Probability distribution of particles created by a black hole

Leonard Parker
Department of Physics, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201
(Received 5 May 1975)

It has been demonstrated by Hawking that a spherical black hole formed by collapse will create particles, and that the average number of outgoing particles created in each mode satisfies a thermal distribution law. Recently, Wald has shown that the probability distribution of the number of outgoing particles in each mode, and not only the average number, obeys the thermal distribution law. In this paper, we give a simpler and more direct derivation of the blackbody probability distribution of the emitted particles.

I. INTRODUCTION

One of the most interesting recent developments in gravitational theory, which may possibly lead to observable consequences, is the prediction that a spherical black hole formed by collapse will create particles.[1-3] Hawking showed that the object will radiate at late times like a body of temperature $T=\kappa/(2\pi k)$, where $\kappa=(4M)^{-1}$ is the surface gravity of the black hole, and k is Boltzmann's constant ($G=c=\hbar=1$). This remarkable process implies that a definite entropy and temperature are associated with a black hole, as anticipated by Bekenstein.[4]

In Refs. 1–3 only the average number or average energy-momentum were considered. Recently, Wald has considered the question of whether the density matrix of the created particles which reach infinity, and not just the average number of particles in each mode, is that of thermal radiation, and has shown that this is indeed the case.[5] In the present paper, we carry out the calculation in a simpler and more direct way. This is accomplished by calculating the matrix elements in the expansion of the initial vacuum state in terms of the basis of the late-time Fock space, in the approximation where backscattering by the static exterior Schwarzschild spacetime is neglected. These results allow one to see directly that the particles are created in pairs with one ingoing and one outgoing member. The methods used here for massless scalar particles can also be applied to massive particles and higher spin (with appropriate changes for statistics). Wald[5] has also considered the change in the probability distribution taking backscattering into account.

In the present section we summarize the notation and results of Ref. 1. In the second section we derive the particle number distribution, and show that it corresponds to thermal radiation. Finally, in the third section, we derive two key equations used in the second section and check the consistency of the various orthonormality and commutation relations.

The Penrose diagram of the spherical collapse process is given in Fig. 1. $\mathscr{I}^-$ represents past null infinity, $\mathscr{I}^+$ represents future null infinity, and h represents the future horizon. The advanced time coordinate v is given (Schwarzschild r and t) by

$$v=t+r+2M\ln\left|\frac{r}{2M}-1\right|, \tag{1}$$

and the retarded time coordinate by

$$u=t-r-2M\ln\left|\frac{r}{2M}-1\right|. \tag{2}$$

The latest advanced time at which a particle can leave $\mathscr{I}^-$, pass through the collapsing body, and return to $\mathscr{I}^+$ is denoted by v_0. The field satisfies the wave equation

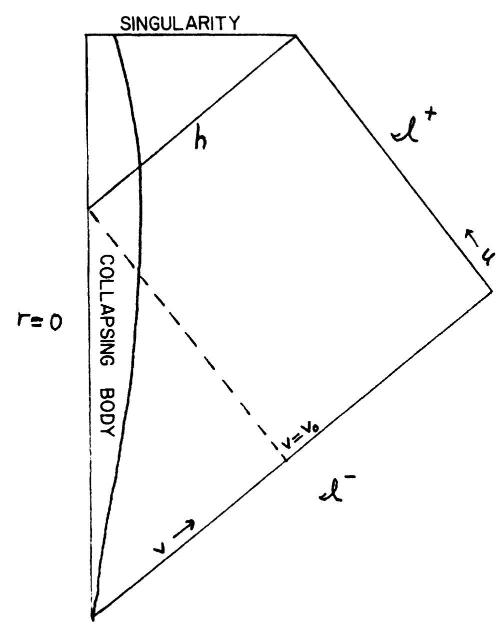

FIG. 1. Penrose diagram of collapse process.

$$\phi_{;ab}g^{ab}=0. \tag{3}$$

The positive-frequency[6] solutions incoming from $\mathscr{I}^-$ are the

$$f_{\omega' lm}=r^{-1}F_{\omega'}(r)(\omega')^{-1/2}e^{i\omega' v}Y_{lm}(\theta,\phi), \tag{4}$$

where $F_{\omega'}(r)$ is a solution of the radial equation. Similarly, the positive-frequency solutions outgoing at $\mathscr{I}^+$ are the

$$p_{\omega lm}=r^{-1}P_\omega(r)\omega^{-1/2}e^{i\omega u}Y_{lm}(\theta,\phi). \tag{5}$$

The complete set of positive-frequency solutions incoming at h (with zero Cauchy data on $\mathscr{I}^+$) are denoted by $q_{\omega lm}$, and have positive Klein-Gordon norm, but are otherwise left unspecified in Ref. 1.

