Current-algebra theorems for $K\pi$ scattering*

D. H. Boal, R. H. Graham, and B. Weisman
Department of Physics, University of Toronto, Toronto, Ontario, Canada
(Received 30 December 1974; revised manuscript received 18 June 1975)

On-mass-shell current-algebra sum rules for $K\pi$ scattering are investigated. At threshold these sum rules relate the combinations $a^{1/2}-a^{3/2}$ and $a^{1/2}+2a^{3/2}$ of $I=\frac{1}{2}$ and $\frac{3}{2}$ S-wave scattering lengths to their soft-meson predictions plus "correction" terms. A dispersion-theoretic approach is used to calculate the corrections to both the soft-koan and soft-pion values for each of these scattering length combinations. The dispersion relations are assumed to be dominated by the $K^*(890)$ and a $J^P=0^+$ κ meson. The correction to the soft-pion value $a^{1/2}-a^{3/2}=0.210\,m_\pi^{-1}$ is found to be very small, thus suggesting that the on-mass-shell current-algebra prediction for $a^{1/2}-a^{3/2}$ is nearly equal to the soft-pion value. Self-consistency requirements of the analysis favor $F_K/F_\pi\simeq1.22$; this implies that a fairly large ($\sim30\%$) correction to the corresponding soft-koan prediction of $a^{1/2}-a^{3/2}\approx0.14\,m_\pi^{-1}$ is needed. The K^* and κ contributions to the latter correction increase this value to $0.177\,m_\pi^{-1}$. Furthermore, since the K^* and κ corrections to and the soft-koan value of $a^{1/2}-a^{3/2}$ decrease with increasing F_K/F_π, a value of $F_K/F_\pi\gtrsim1.3$ seems, at least in the context of the present study, to be rather unlikely.

I. INTRODUCTION

Current-algebra low-energy theorems[1] have proved to be an important tool in probing the structure of elementary particle interactions. Since the relations which result from these low-energy theorems involve amplitudes in which one or more pseudoscalar meson mass has been extrapolated to zero, it becomes necessary to determine the error made when testing them with experimental data (to which, of course, only on-mass-shell amplitudes can be directly related). There have been various estimates of these errors in a number of different applications of the soft-meson current-algebra approach. For the most part these attempts have made use of either kinematic correction factors[2,3] or dispersion relations in the pseudoscalar meson mass.[4]

Recently, however, an on-mass-shell approach to current-algebra sum rules was suggested.[5] The current-algebra theorems are written with all amplitudes on-mass-shell, the price being that terms which vanish in the soft-meson limit now contribute and must be evaluated. These "correction" terms give a measure of the error referred to above.

In their application to πN scattering, Brown *et al.*[5] evaluated the corrections to the Adler-Weisberger[6] and Cheng-Dashen[7] relations and found them to be small, as expected. In a subsequent application to $\pi\pi$ scattering[8] it was found that the corrections to the Adler sum rule[3,9] could be as large as ~25%, but were likely to be smaller. The magnitude and sign of the correction terms lead to upper and lower bounds on the combination $2a^0-5a^2$ of $I=0$ and 2 S-wave scattering lengths. Although on less secure theoretical grounds, the corrected sum rule for the $I=0$ σ term gave rise to results which are consistent with the model of Gell-Mann, Oakes, and Renner (GMOR).[10]

The purpose of the present investigation is to examine the on-mass-shell current-algebra sum rules for $K\pi$ scattering.[11-14] Whereas the earlier on-mass-shell applications provided information on pion mass extrapolation effects, the study of the $K\pi$ sum rules affords an opportunity not only to estimate the corrections to the soft-kaon sum rules, but to compare them with the corresponding corrections to the soft-pion relations *in the same process*. It is possible that this gives a better idea of the relative reliabilities of soft-kaon and soft-pion calculations than can be obtained from a comparison of, say, on-mass-shell current-algebra approaches to πN (see Ref. 15) and KN (see Ref. 16) scattering.

There are two current-algebra sum rules for $K\pi$ scattering; these correspond to the t-channel isospins $I_t=0$ and 1. The on-mass-shell $I_t=1$ relation leads to a corrected Weinberg-Tomozawa sum rule[12,13] for the combination $a^{1/2}-a^{3/2}$ of $I=\frac{1}{2}$ and $\frac{3}{2}$ S-wave scattering lengths. The combination $a^{1/2}+2a^{3/2}$ is related to a σ term in the on-mass-shell version of the $I_t=0$ sum rule. We will express each of these sum rules in a reduced-π form[17] and a reduced-K form, in order to determine the respective soft-pion and soft-kaon corrections.

