,t}=0$, are satisfied as a consequence of the above.

To exclude naked singularities in the non-spherically-symmetric case one should match a regular exterior solution of the above equations to a regular interior solution representing, for example, a charged elastic body. Since our main interest, however, is in the Reissner-Nordström (black hole) solutions, we exclude consideration of the interior problem. For any particular solution of Eqs. (3.4) we let M designate the generic t = constant spacelike hypersurface (with induced metric g_{ij}) of the static space-time. In the Reissner-Nordström case M is a t = constant spacelike slice exterior to the event horizon. In the following we consider perturbations of the space-time $(M\times R,\ {}^4g_{\mu\nu},\ F_{\mu\nu})$, where ${}^4g_{\mu\nu}$ and $F_{\mu\nu}$ are an asymptotically flat Lorentz metric and an electromagnetic field tensor obeying Eqs. (3.4). The local perturbation equations must be supplemented with asymptotic conditions (consistent with asymptotic flatness) and boundary conditions (consistent with regularity at the event horizon or at the boundary of a charged source).

To obtain a variational integral for the perturbation equations we compute the second variation of the exact variational integral (3.1). This method for deriving a variational integral for perturbation problems has been discussed in Refs. 6 and 8. If we designate the perturbation functions (first variations) by

$$h_{ij} \equiv \delta g_{ij}, \quad p^{ij} \equiv \delta \pi^{ij}, \quad A'_\mu \equiv \delta A_\mu, \quad \mathcal{E}^{i\prime} \equiv \delta \mathcal{E}^i, \quad N' \equiv \delta N, \quad N'_i \equiv \delta N_i \tag{3.5}$$

and specialize the background (after taking the variations) to a solution of Eq. (3.4), we obtain the following variational integral for the perturbations:

$$\begin{aligned}
16\pi I_{\text{pert}} = \int_\Omega d^4x \{ & p^{ij} h_{ij,t} + A'_i \mathcal{E}^{i\prime}{}_{,t} \\
& - N\big(g^{-1/2}(p^{ij}p_{ij} - \tfrac{1}{2}p^2) + \tfrac{1}{2}g^{-1/2}g_{ij}(\mathcal{E}^{i\prime}\mathcal{E}^{j\prime} + \mathcal{B}^{i\prime}\mathcal{B}^{j\prime}) + g^{-1/2}(h_{ij} - \tfrac{1}{2}g_{ij}h)\mathcal{E}^i\mathcal{E}^{j\prime} \\
& \quad + \tfrac{1}{4}g^{-1/2}\mathcal{E}^i\mathcal{E}^j(-hh_{ij} + \tfrac{1}{2}g_{ij}h_{kl}h^{kl} + \tfrac{1}{4}g_{ij}h^2) \\
& \quad - \tfrac{1}{2}g^{1/2}[(\tfrac{1}{4}h^2 - \tfrac{1}{2}h^{ij}h_{ij})R - hh_{ij}R^{ij} + 2h_i{}^k h_{kj}R^{ij} + 2h^{ij}h_{|ij} \\
& \quad\quad - 2h^{ij}h_{ki|j}{}^{|k} + \tfrac{3}{2}h^{kj|i}h_{kj|i} + 2h^{kj}h_{kj|i}{}^{|i} - 2h^{kj}h_{ji}{}^{|i}{}_{|k} - h^{ij|k}h_{ki|j} \\
& \quad\quad - 2h^{kj}{}_{|k}h_{ji}{}^{|i} + 2h^{kj}{}_{|k}h_{|j} - \tfrac{1}{2}h^{|i}h_{|i} - hh_{|i}{}^{|i} + hh_{ij}{}^{|ij}]\big) \\
& - N'\mathcal{H}' - N'_i\mathcal{H}^{i\prime} - A'_0\mathcal{E}^{i\prime}{}_{,i}\},
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
&\mathcal{H}' = -g^{1/2}[h_{ij}{}^{|ij} - h_{|i}{}^{|i} - h_{ij}(R^{ij} - \tfrac{1}{2}g^{ij}R)] + \tfrac{1}{2}g^{-1/2}(h_{ij} - \tfrac{1}{2}g_{ij}h)\mathcal{E}^i\mathcal{E}^j + g^{-1/2}g_{ij}\mathcal{E}^i\mathcal{E}^{j\prime}, \\
&\mathcal{H}^{i\prime} = -2p^{ij}{}_{|j} - g^{ik}\epsilon_{klm}\mathcal{E}^l\mathcal{B}^{m\prime}, \quad \mathcal{B}^{i\prime} = \tfrac{1}{2}\epsilon^{ijk}(A'_{k,j} - A'_{j,k}), \quad p = g_{ij}p^{ij}, \quad h = g^{ij}h_{ij},
\end{aligned} \tag{3.7}$$

and in which R_{ij} is the Ricci tensor of g_{ij}, a vertical bar signifies covariant differentiation with respect to g_{ij}, and indices are raised and lowered with this metric and its inverse g^{ij}. The domain of integration Ω may be any smooth domain in $M \times R$ and independent variation (subject to the standard conditions on $\partial\Omega$) of h_{ij}, p^{ij}, A'_i, $\mathcal{E}^{i\prime}$, N', N'_i, and A'_0 produces the linearized ADM equations for the perturbations of the given static background solution.

It will be useful to define the Poisson bracket $\{\ ,\ \}$ for pairs of functions of the canonical variables (h_{ij}, p^{ij}), $(\mathcal{E}^{i\prime}, A'_i)$. Let k_{ij} be an arbitrary, symmetric tensor field and r^{ij} be an arbitrary, symmetric tensor density defined over M. Also, let k_i be an arbitrary vector field and r^i be an arbitrary vector density defined over M. The fundamental Poisson brackets may be expressed as

$$\begin{aligned}
&\left\{ h_{ij}(x^k), \int_M d^3y[p^{mn}k_{mn}] \right\} = k_{ij}(x^k), \\
&\left\{ p^{ij}(x^k), \int_M d^3y[h_{mn}r^{mn}] \right\} = -r^{ij}(x^k), \\
&\left\{ \mathcal{E}^{i\prime}(x^k), \int_M d^3y[A'_m r^m] \right\} = r^i(x^k), \\
&\left\{ A'_i(x^k), \int_M d^3y[\mathcal{E}^{m\prime}k_m] \right\} = -k_i(x^k),
\end{aligned} \tag{3.8}$$

with all the other basic brackets vanishing.

IV. GAUGE TRANSFORMATIONS AND THE PERTURBED CONSTRAINTS

In this section we derive the gauge transformations of the canonical perturbation functions for the case of a static background obeying Eqs. (3.4). We also show that the perturbed constraint functions are the generators of these gauge transformations and that the perturbed constraints have vanishing Poisson brackets with one another.