The field can be expanded in the form (suppressing the l, m indices, and the integration limits which run from 0 to ∞)

$$\phi=\int d\omega'(f_{\omega'}a_{\omega'}+\bar{f}_{\omega'}a^\dagger_{\omega'}) \tag{6}$$

or

$$\phi=\int d\omega(p_\omega b_\omega+\bar{p}_\omega b^\dagger_\omega+q_\omega c_\omega+\bar{q}_\omega c^\dagger_\omega), \tag{7}$$

where $a_{\omega'}$ annihilates a particle incoming from $\mathscr{I}^-$, b_ω annihilates a particle outgoing at $\mathscr{I}^+$, and c_ω annihilates a particle incoming at the future horizon h. The $a_{\omega'}$, $a^\dagger_{\omega'}$ operators obey the standard commutation relations for creation and annihilation operators, as do the b_ω, $b^\dagger_\omega$ operators, and the c_ω, $c^\dagger_\omega$ operators. Furthermore, the b operators commute with the c, $c^\dagger$ operators. Since the $f_{\omega'}$ form a complete orthonormal set, one has

$$p_\omega=\int d\omega'(\alpha_{\omega\omega'}f_{\omega'}+\beta_{\omega\omega'}\bar{f}_{\omega'}) \tag{8}$$

and

$$q_\omega=\int d\omega'(\gamma_{\omega\omega'}f_{\omega'}+\eta_{\omega\omega'}\bar{f}_{\omega'}), \tag{9}$$

where α,β,γ,η are complex numbers. It follows that

$$b_\omega=\int d\omega'(\bar{\alpha}_{\omega\omega'}a_{\omega'}-\bar{\beta}_{\omega\omega'}a^\dagger_{\omega'}) \tag{10}$$

and

$$c_\omega=\int d\omega'(\bar{\gamma}_{\omega\omega'}a_{\omega'}-\bar{\eta}_{\omega\omega'}a^\dagger_{\omega'}). \tag{11}$$

In Eqs. (10) and (11), the indices l, m appear on b_ω, c_ω, and $a_{\omega'}$, but indices $l, -m$ appear on $a^\dagger_{\omega'}$.

Because p_ω has zero Cauchy data on the future horizon, it is natural to consider p_ω propagating backwards in time from $\mathscr{I}^+$. A part $p^{(1)}_\omega$ is backscattered off the static exterior Schwarzschild geometry, reaching $\mathscr{I}^-$ with the original frequency, and without any mixing of positive and negative frequencies. A second part $p^{(2)}_\omega$ enters the collapsing object, and scatters off the time-dependent interior geometry or passes through the collapsing body, reaching $\mathscr{I}^-$ as a superposition of various positive- and negative-frequency parts. Thus, we have on $\mathscr{I}^-$

$$p_\omega=p^{(1)}_\omega+p^{(2)}_\omega, \tag{12}$$

with

$$p^{(1)}_\omega=\alpha^{(1)}_\omega f_\omega, \tag{13}$$

and

$$p^{(2)}_\omega=\int d\omega'(\alpha^{(2)}_{\omega\omega'}f_{\omega'}+\beta^{(2)}_{\omega\omega'}\bar{f}_{\omega'}). \tag{14}$$