The present analysis will parallel the earlier study[8] of $\pi\pi$ sum rules. As before we will employ a dispersive approach to calculate the correction terms. They are first expanded, completely generally, in terms of invariant amplitudes. Unsubtracted dispersion relations[18] are then written for these invariant amplitudes. The dispersion relations are assumed to be saturated by the most important low-lying resonances, in this case the $K^*(890)$ and an $I=\frac{1}{2}$ scalar meson (κ). The contributions of these resonances are determined, as previously, from hard-meson current-algebra calculations[19-23] of the relevant three-point functions. In the present work we will make use of the analysis of Fenster and Hussain.[21,24]

In order for the analytic structure of the sum rules to be correct, the residues of the K^* and κ poles in the correction terms must equal their counterparts in the $K\pi$ scattering amplitude. This requirement provides us with conditions on the parameters. Although we demanded exact equality in the earlier $\pi\pi$ analysis, we felt that it would not be worthwhile to attempt an exact matching of the residues in the present case (although it would have been possible to do so), in view of the uncertainties in several of the parameters. Nevertheless, we did find a solution for which the residues of both K^* poles together with those of both κ poles are very nearly equal. These consistency conditions prove useful in narrowing the range of uncertainty in several of the parameters.

One of the main results of the present calculation is that the K^* and κ corrections to the soft-π sum rule for $a^{1/2}-a^{3/2}$ are both very small for all κ parameters in a plausible, physical range.[25] It is reasonable to suppose that higher-mass contributions to the reduced-π correction terms are also small. Thus, we conclude that the on-mass-shell current-algebra prediction for $a^{1/2}-a^{3/2}$ is nearly equal to its soft-pion limit[12-24] of $\approx 0.21 m_\pi^{-1}$.

Of course, if we know the on-mass-shell value of $a^{1/2}-a^{3/2}$, then we can accurately judge the success of the reduced-K estimates. Now, the residue conditions mentioned above favor the value 1.22 for the ratio of kaon to pion decay constants, F_K/F_π. This leads to a soft-kaon value for $a^{1/2}-a^{3/2}$ of $0.141 m_\pi^{-1}$, from which we infer that the effects of extrapolation in the kaon mass, in this case, are of the order of ~30%. The K^* and κ contributions to the reduced-K correction terms have opposite signs, the K^* contribution being $\sim -0.021 m_\pi^{-1}$ and that of the $\kappa \sim +0.057 m_\pi^{-1}$. Thus, for the best values of the parameters the corrected value of $a^{1/2}-a^{3/2}$ in the reduced-K case is raised from the soft-K value to $\sim 0.177 m_\pi^{-1}$.

An interesting by-product is obtained from the present analysis. Namely, it appears highly unlikely that the ratio F_K/F_π can be greater than 1.3. This result is a consequence of the above consistency requirements on the K^* and κ pole residues.

For the $a^{1/2}+2a^{3/2}$ sum rule we again find small corrections in the reduced-π case, leading to a value for this combination which is close to zero, when use is made of the GMOR value[10] of the σ term. The corresponding reduced-K prediction for $a^{1/2}+2a^{3/2}$ is also very small and agrees well with the reduced-π result.

Previous hard-meson current-algebra studies of $K\pi$ scattering have made use of a number of different approaches. These include simple linear mass extrapolation[14] from the soft-meson point to threshold, more sophisticated mass extrapolation[26] based on the Fubini-Furlan technique,[4] κ dominance[27] of $\langle \pi | \partial^\mu V_\mu^{\Delta S=1}(0) | K\rangle$ [where $V_\mu^{\Delta S=1}(x)$ is the strangeness-changing vector current], and Lagrangian calculations in the tree approximation.[28,29] Most of these calculations have yielded predictions for $a^{1/2}-a^{3/2}$ which are close to the soft-pion value. However, it seems to us that the present analysis justifies the conclusion that the effects of pion mass extrapolation on the determination of $a^{1/2}-a^{3/2}$ are minimal. In addition, the present investigation has dealt more fully with kaon mass extrapolation effects than have the earlier studies.

This paper is organized as follows: In the first part of Sec. II we will develop the formalism required for the reduced-K analysis of the $K\pi$ low-energy theorems. At the end of this section we then simply list the analogous reduced-π expressions, whose derivation parallels that of the reduced-K case. Our results together with details of the numerical analysis will be presented in Sec. III. Section IV will contain our conclusions. We have reserved for the Appendixes a more complete account of the ingredients required to calculate the correction terms. In Appendix A we extend the hard-meson three-point function analysis of Fenster and Hussain to include the κ meson, while in Appendix B the explicit forms of the K^* and κ contributions to the correction terms are given.