The coordinate and electromagnetic gauge transformations of the perturbations are generated by a pair $({}^4X, \Psi)$, where ${}^4X = {}^4X^\alpha \partial/\partial x^\alpha$ is an arbitrary vector field and Ψ an arbitrary function defined on the background space-time $(M \times R, {}^4g_{\mu\nu}, F_{\mu\nu})$. Letting $h_{\mu\nu} = \delta^4 g_{\mu\nu}$, $F'_{\mu\nu} = \delta F_{\mu\nu}$, and $A'_\mu = \delta A_\mu$ represent the metric, field tensor, and vector potential perturbations we write

$$\begin{aligned}
&h_{\mu\nu} \to h_{\mu\nu} + \hat{\delta} h_{\mu\nu}, \\
&F'_{\mu\nu} \to F'_{\mu\nu} + \hat{\delta} F'_{\mu\nu}, \\
&A'_\mu \to A'_\mu + \hat{\delta} A'_\mu
\end{aligned} \tag{4.1}$$

for the gauge transformations induced by some fixed pair $({}^4X, \Psi)$. A standard argument gives

$$\begin{aligned}
&\hat{\delta} h_{\alpha\beta} = (\mathcal{L}_{{}^4X}{}^4g)_{\alpha\beta} = {}^4X_{\alpha;\beta} + {}^4X_{\beta;\alpha}, \\
&\hat{\delta} F'_{\alpha\beta} = (\mathcal{L}_{{}^4X}F)_{\alpha\beta} = {}^4X^\gamma F_{\alpha\beta;\gamma} + {}^4X^\gamma{}_{;\alpha}F_{\gamma\beta} + {}^4X^\gamma{}_{;\beta}F_{\alpha\gamma}, \\
&\hat{\delta} A'_\alpha = (\mathcal{L}_{{}^4X}A)_\alpha + \Psi_{,\alpha} = {}^4X^\beta F_{\beta\alpha} + (\Psi + {}^4X^\beta A_\beta)_{,\alpha},
\end{aligned} \tag{4.2}$$

where $\mathcal{L}_{^4X}$ is the Lie derivative with respect to ^4X and a semicolon signifies covariant differentiation with respect to $^4g_{\alpha\beta}$.

For the following it will be convenient to split ^4X into its contributions normal and tangential to the t = constant hypersurfaces of the background.

Therefore, write

$$^4X = C\,^4n + X^i\partial/\partial x^i , \qquad (4.3)$$

in which $C = -\,^4n_\alpha {}^4X^\alpha = +N\,^4X^0$ is the projection of ^4X along the unit normal $[(^4n_\alpha) = (-N, 0, 0, 0)]$ and $X^i\partial/\partial x^i = g^{ij}X_j\partial/\partial x^i$ (with $X_j = {}^4X_j$) the tangential contribution (intrinsic to the hypersurface). From these definitions and Eqs. (4.2) above we obtain the gauge transformations of h_{ij}

$$\hat{\delta} h_{ij} = {}^4X_{i;j} + {}^4X_{j;i} = X_{i|j} + X_{j|i} , \qquad (4.4)$$

the last equality holding only for a static background. The gauge transformations of $\mathcal{E}^{i\prime}$ and $\mathcal{B}^{i\prime}$ follow from Eqs. (4.2) and from the definitions of these quantities as perturbations of

$$\begin{aligned}\mathcal{E}^i &= (-^4g)^{1/2}F^{0i} , \\ \mathcal{B}^i &= \tfrac{1}{2}\epsilon^{ijk}F_{jk} \\ &= \tfrac{1}{2}\epsilon^{ijk}(A_{k,j} - A_{j,k}) .\end{aligned} \qquad (4.5)$$

A short computation gives

$$\begin{aligned}\hat{\delta}\mathcal{E}^{i\prime} &= (X^k\mathcal{E}^i)_{,k} - X^i{}_{,k}\mathcal{E}^k , \\ \hat{\delta}\mathcal{B}^{i\prime} &= \tfrac{1}{2}\epsilon^{ijk}[(Cg^{-1/2}\mathcal{E}_j)_{,k} - (Cg^{-1/2}\mathcal{E}_k)_{,j}] ,\end{aligned} \qquad (4.6)$$

where we have made use of the background field equations to reexpress certain terms. The gauge transformations of p^{ij} are obtained similarly by reexpressing p^{ij} in terms of the metric perturbations $h_{\alpha\beta}$ and their derivatives and applying Eqs. (4.2). From the variational integral of the preceding section one obtains (by varying p^{ij})

$$h_{ij,t} = 2Ng^{-1/2}(p_{ij} - \tfrac{1}{2}g_{ij}p) + N'_{i|j} + N'_{j|i} \qquad (4.7)$$

or, equivalently

$$p^{ij} = (1/2N)g^{1/2}(g^{ik}g^{jl} - g^{ij}g^{kl}) \times [h_{kl,t} - (N'_{k|l} + N'_{l|k})] . \qquad (4.8)$$

Applying Eqs.(4.2) specialized to a static background [satisfying Eqs. (3.4)] we obtain, after a short computation,

$$\hat{\delta}p^{ij} = g^{1/2}(C^{|ij} - g^{ij}C_{|k}{}^{|k}) - g^{1/2}(R^{ij} - \tfrac{1}{2}g^{ij}R)C - \tfrac{1}{2}g^{-1/2}(\mathcal{E}^i\mathcal{E}^j - \tfrac{1}{2}g^{ij}g_{kl}\mathcal{E}^k\mathcal{E}^l)C , \qquad (4.9)$$

where we have made use of Eqs. (3.4) to reexpress certain terms. Finally for A'_i we obtain from Eq. (4.2)

$$\hat{\delta}A'_i = -Cg^{-1/2}g_{ij}\mathcal{E}^j + (\Psi + {}^4X^\beta A_\beta)_{,i} , \qquad (4.10)$$

where use has been made of Eqs. (3.4) and the condition $\mathcal{B}^i = 0$ to simplify the result.

For the perturbations of arbitrary vacuum spacetimes the general gauge transformations of h_{ij}, p^{ij} were derived in Ref. 12. Those results could readily be generalized to allow for electromagnetic fields in the background and perturbed spacetimes. For simplicity, however, we here consider only the static backgrounds (with $\mathcal{B}^i = 0$).

We now show how the gauge transformations derived above are generated by the perturbed constraint functions $\mathcal{H}'$, $\mathcal{H}^{i\prime}$, and $\mathcal{E}^{i\prime}{}_{,i}$. As before let C and X^i be normal and tangential projections of a vector field ^4X at a t = constant hypersurface σ of the background spacetime. Let χ be a function on σ and define, for given C, X^i, and χ, the generator

$$G(\sigma) = \int_\sigma d^3x(C\mathcal{H}' + X_i\mathcal{H}^{i\prime} + \chi\mathcal{E}^{i\prime}{}_{,i}) . \qquad (4.11)$$

A direct computation of the Poisson brackets $\{h_{ij}, G(\sigma)\}$, etc., gives

$$\begin{aligned}&\{h_{ij}, G(\sigma)\} = \hat{\delta}h_{ij} , \quad \{p^{ij}, G(\sigma)\} = \hat{\delta}p^{ij} , \\ &\{\mathcal{E}^{i\prime}, G(\sigma)\} = \hat{\delta}\mathcal{E}^{i\prime} , \quad \{\mathcal{B}^{i\prime}, G(\sigma)\} = \hat{\delta}\mathcal{B}^{i\prime} ,\end{aligned} \qquad (4.12)$$

where $\hat{\delta}h_{ij}$, etc., are given by Eqs. (4.4), (4.6), and (4.9) provided we identify C and X^i with the corresponding functions used before. For the vector potential we obtain

$$\{A'_i, G(\sigma)\} = -Cg^{-1/2}g_{ij}\mathcal{E}^j + \chi_{,i} , \qquad (4.13)$$

which agrees with Eq. (4.10) if we identify χ with $\Psi + {}^4X^\alpha A_\alpha$ (evaluated at the hypersurface σ).