The part of $p^{(2)}_\omega(v)$ for v near v_0 comes mainly from the terms in Eq. (14) with ω' large with respect to ω. It is this part of $p^{(2)}_\omega(v)$ which is responsible for the steady thermal radiation which remains after the radiation dependent on the details of the collapse has subsided. Hawking assumes that this high-frequency component of $p^{(2)}_\omega(v)$ propagates through the collapsing body by geometrical optics, and obtains for it the following expression,

$$p^{(2)}_\omega(v)=\begin{cases}\omega^{-1/2}r^{-1}P^-_\omega\exp\left[-i\dfrac{\omega}{\kappa}\ln\left(\dfrac{v_0-v}{CD}\right)\right] & \text{for } v_0-v \text{ small and positive}\\ 0 \text{ for } v_0-v \text{ negative,} & \end{cases} \tag{15}$$

where $P^-_\omega=P_\omega(2M)$ is the value of the solution of the radial equation on the horizon in the static Schwarzschild geometry, and C, D are constants which depend on the details of the collapse (there is also a factor of Y_{lm} which we suppress as in Ref. 1).

By taking Fourier components of Eq. (15) with respect to v, Hawking finds that

$$|\beta^{(2)}_{\omega\omega'}|=\exp(-\pi\omega/\kappa)|\alpha^{(2)}_{\omega\omega'}|. \tag{16}$$

It follows that if the vaccum on $\mathscr{I}^-$ is chosen as the state vector, then the average number of created particles per unit frequency near ω reaching $\mathscr{I}^+$ in the thermal component of the radiation is

$$\begin{aligned}\langle n_\omega\rangle&=\int_0^\infty d\omega'|\beta_{\omega\omega'}|^2\\&=\Gamma_\omega[\exp(2\pi\omega/\kappa)-1]^{-1},\end{aligned} \tag{17}$$

where

$$\Gamma_\omega=\int_0^\infty d\omega'(|\alpha^{(2)}_{\omega\omega'}|^2-|\beta^{(2)}_{\omega\omega'}|^2).$$

The factor Γ_ω is formally divergent because there is a steady flow of thermal radiation at late times, corresponding to an infinite number of particles reaching $\mathscr{I}^+$. However, as Hawking shows by considering wave packets on $\mathscr{I}^+$ made from a superposition of p_ω in a small range of ω, the average number $\langle n'_\omega\rangle$ of created particles per unit frequency near ω and *per unit time* reaching $\mathscr{I}^+$ is finite, and can be written as

$$\langle n'_\omega\rangle = \Gamma'_\omega[\exp(2\pi\omega/\kappa)-1]^{-1}\,. \tag{18}$$

This corresponds to thermal radiation of temperature $T=\kappa/(2\pi k)$, where Γ'_ω is the finite absorptivity of the black hole (which is not unity because of backscattering from the exterior spacetime curvature). We now turn to the calculation of the probability of observing n_ω particles outgoing at $\mathscr{I}^+$ in mode ω.

II. PROBABILITY DISTRIBUTION OF OUTGOING PARTICLES

The state of the quantized field (Heisenberg picture) is taken to be the vacuum $|0\rangle$ on $\mathscr{I}^-$, defined by

$$a_{\omega'}|0\rangle = 0 \quad \text{for all } \omega'. \tag{19}$$

Let $|0)$ be the state vector corresponding to no particles entering the future horizon or reaching future null infinity $\mathscr{I}^+$. That state is defined by

$$b_\omega|0)=0, \tag{20}$$

$$c_\omega|0)=0 \tag{21}$$

for all ω. The late-time Fock space is constructed by operating with the $c^\dagger_\omega$ and $b^\dagger_\omega$ on the state $|0)$. In order to find the probability distribution of the particles outgoing at $\mathscr{I}^+$, as well as those incoming at h, we calculate the matrix elements appearing in the expansion of $|0\rangle$ as a superposition of basis states of the late-time Fock space.

Our interest here is primarily in the thermal component of the radiation. Therefore, we make the simplification of ignoring the parts of p_ω and q_ω which are backscattered from the exterior metric and do not contribute to the thermal component of the radiation. In this approximation we find that the outgoing radiation is that of a perfect blackbody. (The backscattering from the exterior spacetime is then taken into account by introducing the absorptivity factor Γ'_ω.) Thus, we drop the superscript (2) on $p^{(2)}_\omega$ and $q^{(2)}_\omega$, and treat relations involving their Fourier components as strictly valid within this approximation. Furthermore, it is convenient to work with ω as a discrete variable, and return to the continuum limit at the end.