II. THE ON-MASS-SHELL SUM RULES

As was stated in the Introduction, there are two methods of calculation that can be used here: Either the pions or kaons can be reduced out of the state vectors. We will follow the reduced-K calculations here in some detail, and then, at the end of this section, we will simply list the results for the reduced-π case.

Reduced K

In the process[30]

$$K_a(q)+\pi_c(p_\alpha)\to K_b(p)+\pi_d(p_\beta), \tag{1}$$

we can use partial conservation of the axial-vector current (PCAC) in the form[31]

$$\partial_\nu A_a^\nu(x) = F_K m_K{}^2 \phi_a(x) \quad (a=4,\ldots,7) \tag{2}$$

to write the on-mass-shell S-matrix element for (1) as follows[32]:

$$S_{bd,ac} = -(2\pi)^{-3}(2p^0 2q^0)^{-(1/2)} \int d^4x\, d^4y\, e^{iqx} e^{-ipy} \langle \pi_d(p_\beta)|\{F_K{}^{-2} p_\mu q_\nu T(\tilde{A}_b^\mu(y)\tilde{A}_a^\nu(x)) + F_K{}^{-2}\delta(y^0-x^0)[A_b^0(y), \partial_\nu A_a^\nu(x)] + iq_\nu F_K{}^{-2}\delta(y^0-x^0)[A_b^0(y), A_a^\nu(x)]\}|\pi_c(p_\alpha)\rangle . \tag{3}$$

In this expression, the operator $\tilde{A}$ is obtained from A by the explicit removal of the pseudoscalar pole, that is,[33]

$$A_K^\mu = \tilde{A}_K^\mu + F_K \partial^\mu \phi_K . \tag{4}$$

Defining the equal-time commutation relations

$$\begin{aligned} \delta(y^0-x^0)[A_b^0(y), A_a^\nu(x)] &= i f_{bac} V_c^\nu(y)\delta^4(x-y) , \\ \delta(y^0-x^0)[A_b^0(y), \partial_\nu A_a^\nu(x)] &= i\sigma_{ab}(y)\delta^4(x-y) , \end{aligned} \tag{5}$$

the expression for the S-matrix element can be rewritten as

$$S_{bd,ac} = -\frac{i(2\pi)^4\delta(q+p_\alpha-p-p_\beta)}{(2\pi)^6(16p^0q^0p_\alpha^0p_\beta^0)^{1/2}} M_{bd,ac}(\nu,\nu_B) , \tag{6}$$

where the matrix element M is given by

$$F_K{}^2 M_{bd,ac}(\nu,\nu_B) \equiv p_\mu q_\lambda R^{\mu\lambda}_{bd,ac}(\nu,\nu_B) + \sigma_{ab,cd}(t) - 2m_K \nu f_{bae} f_{dce} F(t) , \tag{7}$$

with

$$\sigma_{ab,cd}(t) \equiv (2\pi)^3(2p_\alpha^0 2p_\beta^0)^{1/2}\langle \pi_d(p_\beta)|\sigma_{ab}(0)|\pi_c(p_\alpha)\rangle \tag{8}$$

and

$$i\frac{(2\pi)^4\delta(q+p_\alpha-p-p_\beta)}{(2\pi)^3(2p_\alpha^0 2p_\beta^0)^{1/2}} R^{\mu\nu}_{bd,ac}(\nu,\nu_B) \equiv \int d^4x\, d^4y\, e^{-ipy} e^{iqx}\langle \pi_d(p_\beta)|T(\tilde{A}_b^\mu(y)\tilde{A}_a^\nu(x))|\pi_c(p_\alpha)\rangle . \tag{9}$$

$F(t)$ is the pion electromagnetic form factor. In the above, we have introduced the invariants

$$\begin{aligned} \nu &= \frac{-p\cdot(p_\alpha+p_\beta)}{2m_K} = \frac{s-u}{4m_K} , \\ \nu_B &= \frac{p\cdot q}{2m_K} = \frac{t-2m_K{}^2}{4m_K} , \end{aligned} \tag{10}$$

where

$$s = -(q+p_\alpha)^2 ,$$
$$t = -(p-q)^2 , \tag{11}$$

and

$$u = -(q-p_\beta)^2$$

are the usual Mandelstam variables.