It follows from the above considerations that any function of the canonical perturbation variables $(h_{ji}, p^{ij}, \mathcal{E}^{i\prime}, A'_i)$ which commutes with the generators

$$G_{(^4X,\Psi)}(\sigma) = \int_\sigma d^3x[C\mathcal{H}' + X_i\mathcal{H}^{i\prime} + (\Psi + {}^4X^\alpha A_\alpha)\mathcal{E}^{i\prime}{}_{,i}] \qquad (4.14)$$

(for arbitrary ^4X and Ψ) is invariant under simultaneous coordinate and electromagnetic gauge transformations. In particular the constraint functions themselves are gauge-invariant. To show this explicitly we compute the Poisson brackets of $\mathcal{H}'$, $\mathcal{H}^{i\prime}$, and $\mathcal{E}^{i\prime}{}_{,i}$ with $G_{(^4X,\Psi)}(\sigma)$. The only brackets which do not obviously vanish (for the static background case) are, by direct computation,

$$\begin{aligned}\left\{\mathcal{H}', \int_\sigma d^3x X_i\mathcal{H}^{i\prime}\right\} &= (X^i\mathcal{H})_{|i} - g^{-1/2}\mathcal{E}^iX_i\mathcal{E}^k{}_{,k} = 0 , \\ \left\{\mathcal{H}^{i\prime}, \int_\sigma d^3x C\mathcal{H}'\right\} &= C^{|i}\mathcal{H} + Cg^{-1/2}\mathcal{E}^i\mathcal{E}^k{}_{,k} = 0 ,\end{aligned} \qquad (4.15)$$

since $\mathcal{H} = (-g^{1/2}R + \tfrac{1}{2}g^{-1/2}g_{ij}\mathcal{E}^i\mathcal{E}^j) = 0$ and $\mathcal{E}^i{}_{,i} = 0$.

The main results of this section, that the perturbed constraints commute and are the generators of the gauge transformations, may be generalized to the perturbation problem for an arbitrary background solution of the Einstein-Maxwell equations. The proof is a straightforward computation similar to the one given here for the static background case. In the general case, however, the generators $G_{(^4X,\Psi)}(\sigma)$ depend on the choice of hypersurface σ through the time dependence of the background field variables $g_{ij}, \pi^{ij}, \mathcal{E}^i, A_i$. Typically a function of the canonical perturbation functions would have to depend explicitly on the time in order to commute with the generating functions associated with each hypersurface. In particular the perturbed constraints have precisely the required hypersurface dependence through their dependence on the background field variables g_{ij}, etc. In the stationary or static case (in stationary coordinates) a time-independent function of the perturbations commutes with the generators for every t = constant surface if and only if it commutes with those for a single surface. Thus for a stationary background a time-independent canonical transformation is sufficient to accomplish the reduction described in Sec. II whereas, for the general case, a time-dependent canonical transformation would be required. In a variety of practical problems (e.g., perturbations of homogeneous cosmological models) one can perform explicitly the desired time-dependent canonical transformation after expanding the perturbations in suitably chosen tensor harmonics.

V. CANONICAL TRANSFORMATIONS FOR REISSNER-NORDSTRÖM PERTURBATIONS

We now specialize the background to a Reissner-Nordström exterior solution with charge e and mass m ($|e| \leq m$). In standard coordinates the metric exterior to the event horizon at $r = r_+ = m + (m^2 - e^2)^{1/2}$ is given by

$$ds^2 = -N^2dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.1)$$

where

$$N^2 = e^{-2\lambda} = 1 - (2m/r) + (e^2/r^2). \quad (5.2)$$

The other background field variables are

$$\mathcal{E}^r = 2e\sin\theta, \quad \mathcal{E}^\theta = \mathcal{E}^\phi = \mathfrak{B}^i = \pi^{ij} = 0. \quad (5.3)$$

We begin by expanding the perturbation functions in Regge-Wheeler tensor harmonics labeled by the spherical harmonic indices L and M. The odd-parity perturbations (for $L \geq 2$) were treated in detail in Ref. 13 so here we consider only the even-parity case. Since perturbations corresponding to distinct values of (L,M) are not coupled by the perturbation equations (a consequence of the spherical symmetry of the background) we may treat each set separately from the others. Furthermore, it suffices, for each L, to set $M = 0$ since the perturbations for arbitrary M may be obtained by a rotation from those with $M = 0$. In this section we consider perturbations of order $L \geq 2$ for which both electromagnetic and gravitational radiation can occur. The remaining perturbations are discussed in Sec. VII.

Our main approach is to begin with the Regge-Wheeler parametrization for the metric perturbations h_{ij}, N', N_i' and the vector potential A_μ' and to introduce a Regge-Wheeler parameterization for p^{ij} and $\mathcal{E}^{i\prime}$ as well. This can easily be done so that the parametrization is canonical. The idea is then to find a canonical transformation from these variables to a new set in terms of which the perturbed constraints are simply a commuting subset (e.g., a subset of the new momentum functions). This can be accomplished explicitly for the present problem by means of simple transformations. The new canonical variables conjugate to the constraints will be gauge-dependent while the remaining conjugate pairs will be gauge-invariant. This conclusion follows from the general considerations of the preceding section but may readily be checked explicitly for the present problem. Indeed, in practice it is often helpful to construct gauge-invariant functions directly by studying the effect of arbitrary gauge transformations on the Regge-Wheeler expansion functions. Having obtained such quantities one completes the transformation by requiring it to be canonical and afterwards discovers that the perturbed constraints have been thereby simplified. These two approaches are essentially the "conjugates" of one another and are equally useful in practice.

For each $L \geq 2$ we expand the ($M = 0$) electromagnetic perturbations as

$$\begin{aligned}(A_i') &= \left((a_1 + a_{2,r})Y_{L0}, a_2\frac{\partial Y_{L0}}{\partial\theta}, 0\right), \\ (\mathcal{E}^{i\prime}) &= \left(f_1 \sin\theta Y_{L0}, \frac{(f_2 + f_{1,r})}{L(L+1)}\sin\theta\frac{\partial Y_{L0}}{\partial\theta}, 0\right), \\ A_0' &= a_0 Y_{L0},\end{aligned} \quad (5.4)$$

where a_0, a_1, a_2, f_1, and f_2 are real valued functions of r and t and in which we suppress the harmonic label L on these functions to simplify the notation. The parametrization has been chosen so that

$$\int_M d^3x(A_i'\mathcal{E}^{i\prime}{}_{,t}) = \int_{r_+}^\infty dr(a_1 f_{1,t} + a_2 f_{2,t}) \quad (5.5)$$

modulo an inessential boundary term which we have discarded. This parametrization is thus canonical and has been chosen so that the electromagnetic constraint function becomes

$$\mathcal{E}^{i\prime}{}_{,i} = -f_2 \sin\theta Y_{L0}\,. \tag{5.6}$$

Of the canonical variables only a_2 (conjugate to f_2) is affected by electromagnetic gauge transformations.

