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Remarks on possible meson symmetries based on outer product groups

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A previously discussed model of meson symmetries based on the group SU(3)⊗SU(3) is updated by (1) giving rigorous proof of a first-order mass formula obtained previously, (2) deriving a second-order mass formula, and (3) extending the whole treatment to SU(4)⊗SU(4). Tentative assignments are made for the new mesons ψ(3100), ψ(3700), and ψ(4100) in this scheme.

Recent observation of neutral (possibly vector) mesons[1-3] has made it worthwhile to search for suitable symmetries and associated symmetry groups that might be appropriate for mesons. One obvious possibility is to try the group SU(4). However, the acceptability of SU(4) hinges on the existence of a new quantum number, and at the present time there is no conclusive evidence either for or against an additional quantum number for hadrons. In any event it is also apparent that a simple scheme based on SU(4) will not be able to accommodate the three additional mesons that have already been reported. The present experimental situation, although somewhat confusing, leads one to wonder if it is not necessary to consider more general group structures, wherein the new mesons might find a natural place. It is in this spirit that we would like to update and augment a possible symmetry scheme for mesons which was explored in a series a publications quite some time ago.[4-6]

To make this note reasonably self-contained, it is useful to begin with a brief summary of previously established results. The investigation reported in Refs. 4 and 5 arose out of an attempt to understand the nature of the singlet-octet (ω - ϕ) mixing. It was recognized that the existence of a nonet structure implied the possible utility of a group larger than SU(3) being a useful symmetry group for mesons. The group structure considered was the nonchiral SW(3) = SU(3)⊗ SU(3)—the outer product of two SU(3) groups without parity mixing— and the representations advocated for mesons were of the form (d, d^*), d and d^* being *symmetric tensor* representations of the two factor SU(3)'s and d^* being contragredient to d. These representations of SW(3), upon reduction with respect to the "middle" SU(3), consist of a sequence of self-conjugate representations of the latter, with each occurring once, and the sequence starts with an SU(3) singlet. Mathematically, these are the maximal degenerate representations of SW(3). A first-order mass formula was derived, and it was noted that the physical masses satisfy an equal-spacing rule and also display a certain kind of degeneracy with respect to isotopic spin. In particular, for the lowest nontrivial representation $(3, 3^*)$ one recovers *Okubo's* equal-spacing rule as also the degeneracy of the $I=1$, $Y=0$ with one of the physical $I=0$, $Y=0$ masses of a nonet.[7] The properties of the next higher representation $(6, 6^*)$ were worked out in considerable detail in Ref. 5.

The treatment of the SW(3) mass formula detailed in Refs. 4 and 5 suffers from one grave deficiency. In these papers it was assumed that the symmetry-breaking operator is *proportional* to the "hypercharge difference operator" (see below), but the most general form of the symmetry breaking was, in fact, not obtained. In the present note we shall first prove that the form of the mass-splitting operator assumed in Ref. 4 is, indeed, the most general form of the mass-splitting operator to first order in symmetry breaking. Secondly, we will derive a mass formula which is valid to second order in symmetry breaking. Third, we will extend these considerations to the group SW(4) = SU(4) ⊗ SU(4). Indeed, if an additional quantum number is found to exist for hadrons then, from our viewpoint, the symmetry group for mesons must be SU(4) ⊗ SU(4). If no additional quantum number exists then meson symmetry group is SU(3) ⊗ SU(3). Finally, we will make a tentative assignment for the three new mesons.