As is well known, there are two independent isospin amplitudes for $K\pi$ scattering. We can write

$$M_{bd,ac}(\nu,\nu_B) = \delta_{cd}\delta_{ab} M^{(+)}(\nu,\nu_B) + \tfrac{1}{2}[\tau_d,\tau_c]_{ba} M^{(-)}(\nu,\nu_B) . \tag{12}$$

Similar isospin decompositions can be made for the other matrix elements appearing in Eq. (7). Projecting out the even- and odd-isospin amplitudes from Eq. (7) leads to

$$F_K{}^2 M^{(+)}(\nu,\nu_B) = p^\mu q^\lambda R^{(+)}_{\mu\lambda}(\nu,\nu_B) + \sigma(t) , \tag{13}$$

$$F_K{}^2 M^{(-)}(\nu,\nu_B) = p^\mu q^\lambda R^{(-)}_{\mu\lambda}(\nu,\nu_B) - m_K \nu F(t) . \tag{14}$$

The correction terms

$$R^{(\pm)}(\nu,\nu_B) \equiv p^\mu q^\lambda R^{(\pm)}_{\mu\lambda}(\nu,\nu_B) \tag{15}$$

will be determined in a manner analogous to that used in the $\pi\pi$ case.[8] We begin by expanding $R^{(\pm)}_{\mu\lambda}(\nu,\nu_B)$ in terms of the particle momenta. Making use of PT invariance we can write[8]

$$\begin{aligned} R^{(\pm)}_{\mu\lambda} &= A^{(\pm)}P_\mu P_\lambda + B_1^{(\pm)}(P_\mu Q_\lambda + Q_\mu P_\lambda) \\ &\quad + B_2^{(\pm)}(P_\mu\Delta_\lambda - \Delta_\mu P_\lambda) + C_1^{(\pm)}Q_\mu Q_\lambda \\ &\quad + C_2^{(\pm)}(Q_\mu\Delta_\lambda - \Delta_\mu Q_\lambda) + C_3^{(\pm)}\Delta_\mu\Delta_\lambda + C_4^{(\pm)}g_{\mu\lambda} , \end{aligned} \tag{16}$$

where we have defined the combinations

$$P = \tfrac{1}{2}(p_\alpha + p_\beta) ,$$

$$Q = \tfrac{1}{2}(p+q) ,$$

and

$$\Delta = q - p .$$

Using the relation

$$R^{(\pm)}_{\mu\nu}(\nu,\nu_B) = \pm R^{(\pm)}_{\nu\mu}(-\nu,\nu_B) ,$$

one finds that ($i=1,2$; $j=1,\ldots,4$)

$A^{(+)}$, $B_i^{(-)}$, and $C_j^{(+)}$ are even in ν;

$A^{(-)}$, $B_i^{(+)}$, and $C_j^{(-)}$ are odd in ν.

From Eqs. (15) and (16) we find that

$$R^{(\pm)}(\nu,\nu_B)=m_K{}^2\nu^2A^{(\pm)}(\nu,\nu_B)+\nu m_K(m_K{}^2-2m_K\nu_B)B_1^{(\pm)}(\nu,\nu_B)$$
$$+2\nu m_K(m_K{}^2+2m_K\nu_B)B_2^{(\pm)}(\nu,\nu_B)+\tfrac{1}{4}(m_K{}^2-2m_K\nu_B)^2C_1^{(\pm)}(\nu,\nu_B)$$
$$+m_K{}^2(m_K{}^2-4\nu_B{}^2)C_2^{(\pm)}(\nu,\nu_B)-(2m_K\nu_B+m_K{}^2)^2C_3^{(\pm)}(\nu,\nu_B)+2m_K\nu_BC_4^{(\pm)}(\nu,\nu_B)\,. \tag{17}$$

In order to determine the invariant amplitudes we assume that they satisfy unsubtracted fixed-t dispersion relations. Thus, denoting the absorptive parts of $A,\ldots,C_4$ by $a,\ldots,c_4$, we write

$$A^{(\pm)}(\nu,\nu_B)=\frac{1}{\pi}\int_0^\infty d\nu' a^{(\pm)}(\nu',\nu_B)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right), \tag{18}$$

$$B_i^{(\pm)}(\nu,\nu_B)=\frac{1}{\pi}\int_0^\infty d\nu' b_i^{(\pm)}(\nu',\nu_B)\left(\frac{1}{\nu'-\nu}\mp\frac{1}{\nu'+\nu}\right)\quad(i=1,2)\,, \tag{19}$$

$$C_j^{(\pm)}(\nu,\nu_B)=\frac{1}{\pi}\int_0^\infty d\nu' c_j^{(\pm)}(\nu',\nu_B)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right)\quad(j=1,\ldots,4)\,, \tag{20}$$

where we have made use of the crossing properties given above. Although we have assumed unsubtracted dispersion relations for all of the invariant amplitudes, this is probably only rigorously justifiable[34] for the amplitudes $A^{(\pm)}$, $B_i^{(-)}$, and $C_j^{(-)}$.