We expand the metric perturbations of order $(L,0)$ as

$$(h_{ij}) = \begin{bmatrix} e^{2\lambda}H_2 Y_{L0} & h_1\partial Y_{L0}/\partial\theta & 0 \\ \text{sym} & r^2(K+G\partial^2/\partial\theta^2)Y_{L0} & 0 \\ 0 & 0 & r^2(K\sin^2\theta + G\sin\theta\cos\theta\,\partial/\partial\theta)Y_{L0} \end{bmatrix}, \tag{5.7}$$

$$(N'_i) = (H_1 Y_{L0}, h_0\partial Y_{L0}/\partial\theta, 0)\,, \quad N' = -(N/2)H_0 Y_{L0}\,, \tag{5.8}$$

and introduce a similar expansion for the covariant tensor $g^{-1/2}p_{ij}$

$$(g^{-1/2}p_{ij}) = \begin{bmatrix} e^{2\lambda}P_H Y_{L0} & P_h\partial Y_{L0}/\partial\theta & 0 \\ \text{sym} & r^2(P_K+P_G\partial^2/\partial\theta^2)Y_{L0} & 0 \\ 0 & 0 & r^2(P_K\sin^2\theta + P_G\sin\theta\cos\theta\,\partial/\partial\theta)Y_{L0} \end{bmatrix}. \tag{5.9}$$

The parametrization is not yet canonical since

$$\int_M d^3x(p^{ij}h_{ij,t}) = \int_{r_+}^{\infty} dr\{r^2 e^{\lambda}P_H H_{2,t} + 2L(L+1)e^{-\lambda}P_h h_{1,t} + r^2 e^{\lambda}[[2P_K - L(L+1)P_G]K_{,t} + L(L+1)[-P_K + (L(L+1)-1)P_G]G_{,t}]\}\,. \tag{5.10}$$

Canonical form may be readily achieved by a simple reparametrization of the momentum functions.

Under coordinate gauge transformations induced by an arbitrary vector field ${}^4X = C^4 n^\mu\partial/\partial x^\mu + X^i\partial/\partial x^i$ [cf. Eq. (4.3)] the spatial metric perturbations h_{ij} transform according to Eq. (4.4). By taking for X_i an arbitrary vector harmonic of order $(L,0)$,

$$(X_i) = (C_0(r,t)Y_{L0}, C_1(r,t)\partial Y_{L0}/\partial\theta, 0)\,, \tag{5.11}$$

we induce the corresponding gauge transformations of the expansion functions (H_2, h_1, K, G). The explicit form of these transformations motivates the introduction of new functions k_1, k_2, k_3, k_4 defined by the (readily inverted) transformation

$$\begin{aligned} k_1 &= K + re^{-2\lambda}G_{,r} - (2/r)e^{-2\lambda}h_1\,, \\ k_2 &= \tfrac{1}{2}e^{2\lambda}H_2 - [(r/2)e^{2\lambda}K]_{,r} + (r/2)\lambda_{,r}e^{2\lambda}K\,, \\ k_3 &= G\,, \quad k_4 = h_1\,. \end{aligned} \tag{5.12}$$

Under the gauge transformation induced by (5.11) k_1 and k_2 are invariant whereas k_3 and k_4 undergo

$$k_3 \to k_3 + (2/r^2)C_1\,, \quad k_4 \to k_4 + C_0 + C_{1,r} - (2/r)C_1\,. \tag{5.13}$$

By demanding the canonical form

$$\int_M d^3x(p^{ij}h_{ij,t}) = \int_{r_+}^{\infty} dr(p_1 k_{1,t} + p_2 k_{2,t} + p_3 k_{3,t} + p_4 k_{4,t}) \tag{5.14}$$

(modulo an inessential boundary term) we obtain the new momenta $p_1, \ldots, p_4$ defined through

$$\begin{aligned} &P_H = (1/2r^2)e^{\lambda}p_2\,, \quad P_h = [1/2L(L+1)]e^{\lambda}[p_4 - (2/r)e^{-2\lambda}p_1]\,, \\ &[2P_K - L(L+1)P_G] = (1/r^2)e^{-\lambda}[p_1 + (r/2)e^{2\lambda}p_{2,r} + (r/2)\lambda_{,r}e^{2\lambda}p_2]\,, \\ &\{-P_K + [L(L+1)-1]P_G\} = [1/r^2 L(L+1)]e^{-\lambda}[p_3 - (re^{-2\lambda}p_1)_{,r}]\,, \end{aligned} \tag{5.15}$$

which are readily invertible transformations.

In terms of the new canonical variables the perturbed constraints become

$$\mathcal{H}^{r\prime}=p_4\sin\theta\, Y_{L0}\,,$$
$$\mathcal{H}^{\theta\prime}=[2/L(L+1)][(1/r^2)p_3-(e/r^2)L(L+1)a_1-(1/2r^2)(r^2p_4)_{,r}]\sin\theta\,\partial Y_{L0}/\partial\theta\,, \tag{5.16}$$
$$\mathcal{H}'=e^{\lambda}\{-(4re^{-4\lambda}k_2)_{,r}-L(L+1)[2e^{-2\lambda}k_2+r\lambda_{,r}k_1+k_1+(rk_1)_{,r}]+(2e/r^2)[f_1+eL(L+1)k_3]\}\sin\theta\, Y_{L0}\,.$$

The canonical transformation

$$\tilde{k}_3=k_3\,,\quad \tilde{p}_3=p_3-eL(L+1)a_1\,,\qquad \tilde{F}=f_1+eL(L+1)k_3\,,\quad \tilde{p}_F=a_1\,, \tag{5.17}$$

with

$$\int_{r_+}^{\infty}dr(a_1f_{1,t}+p_3k_{3,t})=\int_{r_+}^{\infty}dr(\tilde{p}_F\tilde{F}_{,t}+\tilde{p}_3\tilde{k}_{3,t})\,, \tag{5.18}$$

reduces the $\mathcal{H}^{i\prime}$ constraints to

$$\mathcal{H}^{r\prime}=p_4\sin\theta\, Y_{L0}\,,\qquad \mathcal{H}^{\theta\prime}=[2/r^2L(L+1)][\tilde{p}_3-\tfrac{1}{2}(r^2p_4)_{,r}]\sin\theta\,\partial Y_{L0}/\partial\theta\,, \tag{5.19}$$

which therefore involve only the two momentum functions p_4 and $\tilde{p}_3$. Finally, the transformation defined by

$$q_1=4re^{-4\lambda}k_2+L(L+1)rk_1,\quad F=\tilde{F}\,,$$
$$q_2=[4re^{-4\lambda}k_2+L(L+1)rk_1]_{,r}+L(L+1)[2e^{-2\lambda}k_2+(1+r\lambda_{,r})k_1]-(2e/r^2)\tilde{F} \tag{5.20}$$

reduces $\mathcal{H}'$ to

$$\mathcal{H}'=-e^{\lambda}q_2\sin\theta\, Y_{L0} \tag{5.21}$$

and may be rendered canonical through the definition of new momenta. Demanding that

$$\int_{r_+}^{\infty}dr(p_1k_{1,t}+p_2k_{2,t}+\tilde{p}_F\tilde{F}_{,t})=\int_{r_+}^{\infty}dr(\pi_1q_{1,t}+\pi_2q_{2,t}+\pi_FF_{,t}) \tag{5.22}$$