Let us first consider the scalar product of $|0\rangle$ with the state

$$|n,n) = (n!)^{-1}(c^\dagger_\omega)^n(b^\dagger_\omega)^n|0), \tag{22}$$

containing n particles in mode ω outgoing at $\mathscr{I}^+$, and n particles in mode ω incoming at h. More precisely, c_ω and b_ω represent $c_{\omega,l,-m}$ and $b_{\omega lm}$ (angular momentum is conserved in the pair production process). If there were additional quantum numbers to distinguish particles from antiparticles, then $c^\dagger_\omega$ would be the creation operator for antiparticles incoming at the horizon, which evidently (see the arguments of Hawking in Ref. 1) carry negative energy into the black hole. After the calculation of the matrix element of $|n,n)$ with $|0\rangle$, it will be easy to see that only basis states of the form (22) have a nonvanishing scalar product with $|0\rangle$.

To calculate

$$(n,n|0\rangle = (n!)^{-1}(0|(c_\omega)^n(b_\omega)^n|0\rangle, \tag{23}$$

we first note that as a consequence of Eq. (10)

$$b_\omega|0\rangle = -\int d\omega'\bar{\beta}_{\omega\omega'}a^\dagger_{\omega'}|0\rangle. \tag{24}$$

Furthermore,

$$c^\dagger_\omega|0\rangle = \int d\omega'\gamma_{\omega\omega'}a^\dagger_{\omega'}|0\rangle. \tag{25}$$

[In Eqs. (24) and (25), b_ω carries indices lm, while $c^\dagger_\omega$ and $a^\dagger_\omega$ carry indices $l,-m$.] We will show that there is a very simple relationship between Eqs. (24) and (25) if the solutions q_ω, corrsponding to particles incoming at h, are conveniently chosen.[7]

Following Wald,[5] we effectively generate q_ω from p_ω described by Eq. (15) [as noted earlier, we drop the superscript (2), ignoring backscattering from the exterior geometry; since we are interested only in the steady thermal component of the radiation, we use Eq. (15) to describe p_ω for all v and Eq. (26) below to describe q_ω for all v]. On h the advanced time v runs from v_0 to ∞. Therefore, if we replace v_0-v by $v-v_0$ on the right-hand side of Eq. (15), then the new functions on $\mathscr{I}^-$ describe particles incoming at $\mathscr{I}^-$ for $v>v_0$. In our present approximation, the solution propagates by geometrical optics directly into h with no scattering to $\mathscr{I}^+$. [Thus, we are neglecting an additional part of q_ω which is incoming at $\mathscr{I}^-$ for $v<v_0$ and passes through the collapsing body out toward $\mathscr{I}^+$ in just such a way as to cancel any waves scattered toward $\mathscr{I}^+$, so that nothing reaches $\mathscr{I}^+$ from q_ω. The main long-term effect of that part of q_ω and of the corresponding part of p_ω (i.e., $p^{(1)}_\omega$) is to introduce an absorptivity factor Γ'_ω into the results.] Hence, on $\mathscr{I}^-$ we let

$$\bar{q}_\omega(v) = \begin{cases} \omega^{-1/2}r^{-1}P^-_\omega \exp\left[-i\dfrac{\omega}{\kappa}\ln\left(\dfrac{v-v_0}{CD}\right)\right] & \text{for } v>v_0 \\ 0 \quad \text{for } v<v_0. \end{cases} \tag{26}$$

The bar appears because the p_ω in Eq. (15) have positive Klein-Gordon norm, so that the present $\bar{q}_\omega$ have negative Klein-Gordon norm. The value of $\bar{q}_\omega$ on h can be obtained by propagating Eq. (26) forward by geometrical optics, but we will not need it. As we will show, these wave functions q_ω describe the incoming member of each created pair.