FIRST-ORDER MASS FORMULA

Let us consider symmetry breaking to first order in SW(3). We require the symmetry breaking to be a vector operator of SW(3). Specifically, consider operators T^{μ}_{ν} and S^{μ}_{ν}. T^{μ}_{ν} is a vector (octet) operator under one of the factor SU(3)'s and a scalar under the other SU(3). S^{μ}_{ν} is a vector operator under the other factor SU(3) and scalar under the remainder. Then the SW(3) mass-splitting operator is a linear combination of $(T^3_3 + S^3_3)$ and $(T^3_3 - S^3_3)$. The structure of an octet operator in an irreducible representation of SU(3) is, of course, given by Okubo's formula.[8] In a recent publication Okubo[9] has given an elegant derivation of a set of algebraic identities satisfied by the infinitesimal generators in U(n). In particular, he proves that for the group SU(3), the most general form of the vector operator T^3_3, in a symmetric tensor representation, is

$$T^3_3 = a + bY, \tag{1}$$

where Y is the hypercharge operator. Utilizing the above we see that the SW(3) mass-splitting operator is a linear combination of $(Y_{(1)} + Y_{(2)})$ and $(Y_{(1)} - Y_{(2)})$, $Y_{(1)}$ and $Y_{(2)}$ being the hypercharge operators belonging to the two factor SU(3)'s. The term proportional to $(Y_{(1)} + Y_{(2)})$ cannot contribute to meson masses. Thus we are left with the final form of the SW(3) mass operator:

$$m = m_o + a\mathcal{Y}, \quad \mathcal{Y} = Y_{(1)} - Y_{(2)}, \tag{2}$$

which is, in fact, the form of the mass operator assumed in Ref. 4. Thus the mass formula derived in Ref. 4 is rigorously true. It has been shown before[4,5] that Eq. (2) leads to an equal-spacing rule for the physical masses. In this context it is also worthwhile to recall[5] that the expectation value of the operator $\mathcal{Y}$ for physical states is proportional to the number of "3" indices that appear in the components of a mixed tensor, which describes the physical meson states. Thus mass formula (2) leads to the same result as in a naive quark model.

Because the first-order mass formula involves only two parameters, it is profitable to consider the second-order mass formula.

SECOND-ORDER MASS FORMULA

Derivation of the second-order mass formula proceeds exactly as in the previous case. The second-order symmetry-breaking operator in SU(3) is the T^{33}_{33} component of a tensor and its structure has been analyzed by Okubo[10]. In SW(3) the second-order operator is therefore a linear combination of two such terms T^{33}_{33} and S^{33}_{33}, one pertaining to each of the two factor SU(3)'s. Applying the identities of Ref. 9 one finds that for symmetric tensor representations

$$\begin{aligned} T^{33}_{33} &= a + bY_{(1)} + cY_{(1)}{}^2 . \\ S^{33}_{33} &= a' + b'Y_{(2)} + c'Y_{(2)}{}^2 . \end{aligned} \tag{3}$$

For the terms quadratic in $Y_{(1)}$ and $Y_{(2)}$, we may consider combinations $(Y_{(1)}{}^2 - Y_{(2)}{}^2)$ and $(Y_{(1)}{}^2 + Y_{(2)}{}^2)$. As before, the term $(Y_{(1)}{}^2 - Y_{(2)}{}^2)$ cannot contribute to meson masses. Thus the desired mass formula, valid to second order in symmetry breaking, is

$$\begin{aligned} m &= m_o + a\mathcal{Y} + b(Y^2 + \mathcal{Y}^2), \\ Y &= Y_{(1)} + Y_{(2)}, \ \mathcal{Y} = Y_{(1)} - Y_{(2)} . \end{aligned} \tag{4}$$

In the above, Y is the hypercharge and $\mathcal{Y}$ the "hypercharge difference" operator. Since we have already seen how to compute the expectation value of $\mathcal{Y}$, calculation of that of $\mathcal{Y}^2$ poses no additional problem. Let us apply Eq. (4) to two cases. For a nonet, the second-order formula (4) leads to exactly the same relations as predicted by the first-order formula, Eq. (2), namely:

$$\begin{aligned} &\tilde{m}(0,0) - m(\tfrac{1}{2}, \pm 1) = m(\tfrac{1}{2}, \pm 1) - m(1,0), \\ &m(1,0) = m(0,0). \end{aligned} \tag{5}$$