(modulo a surface term) leads to transformations

$$p_1=L(L+1)[r(\pi_1-\pi_{2,r})+(1+r\lambda_{,r})\pi_2]\,,$$
$$p_2=4re^{-4\lambda}(\pi_1-\pi_{2,r})+2e^{-2\lambda}L(L+1)\pi_2\,, \tag{5.23}$$
$$\tilde{p}_F=\pi_F-(2e/r^2)\pi_2\,,$$

which, in addition to Eqs. (5.20), may be easily inverted.

If, in addition, we introduce the notation

$$q_3=\tilde{k}_3,\quad \pi_3=\tilde{p}_3,\quad q_4=k_4,\quad \pi_4=p_4\,, \tag{5.24}$$

then

$$\int_M d^3x(p^{ij}h_{ij,t}+A_i'\mathcal{E}^{i\prime}{}_{,t})=\int_{r_+}^{\infty}dr(\pi_1q_{1,t}+\pi_2q_{2,t}+\pi_3q_{3,t}+\pi_4q_{4,t}+\pi_FF_{,t}+a_2f_{2,t}) \tag{5.25}$$

(modulo a boundary term) and the new canonical variables have reduced the constraints to the trivial form

$$\mathcal{H}^{r\prime}=\pi_4\sin\theta\, Y_{L0}\,,$$
$$\mathcal{H}^{\theta\prime}=[2/r^2L(L+1)][\pi_3-\tfrac{1}{2}(r^2\pi_4)_{,r}]\sin\theta\,\partial Y_{L0}/\partial\theta\,,$$
$$\mathcal{H}'=-e^{\lambda}q_2\sin\theta\, Y_{L0},\quad \mathcal{E}^{i\prime}{}_{,i}=-f_2\sin\theta\, Y_{L0}\,. \tag{5.26}$$

The canonical pairs (q_1,π_1) and (F,π_F) are invariant under simultaneous coordinate and electromagnetic gauge transformations. The constraint functions π_3, π_4, q_2, and f_2 are also gauge-invariant but must of course vanish for any actual perturbation solution. The remaining canonical variables q_3, q_4, π_2, and a_2 (which are conjugate to the constraint functions) are gauge-dependent. Additional gauge-dependent functions are the perturbed lapse function H_0, the perturbed shift vector functions H_1 and h_0, and the perturbed scalar potential a_0.

VI. THE PERTURBATION HAMILTONIAN

We now specialize the general variational integral (3.6) to the Reissner-Nordström background and substitute the Regge-Wheeler expansions (5.4), (5.7), (5.8), and (5.9) for the perturbations of order $(L,0)$. Reexpressing the result in terms of the new canonical variables $(q_1,\pi_1),\ldots,(q_4,\pi_4)$, (F,π_F), and (f_2,a_2) defined in the preceding section we obtain

$$16\pi I_{\text{pert}}=\int dt\int_{r_+}^{\infty}dr\left(\sum_{a=1}^{4}\pi_aq_{a,t}+\pi_FF_{,t}+a_2f_{2,t}-\mathcal{H}_T\right)\,, \tag{6.1}$$

in which the total Hamiltonian $H_T=\int_{r_+}^{\infty}dr\,\mathcal{H}_T$ is given by

$$H_T=+\tfrac{1}{2}\int_{r_+}^{\infty} dr \Big\{Ne^{-\lambda}L(L+1)(\pi_F)^2+\frac{Ne^{-\lambda}L(L+1)}{r^2(L-1)(L+2)}[(2/L(L+1))\pi_3+r\Lambda\pi_1+2e\pi_F]^2$$
$$+\frac{Ne^{\lambda}}{r^2L(L+1)}[(r\pi_4)^2-4r\pi_4e^{-2\lambda}L(L+1)[r(\pi_1-\pi_{2,r})+(1+r\lambda_{,r})\pi_2]]+4\frac{Ne^{-\lambda}}{r^2}\pi_3\pi_2\Big\}$$
$$+\tfrac{1}{2}\int_{r_+}^{\infty} dr\Big\{\frac{Ne^{-\lambda}[(L-1)(L+2)+(4e^2/r^2)]}{L(L+1)\Lambda^2}[q_2+(2eF/r^2)-q_{1,r}]^2$$
$$-\tfrac{1}{2}Ne^{\lambda}r(q_1)^2[(r\Lambda)_{,r}/(r\Lambda)^2]_{,r}-\frac{Ne^{\lambda}}{\Lambda^2}[(L-1)(L+2)+(4e^2/r^2)](q_1/r)[q_2+(2eF/r^2)]$$
$$+\frac{Ne^{\lambda}}{\Lambda L(L+1)}[-(2m/r)+(2e^2/r^2)][q_2+(2eF/r^2)][q_2+(2eF/r^2)-q_{1,r}]$$
$$+\frac{Ne^{\lambda}}{L(L+1)}(2eF/r^2)[q_2+(2eF/r^2)-q_{1,r}]$$
$$+[2Ne^{\lambda}/(r^3\Lambda)](erF)_{,r}\,[(2re^{-2\lambda}/L(L+1))[q_2+(2eF/r^2)]-q_1]$$
$$-[4Ne^{-\lambda}/(r^2\Lambda L(L+1))](erF)_{,r}q_{1,r}$$
$$+(Ne^{\lambda}/r^2)F^2+\frac{Ne^{-\lambda}}{L(L+1)}(f_2+F_{,r})^2+Ne^{\lambda}[((m/r)-(e^2/r^2))q_2+(2e/r)e^{-2\lambda}f_2][-rq_{3,r}+(2/r)q_4]\Big\}$$
$$+\int_{r_+}^{\infty} dr\{(H_0/2)q_2+H_1\pi_4+h_0[(2/r^2)\pi_3-(1/r^2)(r^2\pi_4)_{,r}]-a_0f_2\}\,, \tag{6.2}$$

in which

$$\Lambda=(L-1)(L+2)+(6m/r)-(4e^2/r^2)\,. \tag{6.3}$$

In deriving (6.2) from (3.6) we have discarded a number of boundary terms which do not affect the equations of motion.