Denoting the absorptive part of R by r one has

$$r^{\mu\lambda}_{bd,ac}=-\left[\pi(2\pi)^6\sum_k(2p^0_\alpha 2p^0_\beta 2k^0 2k^0)^{1/2}\langle\pi_d(p_\beta)|\tilde{A}^\mu_b(0)|k\rangle\langle k|\tilde{A}^\lambda_a(0)|\pi_c(p_\alpha)\rangle\delta(s+k^2)\right]$$
$$+\left[\pi(2\pi)^6\sum_{k'}(2p^0_\alpha 2p^0_\beta 2k'^0 2k'^0)^{1/2}\langle\pi_d(p_\beta)|\tilde{A}^\lambda_a(0)|k'\rangle\langle k'|\tilde{A}^\mu_b(0)|\pi_c(p_\alpha)\rangle\delta(u+k'^2)\right]. \tag{21}$$

We assume that r is saturated by the $K^*(890)$ and an $I=\frac{1}{2}$ scalar meson, κ. The matrix elements contributing to Eq. (21) in this approximation are evaluated in Appendix A. The absorptive parts $a^{(\pm)},\ldots,c_4^{(\pm)}$ are obtained from the expansion for $r_{\mu\nu}^{(\pm)}$ which corresponds to Eq. (16). The resultant K^* and κ contributions to the amplitudes $A^{(\pm)},\ldots,C_4^{(\pm)}$ are given in Appendix B.

Once expressions for $R^{(\pm)}(\nu,\nu_B)$ have been obtained from Eq. (17), they may be substituted in Eqs. (13) and (14). This, however, still leaves $F(t)$ and $\sigma^{(+)}_{ab,cd}$ to be evaluated, before the full expression for $M^{(\pm)}$ may be obtained. Since this cannot be done within our theoretical framework, we choose rather to evaluate the expressions at $t=0$ (i.e., $\nu_B=-m_K/2$). We may thus use the fact that $F(0)=1$ to help determine $M^{(-)}$ at $t=0$. The GMOR model[10] will be used to evaluate $\sigma^{(+)}(0)$.

The $K\pi$ S-wave scattering lengths are most directly obtained by taking $\nu=m_\pi$, $t=0$ (threshold) in Eqs. (13) and (14). The threshold value of the amplitude with definite isospin is related to the corresponding S-wave scattering length by

$$a^I=\frac{-1}{8\pi(m_\pi+m_K)}M^I(\nu=m_\pi,\,t=0)\,, \tag{22}$$

where I is the s-channel isotopic spin. From the relations

$$M^{(-)}=\tfrac{1}{3}(M^{1/2}-M^{3/2})\,,\qquad M^{(+)}=\tfrac{1}{3}(M^{1/2}+2M^{3/2})\,, \tag{23}$$

and Eq. (17) one can now evaluate Eqs. (13) and (14) at threshold and $t=0$ to obtain

$$a^{1/2}-a^{3/2}=\frac{-3}{8\pi F_K{}^2(m_\pi+m_K)}\left[m_\pi{}^2m_K{}^2A^{(-)}\left(m_\pi,-\frac{m_K}{2}\right)+2m_\pi m_K{}^3B_1^{(-)}\left(m_\pi,-\frac{m_K}{2}\right)\right.$$
$$\left.+m_K{}^4C_1^{(-)}\left(m_\pi,-\frac{m_K}{2}\right)-m_K{}^2C_4^{(-)}\left(m_\pi,-\frac{m_K}{2}\right)-m_\pi m_K\right], \tag{24}$$

$$a^{1/2}+2a^{3/2}=\frac{-3}{8\pi F_K{}^2(m_\pi+m_K)}\left[m_\pi{}^2m_K{}^2A^{(+)}\left(m_\pi,-\frac{m_K}{2}\right)+2m_\pi m_K{}^3B_1^{(+)}\left(m_\pi,-\frac{m_K}{2}\right)\right.$$
$$\left.+m_K{}^4C_1^{(+)}\left(m_\pi,-\frac{m_K}{2}\right)-m_K{}^2C_4^{(+)}\left(m_\pi,-\frac{m_K}{2}\right)+\sigma(0)\right]. \tag{25}$$

We will refer to Eqs. (24) and (25) as threshold sum rules.