In the above $m(1, Y)$ stands for the (mass)^2 of a meson with isospin I and hypercharge Y. The two $I = 0 = Y$ states in a nonet are denoted by $\tilde{m}(0,0)$ $(=\phi)$ and $m(0,0)$ $(=\omega)$. Let us now consider the 36 plet $(6, 6^*)$, which has the SU(3) reduction $1 \oplus 8 \oplus 27$. It is shown in Ref. 4 that the first-order mass formula Eq. (2) leads to the following rules for the 36-plet masses:

$$\begin{aligned} &m(2,0) = m(1,0) = L_1, \\ &\tilde{m}(1,0) = m(1,\pm 2) = L_2, \\ &m(\tfrac{1}{2},\pm 1) = m(\tfrac{3}{2},\pm 1), \\ &L_3 - \tilde{m}(\tfrac{1}{2},\pm 1) = \tilde{m}(\tfrac{1}{2},\pm 1) - L_2 \\ &\qquad = L_2 - m(\tfrac{1}{2},\pm 1) \\ &\qquad = m(\tfrac{1}{2},\pm 1) - L_1, \end{aligned} \tag{6}$$

where the three $I = 0 = Y$ physical states that appear in a 36-plet are denoted by L_1, L_2, and L_3. The second-order mass formula Eq. (4) when applied to the 36-plet yields the following relations:

$$\begin{aligned} &m(2,0) = m(1,0) = L_1, \\ &\tilde{m}(1,0) = L_2, \\ &m(\tfrac{1}{2},\pm 1) = m(\tfrac{3}{2},\pm 1), \\ &L_2 - m(\tfrac{1}{2},\pm 1) = m(\tfrac{1}{2},\pm 1) - L_1, \\ &L_3 - \tilde{m}(\tfrac{1}{2},\pm 1) = \tilde{m}(\tfrac{1}{2}\pm 1) - L_2, \\ &L_3 - m(1,\pm 2) = m(1,\pm 2) - L_1 \end{aligned} \tag{7}$$

Thus the second-order mass formula gives a modified equal-spacing rule in which groups of particles still satisfy an equal-spacing rule amongst themselves, but the spacing is not universal. The degeneracy of the masses with respect to isospin, displayed by Eqs. (5)–(7), is of more general validity. Indeed, this degeneracy persists to all orders in symmetry breaking[5] (neglecting electromagnetic corrections).

MODELS BASED ON NONCHIRAL SU(4) ⊗ SU(4)

As explained earlier, if an additional quantum number is discovered for hadrons, the corresponding outer-product group will be SW(4) = SU(4) ⊗ SU(4). We will hereafter designate this new quantum number by z and call it paracharge. The maximal degenerate representations of SW(4) can be constructed out of the symmetric tensor representation (and its conjugate) of the factor SU(4)'s. The lowest nontrivial representation is $(4, 4^*)$, which has the SU(4) content of $1 \oplus 15$. The next-higher representation is $(10, 10^*)$ with the SU(4) reduction of $1 \oplus 15 \oplus 84$ and so on. Mass formula for SW(4) is derived in a manner analogous to that in SW(3). The structure of the mass-splitting operator in SU(4) has been obtained by Okubo [11] and by the present author. [12] Using these results and the identities in Ref. 9 the SW(4) mass formula can be written down at once. [13] The second-order mass formula for the maximal degenerate representation of SW(4) is

$$m = m_o + a\mathcal{Y} + b\mathfrak{z} + c(Y^2 + \mathcal{Y}^2) + d(z^2 + \mathfrak{z}^2) + e(Yz + \mathcal{Y}\mathfrak{z}) . \tag{8}$$