Variation of the lapse function H_0, the shift functions H_1 and h_0, and the scalar potential a_0 gives the initial value equations

$$q_2=\pi_4=\pi_3=f_2=0\,. \tag{6.4}$$

These are conserved in time as a consequence of the Hamilton equations:

$$\begin{aligned} q_{2,t}&=-2(Ne^{-\lambda}\pi_4)_{,r}-(2/r)Ne^{-\lambda}(1+r\lambda_{,r})\pi_4+(2/r^2)Ne^{-\lambda}\pi_3\,,\\ \pi_{3,t}&=-\tfrac{1}{2}[rq_2((m/r)-(e^2/r^2))+2ef_2e^{-2\lambda}]_{,r}\,,\\ \pi_{4,t}&=-[(q_2/r)((m/r)-(e^2/r^2))+(2e/r^2)f_2e^{-2\lambda}]\,,\\ f_{2,t}&=0\,. \end{aligned} \tag{6.5}$$

Of particular interest are the Hamilton equations for the gauge-invariant functions q_1, π_1, F, π_F. These take a simpler form in terms of the new canonical variables

$$\begin{aligned} &Q=(q_1/\Lambda)(L-1)^{1/2}(L+2)^{1/2}\,,\quad H=F-(2e/r)(q_1/\Lambda)\,,\\ &P_Q=(L-1)^{-1/2}(L+2)^{-1/2}[\Lambda\pi_1+(2e/r)\pi_F]\,,\quad P_H=\pi_F\,. \end{aligned} \tag{6.6}$$

Q and H are the functions defined by Eqs. (8) and (9) of Ref. 1 and P_Q and P_H are their conjugate momenta. For the present purpose we may discard all terms of H_T containing the constraints since such terms do not affect the Hamilton equations for Q, P_Q, H, P_H. The reduced Hamiltonian becomes

$$H_T^*=+\tfrac{1}{2}\int_{r_+}^{\infty} dr\{Ne^{-\lambda}L(L+1)[(P_H)^2+(P_Q)^2]\}$$
$$+\tfrac{1}{2}\int_{r_+}^{\infty} dr\{[Ne^{-\lambda}/L(L+1)][(H_{,r})^2+(Q_{,r})^2]$$
$$+[1/L(L+1)][(H^2+Q^2)V+(3mH^2-3mQ^2-4e(L-1)^{1/2}(L+2)^{1/2}QH)S]\}\,, \tag{6.7}$$

in which

$$V=[Ne^{-\lambda}/(r\Lambda)^2][(8e^2/r^2)-(6m/r)]^2+[8e^2Ne^{-\lambda}/(r^4\Lambda)]$$
$$+\frac{1}{r^2\Lambda}L(L+1)(L-1)(L+2)+(3m/r^3)+(4e^2/r^4\Lambda)[2-(6m/r)+(4e^2/r^2)] \tag{6.8}$$

and

$$S=[L(L+1)/r^3\Lambda]+(2Ne^{-\lambda}/r^3\Lambda^2)[(L-1)(L+2)+(4e^2/r^2)]-(1/r^3\Lambda)[(2m/r)-(2e^2/r^2)], \tag{6.9}$$

and where the asterisk signifies that the constraints and a boundary term, resulting from an integration by parts, have been discarded from H_T. Since V and S are the same functions defined in Eqs. (12) and (13) of Ref. 1 it is easily shown that Hamilton's equations may be combined to reproduce Eq. (11) of Ref. 1:

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^{*2}}\right)\begin{pmatrix}H\\Q\end{pmatrix}+Ne^{-\lambda}(V+S\,T)\begin{pmatrix}H\\Q\end{pmatrix}=0, \tag{6.10}$$

with

$$\frac{dr}{dr^*}=Ne^{-\lambda}=1-(2m/r)+(e^2/r^2) \tag{6.11}$$

and

$$T=\begin{pmatrix}+3m & -2e(L-1)^{1/2}(L+2)^{1/2}\\ -2e(L-1)^{1/2}(L+2)^{1/2} & -3m\end{pmatrix}. \tag{6.12}$$

As was shown in Ref. 1, Eqs. (6.10) may be decoupled by the orthogonal transformation which diagonalizes T. Equivalently, if A is an orthogonal matrix for which

$$ATA^{\rm tr}=\begin{pmatrix}+\sigma & 0\\ 0 & -\sigma\end{pmatrix},$$
$$\sigma=[9m^2+4e^2(L-1)(L+2)]^{1/2} \tag{6.13}$$

then the canonical transformation

$$\begin{pmatrix}R_+\\R_-\end{pmatrix}=A\begin{pmatrix}H\\Q\end{pmatrix},\quad \begin{pmatrix}P_+\\P_-\end{pmatrix}=A\begin{pmatrix}P_H\\P_Q\end{pmatrix} \tag{6.14}$$

diagonalizes the reduced Hamiltonian H_T^*:

$$H_T^*=+\tfrac{1}{2}\int_{r_+}^{\infty}dr\left\{Ne^{-\lambda}L(L+1)[(P_+)^2+(P_-)^2]+\frac{Ne^{-\lambda}}{L(L+1)}[(R_{+,r})^2+(R_{-,r})^2]+[1/L(L+1)][V_+(R_+)^2+V_-(R_-)^2]\right\} \tag{6.15}$$

with

$$V_\pm=V\pm\sigma S\,. \tag{6.16}$$

Evidently H_T^* is a positive-definite function of the gauge-invariant canonical variables provided V_+ and V_- are positive-definite on the interval (r_+,∞) for each $L\geq 2$ and for all e and m such that $|e|\leq m$. Proof that V_+ and V_- are indeed definite is straightforward algebra which we relegate to the Appendix.

The positive definiteness of H_T^* may be used as the basis for a stability argument similar to that given in Ref. 13 for the odd-parity perturbations. The conservation equation obeyed by H_T^* is

$$\frac{dH_T^*}{dt}=[Ne^{-\lambda}(P_+R_{+,r^*}+P_-R_{-,r^*})]\Big|_{r_+}^{\infty}=[1/L(L+1)](R_{+,t}R_{+,r^*}+R_{-,t}R_{-,r^*})\Big|_{r_+}^{\infty}, \tag{6.17}$$

which expresses the time derivative of the Hamiltonian in terms of an energy flux through the boundaries at the event horizon and at spatial infinity. For solutions with boundary conditions of ingoing radiation at the event horizon and outgoing radiation at infinity this flux obeys $dH_T^*/dt\leq 0$ so that H_T^* remains bounded by its initial value on any time interval throughout which the outgoing radiation boundary conditions apply. In particular this result may be used to prove the nonexistence of unstable (exponentially growing) normal-mode solutions obeying the outgoing radiation boundary conditions. This argument is essentially equivalent to that given in Ref. 1.

VII. THE $L=1$ PERTURBATIONS

The $L=0$ and $L=1$ perturbations require a special treatment and were not discussed in Refs. 1 or 13. However, the $L=0$ perturbations are spherically symmetric and merely allow for small changes of the charge and mass parameters within the Reissner-Nordström family of solutions. In this section we shall discuss the $L=1$ perturbations for both the odd- and even-parity classes.