For convenience, we define

$$D(\nu, \nu_B) = 3M^{(-)}(\nu, \nu_B)\,. \tag{26}$$

Then

$$D(\nu, \nu_B) = \frac{2\nu}{\pi}\int_{m_\pi}^{\infty} \frac{d\nu' \,\mathrm{Im}D(\nu', \nu_B)}{\nu'^2 - \nu^2}. \tag{27}$$

If we use the optical theorem in the form

$$\sigma^I_{\text{tot}} = \frac{-1}{2q_c\sqrt{s}}\mathrm{Im}M^I(t=0)\,, \tag{28}$$

where q_c is the center-of-mass momentum, then at threshold we have

$$D\left(m_\pi, -\frac{m_K}{2}\right) = \frac{2m_\pi}{\pi}\int_{m_\pi}^{\infty} \frac{d\nu'(-2q_c\sqrt{s})[\sigma^{1/2}_{\text{tot}}(\nu') - \sigma^{3/2}_{\text{tot}}(\nu')]}{\nu'^2 - m_\pi^2}. \tag{29}$$

Introducing p^L_π as the laboratory momentum of the pion (assuming the K is stationary), then

$$\sqrt{s}\,q_c = m_K p^L_\pi\,. \tag{30}$$

Further, letting $\omega^2 \equiv p^{L2}_\pi + m_\pi^2$ ($\omega = \nu$ at $t=0$) and inserting the definition for the scattering lengths, we obtain

$$a^{1/2} - a^{3/2} = \frac{m_\pi}{2\pi^2(m_\pi + m_K)}\int \frac{m_K dp^L_\pi}{\omega}(\sigma^{1/2}_{\text{tot}} - \sigma^{3/2}_{\text{tot}})\,. \tag{31}$$

This expression for the scattering lengths will be evaluated later as a consistency check on the theory.

A second approach to Eq. (14) is possible, which involves taking the derivative of both sides of that equation with respect to ν and then setting $\nu = 0$. Let us first concentrate on $(\partial/\partial\nu)M^{(-)}(\nu, \nu_B)|_{\nu=0}$. From Eqs. (26) and (27) one finds

$$F_K{}^2\frac{\partial}{\partial\nu}M^{(-)}\left(\nu, -\frac{m_K}{2}\right)\bigg|_{\nu=0} = \frac{2F_K{}^2}{\pi}\int_{m_\pi}^{\infty}\frac{d\nu'\,\mathrm{Im}M^{(-)}(\nu', -m_K/2)}{\nu'^2}. \tag{32}$$

Applying the optical theorem, this equation becomes

$$F_K{}^2\frac{\partial}{\partial\nu}M^{(-)}\left(\nu, -\frac{m_K}{2}\right)\bigg|_{\nu=0} = -\frac{4F_K{}^2}{3\pi}\int\frac{m_K dp^L_\pi(\sigma^{1/2}_{\text{tot}} - \sigma^{3/2}_{\text{tot}})}{\omega} + \frac{4F_K{}^2 m_\pi{}^2 m_K}{3\pi}\int\frac{dp^L_\pi(\sigma^{1/2}_{\text{tot}} - \sigma^{3/2}_{\text{tot}})}{\omega^3}. \tag{33}$$

The second integral on the right-hand side is negligible relative to the first, owing to the fact that it is much more rapidly convergent.[35] Thus, the second term is dropped, and a comparison of Eqs. (14), (31), and (33) leads to

$$a^{1/2} - a^{3/2} = -\left(\frac{3m_\pi}{8\pi F_K{}^2(m_\pi + m_K)}\right)\left[\frac{\partial}{\partial\nu}R^{(-)}\left(\nu, -\frac{m_K}{2}\right)\bigg|_{\nu=0} - m_K\right]. \tag{34}$$

Using Eq. (17), the above relation becomes

$$-\frac{2F_K{}^2}{3\pi}\frac{4\pi^2(m_\pi + m_K)}{m_\pi}(a^{1/2} - a^{3/2}) = 2m_K{}^3 B_1^{(-)}\left(0, -\frac{m_K}{2}\right) + m_K{}^4\frac{\partial}{\partial\nu}C_1^{(-)}\left(\nu, -\frac{m_K}{2}\right)\bigg|_{\nu=0} - m_K{}^2\frac{\partial}{\partial\nu}C_4^{(-)}\left(\nu, -\frac{m_K}{2}\right)\bigg|_{\nu=0} - m_K\,. \tag{35}$$

Equation (35) will be called a $\nu = 0$ sum rule.