By taking Fourier components with respect to v of $\bar{q}_\omega$ in Eq. (26), one can now find $\bar{\gamma}_{\omega\omega'}$ and compare it with $\beta_{\omega\omega'}$ obtained from Eq. (15). One finds that

$$\beta_{\omega\omega'} = -\exp(-\omega\pi/\kappa)\bar{\gamma}_{\omega\omega'}. \tag{27}$$

We give the derivation of Eq. (27) in the next section, after completing the calculation of the probability amplitudes. It follows from Eqs. (24), (25), and (27) that

$$b_\omega|0\rangle = \exp(-\omega\pi/\kappa)c_\omega^\dagger|0\rangle, \tag{28}$$

where b_ω carries indices lm, while $c_\omega^\dagger$ carries l, $-m$. The b_ω commute with the c_ω and $c_\omega^\dagger$ because of the orthogonality of p_ω with q_ω and $\bar{q}_\omega$ in the Klein-Gordon product. Therefore, we can write Eq. (23) as

$$\begin{aligned}(n, n|0\rangle &= \exp(-n\pi\omega/\kappa)(n!)^{-1}(0|(c_\omega)^n(c_\omega^\dagger)^n|0\rangle \\ &= \exp(-n\pi\omega/\kappa)(n!)^{-1/2}(0, n|(c_\omega^\dagger)^n|0\rangle,\end{aligned} \tag{29}$$

where $(0, n|$ is the state with 0 particles outgoing at $\mathscr{I}^+$ and n particles incoming at h. By repeatedly using the relation

$$(0, n|c_\omega^\dagger = n^{1/2}(0, n-1|, \tag{30}$$

one finally obtains

$$(n, n|0\rangle = \exp(-n\pi\omega/\kappa)(0|0\rangle. \tag{31}$$

This is the most general nonvanishing matrix element of a basis vector (containing only particles in mode ω) of the late-time Fock space with the initial vacuum. The most general such matrix element involves

$$(0|(c_\omega)^n(b_\omega)^m|0\rangle. \tag{32}$$

If $m > n$, then one uses Eq. (28) to replace each b_ω by $c_\omega^\dagger$, and immediately sees that the matrix element vanishes because one is left with $(0|c_\omega^\dagger = 0$. If $m < n$ in Eq. (32), then another relation derived in Sec. III must be used, namely

$$\eta_{\omega\omega'} = -\exp(-\omega\pi/\kappa)\bar{\alpha}_{\omega\omega'}. \tag{33}$$

One then has

$$c_\omega|0\rangle = \exp(-\omega\pi/\kappa)b_\omega^\dagger|0\rangle. \tag{34}$$

We can now move the $(c_\omega)^n$ to the right of $(b_\omega)^m$ in Eq. (32), and replace $(c_\omega)^n|0\rangle$ by $\exp(-n\omega\pi/\kappa)\times(b_\omega^\dagger)^n|0\rangle$. The matrix element (32) then vanishes because one is left with $(0|b_\omega^\dagger = 0$. Hence, for a given mode ω, the only nonvanishing matrix elements in the expansion of $|0\rangle$ are of the form evaluated in Eq. (31), corresponding to the creation of n pairs, with one member outgoing at $\mathscr{I}^+$ (described by wave function p_ω) and the other incoming at h (described by wave function q_ω).

Since the creation and annihilation operators for different ω commute, one finds that the most general nonvanishing matrix element is of the form

$$(\{n_\omega\}, \{n_\omega\}|0\rangle = \prod_\omega \exp(-\pi\omega n_\omega/\kappa)(0|0\rangle, \tag{35}$$

where $(\{n_\omega\}, \{n_\omega\}|$ is the basis state with n_ω particles outgoing at $\mathscr{I}^+$ and n_ω particles incoming at h in mode ω, for a set of various different modes ω (e.g., $\{n_\omega\} = \{n_{\omega_1}, n_{\omega_2}, n_{\omega_3}\}$ with n_{ω_j} representing the number of pairs in mode ω_j).

It is already evident from Eqs. (31) and (35) that one will obtain a blackbody probability distribution for the number of particles in each mode outgoing at $\mathscr{I}^+$, and thus that the emission is that of a blackbody. To find $|(0|0\rangle|^2$, we use Eq. (35) to write

$$1 = \langle 0|0\rangle = |(0|0\rangle|^2 \sum_{\{n_\omega\}} \prod_\omega \exp(-\mu n_\omega \omega), \tag{36}$$

where the sum is over all the sets $\{n_\omega\}$, and

$$\mu = 2\pi/\kappa. \tag{37}$$

The sum and product can be rearranged to give

$$\begin{aligned}1 &= |(0|0\rangle|^2 \prod_\omega \sum_{n=0}^{\infty} \exp(-\mu n\omega) \\ &= |(0|0\rangle|^2 \prod_\omega [1 - \exp(-\mu\omega)]^{-1}.\end{aligned}$$