The odd-parity perturbations (for $L\geq 2$) were discussed in Ref. 13. For $L=1$, the Regge-Wheeler expansions simplify since the angular coefficients of h_2 and p_2 vanish. Therefore, these two functions drop out leaving only the two conjugate pairs (h_1,p_1), (E,A) and the shift function h_0. The canonical transformation

$$k_1=(2/r^2)h_1\,,\quad \pi_1=(r^2/2)p_1-2eA\,,$$
$$f=E+(4e/r^2)h_1\,,\quad \pi_f=A \tag{7.1}$$

leads to the variational integral

$$16\pi I_{\rm pert}=\int dt\int_{r_+}^{\infty}dr(\pi_1 k_{1,t}+\pi_f f_{,t}-\mathcal{H}_T)\,, \tag{7.2}$$

with

$$\begin{aligned}H_T&=\int_{r_+}^{\infty}dr\,\mathcal{H}_T\\&=\tfrac{1}{4}\int_{r_+}^{\infty}dr\{Ne^{\lambda}[(2/r^2)\pi_1+(4e/r^2)\pi_f]^2\}\\&\quad+\int_{r_+}^{\infty}dr\{N[(2e^{\lambda}/r^2)(\pi_f)^2+e^{-\lambda}(\pi_{f,r})^2]\}\\&\quad+\int_{r_+}^{\infty}dr[Ne^{-\lambda}(f/2)^2]\\&\quad-2\int_{r_+}^{\infty}dr[(h_0/r^2)\pi_{1,r}]\,.\end{aligned} \tag{7.3}$$

Variation of the shift function h_0 gives the constraint $\pi_{1,r}=0$. Since k_1 is cyclic we have $\pi_{1,t}=0$ and thus $\pi_1=c=\text{constant}$. If one considers purely stationary perturbations then Hamilton's equations give

$$\begin{aligned}0&=f_{,t}\\&=(2e/r^2)[(2/r^2)\pi_1+(4e/r^2)\pi_f]\\&\quad+(4/r^2)\pi_f-2(Ne^{-\lambda}\pi_{f,r})_{,r}\,,\end{aligned} \tag{7.4}$$

which has the regular solution

$$\pi_f=(2e/r)\delta a\,,\quad \pi_1=c=-6m\delta a \tag{7.5}$$

and (from the $k_{1,t}=0$ equation)

$$h_0=(-2m/r)\delta a+(e^2/r^2)\delta a\,. \tag{7.6}$$

This solution corresponds to perturbing the Reissner-Nordström black hole to a slowly rotating Kerr-Newman black hole (with rotation parameter δa).

In addition to the stationary perturbations there are solutions corresponding to electromagnetic radiation. The wave equation satisfied by these perturbations (e.g., the equation for π_f) may be readily derived from the variational integral (7.2). We only note here that for any actual perturbation solution (which has $\pi_1=c$) the Hamiltonian H_T is a positive-definite function of f and π_f. This result allows a stability argument of the type previously discussed provided suitable boundary conditions are imposed upon the perturbations.

Turning to the even-parity perturbations we begin with the Regge-Wheeler expansions (5.4) and (5.7)–(5.9) and observe that, when $L=1$, the angular coefficients of K and $-G$ and of P_K and $-P_G$ become equal. Therefore, we may set $G=P_G=0$ to exclude superfluous variables. Define new canonical variables q_1, q_2, q_3, and H by

$$\begin{aligned}q_1&=2re^{-2\lambda}k_2\,,\\q_2&=\Lambda e^{2\lambda}k_1-(2e/r^2)[f_1-(4e^{-2\lambda}/\Lambda)ek_2]\\q_3&=K\,,\quad H=f_1-(4e^{-2\lambda}/\Lambda)ek_2\,,\end{aligned} \tag{7.7}$$

where

$$\Lambda=+(6m/r)-(4e^2/r^2)\,, \tag{7.8}$$

and in which k_1 and k_2 are defined by

$$\begin{aligned}k_1&=(2/r)e^{-2\lambda}h_1-K+2re^{-2\lambda}[(e^{-2\lambda}/\Lambda)k_2]_{,r}\\&\quad+(2e^{-2\lambda}/\Lambda)k_2\,,\\k_2&=H_2-e^{-\lambda}[re^{\lambda}K]_{,r}\\&\quad-e^{2\lambda}[(2/r)e^{-2\lambda}h_1-K]\,.\end{aligned} \tag{7.9}$$

With new momenta $\pi_1, \pi_2, \pi_3, \pi_H$ defined by

$$\begin{aligned}\pi_1&=re^{3\lambda}P_H[\tfrac{1}{2}-(1/\Lambda)]\\&\quad+(1/r\Lambda)[(r^3e^{\lambda}P_H)_{,r}+(2r^2e^{-\lambda}P_h)_{,r}]\\&\quad-(2e^{\lambda}/\Lambda)P_h+(2e/r\Lambda)a_1\,,\\\pi_2&=(r^2e^{\lambda}/\Lambda)P_H+(2re^{-\lambda}/\Lambda)P_h\,,\\\pi_3&=-re^{\lambda}(r^2P_H)_{,r}+2re^{\lambda}P_h+2r^2e^{\lambda}P_K\,,\\\pi_H&=(2e^{\lambda}/\Lambda)eP_H+(4e^{-\lambda}/r\Lambda)eP_h+a_1\,,\end{aligned} \tag{7.10}$$

we have

$$\int_M d^3x(p^{ij}h_{ij,t}+A_i'\mathcal{E}^{i\prime}{}_{,t})=\int_{r_+}^{\infty}dr\left(\sum_{a=1}^{3}\pi_a q_{a,t}+\pi_H H_{,t}+a_2 f_{2,t}\right) \tag{7.11}$$

(modulo a boundary term). Thus the new variables are canonical and, as is easily shown, the transformations are invertible.

In terms of the new variables the perturbed constraints become simply

$$\begin{aligned}\mathcal{H}^{r\prime}&=(2/r)e^{-2\lambda}\pi_3\sin\theta\,Y_{10}\,,\\\mathcal{H}^{\theta\prime}&=-(1/r^2)(\pi_3+r\Lambda\pi_1)\sin\theta\,\partial Y_{10}/\partial\theta\\\mathcal{H}'&=-e^{\lambda}q_2\sin\theta\,Y_{10}\,,\\\mathcal{E}^{i\prime}{}_{,i}&=-f_2\sin\theta\,Y_{10}\,,\end{aligned} \tag{7.12}$$