Reduced π

The equations and results for the reduced-π case are very similar to those for the reduced-K case except for a few changes in notation. The momenta and isospin are assigned according to

$$\pi_a(q) + K_c(p_\alpha) \to \pi_b(p) + K_d(p_\beta)\,. \tag{36}$$

Using PCAC in the form

$$\partial_\nu A^\nu_a(x) = F_\pi m_\pi{}^2\phi_a(x) \quad (a = 1, 2, 3)\,, \tag{37}$$

the S-matrix element becomes

$$S_{bd,ac} = \frac{-i(2\pi)^4\delta(q + p_\alpha - p - p_\beta)}{(2\pi)^6(16p^0q^0p^0_\alpha p^0_\beta)^{1/2}}M'_{bd,ac}(\nu, \nu'_B)\,, \tag{38}$$

where

$$F_\pi^2 M'_{bd,ac}(\nu, \nu'_B) = p_\mu q_\lambda R'^{\mu\lambda}_{bd,ac}(\nu, \nu'_B) + \sigma'_{ab,cd}(t) - 2m_K \nu f_{bae} f_{dce} G(t), \quad (39)$$

and $G(t)$ is the kaon isovector form factor, defined by

$$(2\pi)^3 (4p_0 p'_0)^{1/2} \langle K_b(p') | V^\mu_c(0) | K_a(p) \rangle = f_{abc} G(t)(p+p')^\mu . \quad (40)$$

In the above, ν retains the same definition as in the reduced-K case, but ν'_B is defined as

$$\nu'_B = \frac{p \cdot q}{2m_\pi}. \quad (41)$$

We have used primes to distinguish the reduced-π quantities from those corresponding to the reduced-K case.

The form of the low-energy theorems in the reduced-π case can be obtained from Eqs. (13) and (14) by letting $F_K \to F_\pi$, and so we have

$$F_\pi^2 M'^{(+)}(\nu, \nu'_B) = p^\mu q^\lambda R'^{(+)}_{\mu\lambda}(\nu, \nu'_B) + \sigma'(t), \quad (42)$$

$$F_\pi^2 M'^{(-)}(\nu, \nu'_B) = p^\mu q^\lambda R'^{(-)}_{\mu\lambda}(\nu, \nu'_B) - m_K \nu G(t). \quad (43)$$

$R'_{\mu\nu}$ is expanded in terms of invariant amplitudes $A', \ldots, C'_4$, which are assumed to satisfy unsubtracted dispersion relations. R' can be expressed as

$$\begin{aligned} p^\mu q^\lambda R'^{(\pm)}_{\mu\lambda}(\nu, \nu'_B) = {} & m_K^2 \nu^2 A'^{(\pm)}(\nu, \nu'_B) + \nu m_K (m_\pi^2 - 2m_\pi \nu'_B) B'^{(\pm)}_1(\nu, \nu'_B) + 2\nu m_K (m_\pi^2 + 2m_\pi \nu'_B) B'^{(\pm)}_2(\nu, \nu'_B) \\ & + \tfrac{1}{4}(m_\pi^2 - 2m_\pi \nu'_B)^2 C'^{(\pm)}_1(\nu, \nu'_B) + m_\pi^2 (m_\pi^2 - 4\nu'^2_B) C'^{(\pm)}_2(\nu, \nu'_B) - (m_\pi^2 + 2m_\pi \nu'_B)^2 C'^{(\pm)}_3(\nu, \nu'_B) \\ & + 2m_\pi \nu'_B C'^{(\pm)}_4(\nu, \nu'_B). \end{aligned} \quad (44)$$

The absorptive part of R', denoted by r', is

$$\begin{aligned} r'^{\mu\lambda}_{bd,ac}(\nu, \nu'_B) = {} & -\left[\pi(2\pi)^6 \sum_k (2p^0_\alpha 2p^0_\beta 2k^0 2k^0)^{1/2} \langle K_d(p_\beta) | \tilde{A}^\mu_b(0) | k \rangle \langle k | \tilde{A}^\lambda_a(0) | K_c(p_\alpha) \rangle \delta(s+k^2) \right] \\ & + \left[\pi(2\pi)^6 \sum_{k'} (2p^0_\alpha 2p^0_\beta 2k'^0 2k'^0)^{1/2} \langle K_d(p_\beta) | \tilde{A}^\lambda_a(0) | k' \rangle \langle k' | \tilde{A}^\mu_b(0) | K_c(p_\alpha) \rangle \delta(u+k'^2) \right]. \end{aligned} \quad (45)$$