Hence

$$|(0|0\rangle|^2 = \prod_\omega (1 - e^{-\mu\omega}) = \exp\Big[\sum_\omega \ln(1 - e^{-\mu\omega})\Big]. \tag{38}$$

It follows that

$$|(\{n_\omega\}, \{n_\omega\}|0\rangle|^2 = \prod_\omega \exp(-\mu\omega n_\omega)[1 - \exp(-\mu\omega)]. \tag{39}$$

Therefore, the probabilities for occupation of each mode are independent, and we have

$$P_n(\omega) = \exp(-\mu\omega n)[1 - \exp(-\mu\omega)] \tag{40}$$

for the probability of finding n pairs in mode ω. Since only the outgoing member of each pair is observed at $\mathscr{I}^+$, the outgoing radiation is described by a density matrix or mixture, with $P_n(\omega)$ being the probability of finding n particles outgoing at $\mathscr{I}^+$ in mode ω. This is identical with the density matrix of blackbody radiation of temperature

$$T = (k\mu)^{-1} = \kappa/(2\pi k). \tag{41}$$

The average number of particles outgoing at $\mathscr{I}^+$ in mode ω will then be

$$\langle n_\omega\rangle = \sum_{n=1}^{\infty} nP_n(\omega) = [\exp(+\mu\omega)-1]^{-1}. \tag{42}$$

Because we have neglected backscattering from the exterior Schwarzschild spacetime, the black hole behaves like a perfect blackbody. Wald[5] has considered in detail the case when backscattering is taken into account ($\Gamma_\omega \neq 1$), and has shown how Eq. (40) is modified, and that the black hole behaves like a blackbody when placed in a cavity at the same temperature.[8]

III. RELATIONS AMONG COMPONENTS OF WAVE FUNCTIONS

In this section, we derive Eqs. (27) and (33), which were used in obtaining the blackbody distribution. We also check the consistency of the various orthonormality and commutation relations. As noted in the previous section, we neglect backscattering from the geometry [and drop the superscript (2)].

Taking Fourier components with respect to v on $\mathscr{I}^-$, as in Ref. 1, we have

$$\alpha_{\omega\omega'} = (2\pi)^{-1}\int_{-\infty}^{\infty} dv(\omega')^{1/2}\exp(-i\omega' v) r p_\omega(v), \tag{43}$$

or using Eq. (15)

$$\alpha_{\omega\omega'} = (2\pi)^{-1}(\omega'/\omega)^{1/2}P_\omega^- \times \int_{-\infty}^{v_0} dv\exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0-v}{CD}\right)\right]e^{-i\omega' v}. \tag{44}$$

Replacing v by $(-v+2v_0)$ as variable of integration, one finds

$$\alpha_{\omega\omega'} = (2\pi)^{-1}(\omega'/\omega)^{1/2}P_\omega^- e^{-2i\omega' v_0} \times \int_{v_0}^{\infty} dv\exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v-v_0}{CD}\right)\right]e^{i\omega' v}. \tag{45}$$

By comparison with Eq. (26), we see that the right-hand side of Eq. (45) is proportional to the coefficient of $\bar{f}_{\omega'}$ in the expansion of $\bar{q}_\omega$, that is,

$$\alpha_{\omega\omega'} = \exp(-2i\omega' v_0)\bar{\gamma}_{\omega\omega'}. \tag{46}$$

Equation (44) has been integrated by Hawking (Ref. 1), with the result

$$\alpha_{\omega\omega'} = (CD)^{i\omega/\kappa}P_\omega^-\Gamma\left(1-\frac{i\omega}{\kappa}\right) \times\exp[i(\omega-\omega')v_0]\left(\frac{\omega'}{\omega}\right)^{1/2}(-i\omega')^{-1+i\omega/\kappa}. \tag{47}$$