while the Hamiltonian becomes

$$H_T = \int_{r_+}^{\infty} dr\{Ne^{-\lambda}(\pi_H)^2 - (2/r)Ne^{-\lambda}\Lambda\pi_1\pi_2$$
$$- (2/r)Ne^{-\lambda}\pi_3[\pi_1 e^{-2\lambda} + (\pi_2/r) - (2e/r\Lambda)e^{-2\lambda}(\pi_H - (2e/r^2)\pi_2) - (1/r\Lambda)e^{-2\lambda}(r\Lambda\pi_2)_{,r}]\}$$
$$+ \tfrac{1}{4}\int_{r_+}^{\infty} dr\, Ne^{-\lambda}(H_{,r})^2$$
$$+ \int_{r_+}^{\infty} dr\{[Ne^{\lambda}/(r\Lambda)^2](H)^2[18(m^2/r^2) - 36(m^2e^2/r^4) + 32(me^4/r^5) - 8(e^6/r^6)]\}$$
$$+ \tfrac{1}{2}\int_{r_+}^{\infty} dr(Ne^{\lambda}/\Lambda^2)q_2\{(2e/r^2)e^{-2\lambda}[eq_2 + (6m/r)H + r\Lambda H_{,r} + (2e/r)q_1]$$
$$+ \Lambda[-(m/r) + (e^2/r^2)][+(4e^2/r^3\Lambda)q_1 + (2e/r^2)H - e^{2\lambda}\Lambda q_3 + (1/r)e^{2\lambda}q_1]$$
$$- (4e^2/r^3)q_1 + (e/r^2)\Lambda^2[H + (2e/r\Lambda)q_1]\}$$
$$+ \tfrac{1}{4}\int_{r_+}^{\infty} dr\, Ne^{-\lambda}f_2[f_2 + 2H_{,r} + (4e/r)e^{2\lambda}q_3 + (4e/r\Lambda)(q_2 + (2e/r^2)H) - (4e/r^2\Lambda)e^{2\lambda}q_1]$$
$$+ \int_{r_+}^{\infty} dr[\tfrac{1}{2}H_0 q_2 - a_0 f_2 + (2/r)e^{-2\lambda}H_1\pi_3 - (2/r^2)h_0(\pi_3 + r\Lambda\pi_1)]\,. \tag{7.13}$$

Variation of the perturbed lapse function H_0, shift functions H_1 and h_0, and scalar potential a_0 reproduces the constraints (7.12). It is easy to show that the constraints are conserved in time provided they are satisfied initially.

The unconstrained, gauge-invariant functions are H and π_H which comprise a canonical pair. Hamilton's equations for these functions may be obtained from H_T or, equivalently, from the reduced Hamiltonian H_T^* obtained from H_T by setting the constraints f_2, q_2, π_1, π_3 to zero. It is easily seen that H_T^* is a positive-definite function of H and π_H provided the quantity $[18(m^2/r^2) - 36(m^2e^2/r^4) + 32(me^4/r^5) - 8(e^6/r^6)]$ is positive-definite on (r_+, ∞) for all e and m such that $|e| \leq m$. A proof of this inequality is given in the appendix. The corresponding stability result follows as in the preceding sections.

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APPENDIX

To complete the argument of Sec. VI we need to show that V_+ and V_-, defined by Eq. (6.16), are positive-definite on (r_+, ∞) for each $L \geq 2$ and for all e and m such that $|e| \leq m$. The function S defined by Eq. (6.9) may be written as

$$S = (1/r^3\Lambda)[L(L+1) - (2m/r) + (2e^2/r^2)] + (2Ne^{-\lambda}/r^3\Lambda^2)[(L-1)(L+2) + (4e^2/r^2)]\,, \tag{A1}$$

and is thus manifestly positive-definite for the given range of variables. Therefore, it is sufficient to prove that $V_- = V - \sigma S$ is positive-definite since the definiteness of $V_+ = V + \sigma S$ will then be trivial. V_- may be reexpressed as

$$V_- = +[Ne^{-\lambda}/(r\Lambda)^2][(8e^2/r^2) - (6m/r)]^2 + (8Ne^{\lambda}/r^2\Lambda)(e^2/r^2)^2$$
$$+ (\sigma/r^3)(2mr - 2e^2)[(Ne^{\lambda}/r^2\Lambda) + 6Ne^{-\lambda}/(r\Lambda)^2] + [4Ne^{-\lambda}/(r\Lambda)^2](e^2/r^2)C_1 + (Ne^{\lambda}/r^2\Lambda)C_2\,, \tag{A2}$$

where

$$C_1 = 4\Lambda - (\sigma/r)\,, \quad \sigma = [9m^2 + 4e^2(L-1)(L+2)]^{1/2}$$
$$C_2 = [L(L+1)(L-1)(L+2) - (8e^2m/r^3) + (3m/r)\Lambda - (\sigma/r)(L(L+1) + 2e^{-2\lambda})]\,, \tag{A3}$$
$$\Lambda = (L-1)(L+2) + (6m/r) - (4e^2/r^2)\,.$$

Thus it suffices to show that $C_1 \geq 0$ and $C_2 \geq 0$. Since $\sigma \leq m(2L+1)$ (with equality holding for $|e| = m$) we have

$$\begin{aligned} C_1 &\geq 4\Lambda - (2L+1)(m/r) \\ &\geq 4[(L-1)(L+2) + (6m/r) - (4m^2/r^2)] - (2L+1)(m/r) \\ &= 4(L^2-2) + 2L[1-(m/r)] + [2L-(m/r)] + (4/r^2)(6mr - 4m^2) \\ &> 0 . \end{aligned} \tag{A4}$$

Similarly,

$$\begin{aligned} C_2 &\geq L(L+1)(L-1)(L+2) - (8m^3/r^3) + (3m/r)[(L-1)(L+2) + (6m/r) - (4m^2/r^2)] \\ &\quad - (2L+1)(m/r)[L(L+1) + 2 - (4m/r) + (2m^2/r^2)] \\ &= L^2(L^2-1) - 4(L+2) + (2L^3 + 2L + 8)[1-(m/r)] + (m^2/r^2)[(20 + 2(2L+1))(1 - m/r) + 4L], \end{aligned} \tag{A5}$$

which is manifestly positive-definite for $L \geq 3$. For $L=2$ Eq. (A5) reduces to

$$C_2(L=2) \geq (2/r^3)(12r^3 - 14mr^2 + 19m^2r - 15m^3) \equiv (2/r^3)D(r). \tag{A6}$$

However, $D(m) = 2m^3$ and

$$dD(r)/dr = 8r^2 + 28r^2[1-(m/r)] + 19m^2, \tag{A7}$$

which is greater than zero for $r \geq m$. Therefore, $C_2 > 0$ and consequently V_- is positive-definite.

To obtain the corresponding stability result for the $L=1$ perturbations it suffices to show that

$$C_3 \equiv 18m^2r^4 - 36m^2e^2r^2 + 32me^4r - 8e^6 \tag{A8}$$

is positive-definite on (r_+, ∞) for all $|e| \leq m$. But

$$dC_3/dr = 72m^2r(r^2 - e^2) + 32me^4 \tag{A9}$$

is greater than zero on (m, ∞). Therefore, one need only show that $C_3(m) > 0$. However,

$$C_3(m) = m^6[18 - 36(e/m)^2 + 24(e/m)^4 + 8(e/m)^4(1-(e/m)^2)], \tag{A10}$$

and it is easy to show that $18 - 36y + 24y^2$ is positive-definite for all real y.

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