In the above, $\mathfrak{z}$ is the paracharge difference operator $\mathfrak{z} = z_1 - z_2$, z_1 and z_2 being the paracharge operator of the two commuting factor SU(4)'s. z is the paracharge operator for physical meson states $z = z_1 + z_2$. If we neglect the terms proportional to c, d, and e in Eq. (8), we are left with the first-order mass formula. The expectation value of the operator $\mathfrak{z}$ in physical meson states is equal to the number of index "4" (in other words, the number of "charmed quarks," to use popular terminology) that appear in a mixed tensor, whose components represent the physical meson states. Thus the first-order mass formula yields a two-dimensional equal-spacing rule and the second-order formula a modified equal-spacing rule, exactly as was the case with SW(3). As an example, consider the splitting of the 16-plet $(4, 4^*)$. Setting $c = d = e = o$ in Eq. (8) we obtain the prediction of the first-order splitting for the 16-plet:

$$\begin{aligned} &m'(0,0,0) + m(1,0,0) = 2m(\tfrac{1}{2},1,0) , \\ &m(1,0,0) = m(0,0,0) , \end{aligned} \tag{9a}$$

and

$$\begin{aligned} &m''(0,0,0) + m'(0,0,0) = 2m(0,0,\pm 1) , \\ &m''(0,0,0) + m(1,0,0) = 2m(\tfrac{1}{2},\mp 1,\pm 1) . \end{aligned} \tag{9b}$$

In the above $m(I, Y, z)$ stands for the $(\text{mass})^2$ of a meson with isospin I, hypercharge Y, and paracharge z. The three neutral physical states (i.e., states after taking singlet–15-plet mixing into account) that are present in $(4, 4^*)$ are denoted as $m(0,0,0)$, $m'(0,0,0)$, and $m''(0,0,0)$. Equation (9a) is the usual nonet mass formula. [7] Equation (9b) is the same as a mass formula derived previously by Okubo *et al.*[14] and Gaillard *et al.*,[14] as can be seen by writing it in a slightly different way, namely,

$$\frac{m(\frac{1}{2},\mp 1,\pm 1) - m(1,0,0)}{m(\frac{1}{2},\pm 1,0,) - m(1,0,0)} = \frac{m(0,0,\pm 1) - m(\frac{1}{2},\pm 1,0)}{m(\frac{1}{2},\pm 1,0) - m(1,0,0)} = \frac{1}{2}\frac{m''(0,0,0) - m(1,0,0)}{m(\frac{1}{2},\pm 1,0) - m(1,0,0)} . \tag{10}$$

The full second-order formula Eq. (8) when applied to the $(4, 4^*)$ representation yields two relations, the first of which is still Eq. (9a) and the second of which is

$$m''(0,0,0) - m(\tfrac{1}{2},\mp 1,\pm 1) = m(0,0,\pm 1) - m(\tfrac{1}{2},\pm 1,0) , \tag{11}$$

which is a weaker version of Eq. (10). That the second-order formula leaves the nonet spacing rule (9a) intact is only expected as $(4, 4^*)$ of SW(4) contains $(3, 3^*)$ of SW(3), and we have previously seen that the nonet rule is satisfied to second-order breaking of SW(3). Exactly similar calculations can be done with any other maximal degenerate representation of SW(4).

TENTATIVE ASSIGNMENTS FOR ψ(3100), ψ(3700), AND ψ(4100)