To analytically continue this expression to negative values of ω' (so as to obtain $\beta_{\omega\omega'}$), we recall that $\alpha_{\omega\omega'}$ is the Fourier transform of a function $p_\omega(v)$ which is zero for $v>v_0$. Consequently, $\alpha_{\omega\omega'}$ is analytic in the upper half ω' plane, and the branch cuts must be chosen in the lower half-plane. One finds

$$(-\omega')^{1/2}(i\omega')^{-1+i\omega/\kappa} = -i(\omega')^{1/2}(-i\omega')^{-1+i\omega/\kappa}\exp(-\omega\pi/\kappa). \tag{48}$$

It follows that

$$\begin{aligned}\beta_{\omega\omega'} &= -i\alpha_{\omega,-\omega'}\\ &= -(CD)^{i\omega/\kappa}P_\omega^-\Gamma\left(1-\frac{i\omega}{\kappa}\right)e^{i(\omega+\omega')v_0} \times\left(\frac{\omega'}{\omega}\right)^{1/2}(-i\omega')^{-1+i\omega/\kappa}e^{-\omega\pi/\kappa}.\end{aligned} \tag{49}$$

Comparing with Eq. (47), we obtain

$$\beta_{\omega\omega'} = -\exp(2i\omega' v_0)\exp(-\omega\pi/\kappa)\alpha_{\omega\omega'}. \tag{50}$$

Finally, substituting Eq. (46) into (50), one obtains

$$\beta_{\omega\omega'} = -\exp(-\omega\pi/\kappa)\bar{\gamma}_{\omega\omega'}, \tag{51}$$

which is the same as Eq. (27) used in Sec. II.

To derive Eq. (33), we make use of Eq. (15) as before to obtain

$$\begin{aligned}\beta_{\omega\omega'} &= (2\pi)^{-1}\left(\frac{\omega'}{\omega}\right)^{1/2}P_\omega^-\int_{-\infty}^{v_c} dv\exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0-v}{CD}\right)\right]\exp(i\omega' v)\\ &= (2\pi)^{-1}\left(\frac{\omega'}{\omega}\right)^{1/2}P_\omega^- e^{2i\omega' v_0}\int_{v_0}^{\infty} dv\exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v-v_0}{CD}\right)\right]\exp(-i\omega' v).\end{aligned} \tag{52}$$

Comparing with Eq. (26) for $\bar{q}_\omega$, we see that

$$\beta_{\omega\omega'} = \exp(2i\omega' v_0)\bar{\eta}_{\omega\omega'}. \tag{53}$$

The using Eq. (50) to replace β by α, one finds

$$\eta_{\omega\omega'} = -\exp(-\omega\pi/\kappa)\bar{\alpha}_{\omega\omega'}, \tag{54}$$

which is the same as Eq. (33) of Sec. II.

Next we check the consistency of the orthonormality and commutation relations within the present approximation of neglecting backscattering. The orthonormality of the q_ω and the commutation relations of the c_ω, $c_\omega^\dagger$ operators require that

$$\delta(\omega_1-\omega_2)=\int_0^\infty d\omega'(\bar{\gamma}_{\omega_1\omega'}\gamma_{\omega_2\omega'}-\eta_{\omega_1\omega'}\eta_{\omega_2\omega'}) \tag{55}$$

and

$$0=\int_0^\infty d\omega'(-\gamma_{\omega_1\omega'}\eta_{\omega_2\omega'}+\eta_{\omega_1\omega'}\gamma_{\omega_2\omega'}). \tag{56}$$

From Eqs. (51), (54), and (50), we find

$$\begin{aligned}\int d\omega'(\bar{\gamma}_{\omega_1\omega'}\gamma_{\omega_2\omega'}-\bar{\eta}_{\omega_1\omega'}\eta_{\omega_2\omega'}) &= \int d\omega'\left(e^{2\omega\pi/\kappa}\beta_{\omega_1\omega'}\bar{\beta}_{\omega_2\omega'}-e^{-2\omega\pi/\kappa}\alpha_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}\right)\\ &=\int d\omega'(\alpha_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}-\beta_{\omega_1\omega'}\bar{\beta}_{\omega_2\omega'}),\end{aligned} \tag{57}$$

and the latter integral gives $\delta(\omega_1-\omega_2)$ in the approximation in which backscattering is negligible. Similarly, Eqs. (51) and (54) imply

$$\begin{aligned}\int d\omega'(-\gamma_{\omega_1\omega'}\eta_{\omega_2\omega'}+\eta_{\omega_1\omega'}\gamma_{\omega_2\omega'}) &= \int d\omega'(-\bar{\beta}_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}+\bar{\alpha}_{\omega_1\omega'}\bar{\beta}_{\omega_2\omega'})\\ &=0.\end{aligned} \tag{58}$$