We shall consider possible assignment of the new mesons based on both the groups SW(3) and SW(4). In SW(3) we may assign the three new mesons to a 36-plet representation $(6, 6^*)$. Notice that if the new mesons have $J^P = 1^-$, then we will have two vector-meson multiplets; the 36-plet occurring in addition to the usual nonet of ϕ, ρ, ω, and k^*. The insertion of the masses of mesons $\psi(3100)$, $\psi(3700)$, and $\psi(4100)$ as input then completely fixes the parameters of the second-order mass formula, Eq. (7), and the following prediction results for the remaining states in the 36-plet: three states with $I = 2$, $Y = 0$; $I = 1$, $Y = 0$, and $I = 0$, $Y = 0$ should occur at 3.1 GeV [one of these three is $\psi(3100)$]; two states with $I = 1$, $Y = 0$, and $I = 0$, $Y = 0$ should occur at 3.7 GeV [one of these is $\psi(3700)$]; two states with $I = \frac{3}{2}$, $Y = \pm 1$, and $I = \frac{1}{2}$, $Y = \pm 1$ should occur at 3.41 GeV; a state with $I = 1$, $Y = \pm 2$ should occur at 3.63 GeV; a state with $I = \frac{1}{2}$, $Y = \pm 1$ at 3.9 GeV; and finally a state with $I = 0 = Y$ at 4.1 GeV, which is, of course, $\psi(4100)$. This is our model I.

For models based on SU(4) ⊗ SU(4) one is at first tempted to assign the known vector mesons together with the three new mesons to a 100-plet representation (10, 10 *) of SW(4). But the particle spectrum now is probably too rich. So we do not consider this case any further. A more economic scheme obtains if we assign the $\psi(3100)$ together with the known vector nonet to a 16-plet (4, 4 *) and the $\psi(3700)$, $\psi(4100)$ together with $\rho'(1600)$ and $\omega'(1670)$ to a different 16-plet.[15] This scheme is possible only if all the three new mesons are isoscalar and have $J^P = 1^-$. Then the mass formula Eq. (9) predicts the spectrum of remaining vector mesons: a state with $I=\frac{1}{2}$, $Y=-1$, $z=1$ at 2258 MeV, a state with $I=0$, $Y=0$, $z=1$ at 2308 MeV, a state with $I=\frac{1}{2}$, $Y=1$, $z=0$ ($k^{*\prime}$) at 2850 MeV, a state with $I=\frac{1}{2}$, $Y=-1$, $z=1$ at 3112 MeV, and a state with $I=0$, $Y=0$, $z=1$ at 3905 MeV. Here the last three states belong to 16-plet containing $\psi(3700)$, $\psi(4100)$, $\rho'(1600)$, and $\omega'(1670)$, while the first two states belong to 16-plet containing ρ, ω, ϕ, k^*, and $\psi(3100)$. Of these predictions, that referring to the state $I=\frac{1}{2}$, $Y=1$, $z=0$ is more reliable, as it is valid to second-order symmetry breaking. This is our model II.

A third model is conceivable, where one introduces a 16 plet of SW(4) containing $\psi(3100)$, $\psi(3700)$, and $\psi(4100)$ in addition to the known nonet [of SW(3)] of vector mesons. The masses of the remaining members of 16-plet can be easily calculated from Eq. (9).

REMARKS

(1) In the present scheme the physical meson states are denoted by a nontraceless mixed tensor $G^{\mu\lambda\cdots}_{\nu\sigma\cdots}$ which possesses complete symmetry with respect to permutation of upper indices and of lower indices. The upper and lower indices are identified with the tensor indices corresponding to the two factor groups in the outer product. Thus G^{μ}_{ν} and $G^{\mu\lambda}_{\nu\sigma}$ ($\mu,\nu,\lambda,\sigma=1,2,3$) denote the SW(3) nonet and 36-plet, respectively, and G^{μ}_{ν} ($\mu,\nu=1,2,3,4$) denote the 16-plet of SW(4). Contact with the conventional quark picture is established if we identify the upper (lower) indices with quark (antiquark) indices.[5] Thus the group-theoretic content of the statement "meson is a bound state of quarks and antiquarks" is expressed rather naturally in the present scheme.

(2) In SW(3), the $J^P=\frac{1}{2}^+$ baryons are assigned[5] to the representation (8, 1). In SW(4), the $J^P=\frac{1}{2}^+$ baryons are assigned to the representation (20′, 1) and $J^P=\frac{3}{2}^+$ baryons to (20, 1). For baryon mass relations then nothing further follows beyond the predictions of SU(4).

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