Finally, the requirement that c_ω commute with both b_ω and $b_\omega^\dagger$ yields the relations

$$0=\int d\omega'(-\bar{\gamma}_{\omega_1\omega'}\bar{\beta}_{\omega_2\omega'}+\bar{\eta}_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}) \tag{59}$$

and

$$0=\int d\omega'(\bar{\gamma}_{\omega_1\omega'}\alpha_{\omega_2\omega'}-\bar{\eta}_{\omega_1\omega'}\beta_{\omega_2\omega'}). \tag{60}$$

In the same way as before, one finds that

$$\begin{aligned}\int d\omega'(-\bar{\gamma}_{\omega_1\omega'}\bar{\beta}_{\omega_2\omega'}+\bar{\eta}_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}) &= \int d\omega'\left(e^{\omega\pi/\kappa}\beta_{\omega_1\omega'}\bar{\beta}_{\omega_2\omega'}-e^{-\omega\pi/\kappa}\alpha_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}\right)\\ &=\int d\omega'\left(e^{-\omega\pi/\kappa}\alpha_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}-e^{-\omega\pi/\kappa}\alpha_{\omega_1\omega'}\bar{\alpha}_{\omega_2\omega'}\right)\\ &=0.\end{aligned} \tag{61}$$

Similarly, one finds that

$$\int d\omega'(\bar{\gamma}_{\omega_1\omega'}\alpha_{\omega_2\omega'}-\bar{\eta}_{\omega_1\omega'}\beta_{\omega_2\omega'})=(e^{2\omega\pi/\kappa}-1)\int d\omega'\beta_{\omega_1\omega'}\beta_{\omega_2\omega'}e^{-2i\omega' v_0}. \tag{62}$$

Substituting $\beta_{\omega\omega'}$ from Eq. (49), we find that

$$\begin{aligned}\int d\omega'(\bar{\gamma}_{\omega_1\omega'}\alpha_{\omega_2\omega'}-\bar{\eta}_{\omega_1\omega'}\beta_{\omega_2\omega'}) &\propto \int_0^\infty d\omega'(\omega')^{-1+2i\omega/\kappa}\\ &\propto\int_{-\infty}^\infty du\,\exp(2i\omega u/\kappa)\\ &\propto\delta(2\omega/\kappa)=0,\end{aligned} \tag{63}$$

since $\omega\neq 0$ (the divergence at $\omega=0$ probably reflects the fact that the approximations used break down for small ω). This completes our check on the consistency of the conclusions reached in Sec. II.

ACKNOWLEDGMENTS

I thank Dr. Lawrence Ford and Dr. Robert Wald for helpful discussions.

[1]S. W. Hawking, Commun. Math. Phys. 43, 199 (1975)

[2]S. W. Hawking, Nature 248, 30 (1974).

[3]Other relevant references are cited in the review article by B. S. DeWitt, Phys. Rep. 19C, 295 (1975).

[4]J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).

[5]R. Wald, Commun. Math. Phys. (to be published). I am grateful to Dr. Wald for informing me of his result prior to his report, particularly with regard to use of the wave function q_ω [Eq. (26) below]. The results presented here were otherwise obtained independently of Dr. Wald's report.

[6]We follow here the somewhat unconventional definition of positive frequency used in Ref. 1 [using $\exp(+i\omega t)$ rather than $\exp(-i\omega t)$ for positive frequency does not alter any results].

[7]Wald (Ref. 5) has shown that the density matrix or operator characterizing the outgoing radiation at $\mathscr{I}^+$ is independent of the particular set of wave functions used to describe the unobserved particles incoming at h.

[8]Since completion of this work I have been informed that the blackbody probability distribution has also been obtained by S. W. Hawking (unpublished). J. D. Bekenstein [Phys. Rev. D (to be published)], using a statistical approach based on information theory, has also obtained the probability distribution, including the effects of backscattering.