Repeating the arguments of the reduced-K case, we easily find the threshold sum rules to be

$$\begin{aligned} a^{1/2} - a^{3/2} = \frac{-3}{8\pi F_\pi^2 (m_\pi + m_K)} \Big[& m_\pi^2 m_K^2 A'^{(-)}\left(m_\pi, -\frac{m_\pi}{2}\right) + 2m_\pi^3 m_K B'^{(-)}_1\left(m_\pi, -\frac{m_\pi}{2}\right) \\ & + m_\pi^4 C'^{(-)}_1\left(m_\pi, -\frac{m_\pi}{2}\right) - m_\pi^2 C'^{(-)}_4\left(m_\pi, -\frac{m_\pi}{2}\right) - m_\pi m_K \Big], \end{aligned} \quad (46)$$

$$\begin{aligned} a^{1/2} + 2a^{3/2} = \frac{-3}{8\pi F_\pi^2 (m_\pi + m_K)} \Big[& m_\pi^2 m_K^2 A'^{(+)}\left(m_\pi, -\frac{m_\pi}{2}\right) + 2m_\pi^3 m_K B'^{(+)}_1\left(m_\pi, -\frac{m_\pi}{2}\right) \\ & + m_\pi^4 C'^{(+)}_1\left(m_\pi, -\frac{m_\pi}{2}\right) - m_\pi^2 C'^{(+)}_4\left(m_\pi, -\frac{m_\pi}{2}\right) + \sigma'(0) \Big]. \end{aligned} \quad (47)$$

The $\nu = 0$ sum rule analog of Eq. (35) is found to be

$$\begin{aligned} a^{1/2} - a^{3/2} = -\left(\frac{m_\pi}{m_\pi + m_K}\right) \frac{3}{8\pi F_\pi^2} \Big[& 2m_\pi^2 m_K B'^{(-)}_1\left(0, -\frac{m_\pi}{2}\right) + m_\pi^4 \frac{\partial}{\partial \nu} C'^{(-)}_1\left(\nu, -\frac{m_\pi}{2}\right)\Big|_{\nu=0} \\ & - m_\pi^2 \frac{\partial}{\partial \nu} C'^{(-)}_4\left(\nu, -\frac{m_\pi}{2}\right)\Big|_{\nu=0} - m_K \Big]. \end{aligned} \quad (48)$$

III. NUMERICAL ANALYSIS AND RESULTS

One of the aims of the present calculation is to obtain a reasonably accurate value for the scattering lengths in the $K\pi$ process. Regrettably, some of the data needed for the calculation are not known with great precision (for example, the κ mass and width). However, a number of "consistency conditions" can be imposed to obtain considerable control over the parameters.

For example, the pole structure of the correction term was examined for consistency. This was done by requiring that the residues of a particular pole (either K^* or κ) in R and M (evaluated by the narrow-resonance approximation) be equal. The necessity of this condition is apparent from Eqs. (13) and (14), since the form factors $F(t)$ and $\sigma(t)$ cannot exhibit these poles (nor any ν dependence). It was found that residue equality was only possible if the saturated form of the first Weinberg sum rule[36] (see below) for K_A and K was relaxed[37] by ~10%. This then allowed approximate residue equality for both the K^* and the κ poles simultaneously in both the reduced-π and reduced-K approaches.

Similarly, additional relations amongst the parameters were obtained by evaluating the expressions for the κ and K^* decay widths from the three-point functions, determined in Appendix A, and then setting these widths equal to those used in the evaluation of $M(\nu, \nu_B)$ by the narrow-resonance approximation. Values of the parameters were sought which maximally satisfy these requirements together with the above residue conditions.

The above calculations were repeated and refined until a self-consistent model was obtained.

Parameters

Inspection of the general expressions obtained for $R^{\mu\lambda}_{bd,ac}(\nu, \nu_B)$ in Appendix B shows that there are a considerable number of parameters required to describe the correction terms. In Table I we list those particle masses and widths used as inputs. These quantities were not adjusted in the course of the calculations. This would certainly have been possible in the cases of the A_1 and K_A "resonances," but was not found to be necessary.

We make use of the first Weinberg sum rules[36] to eliminate several of the parameters. They are written in the slightly modified form[38]

$$\begin{aligned}\frac{g_\rho^2}{m_\rho^2} &= \frac{g_{A_1}^2}{m_{A_1}^2} + F_\pi^2 \\ &= \frac{g_{K^*}^2}{m_{K^*}^2} + F_\kappa^2 \\ &= \rho^2 \frac{g_{K_A}^2}{m_{K_A}^2} + F_K^2 . \end{aligned} \quad (49)$$

In the last line of Eq. (49) we have introduced a parameter ρ^2, which is usually taken to have the value $\rho^2 = 1.0$; in our case ρ^2 turns out to be slightly less[39] than 1.0.

To evaluate the above sum rules, we need to know g_ρ^2/m_ρ^2. We write

$$\frac{g_\rho^2}{m_\rho^2} = 2\xi^2 F_\pi^2 , \quad (50)$$

which reduces to the KSRF[40] relation for $\xi^2 = 1$. However, we find from the experimental value[41] of the $\rho \to e^+e^-$ rate that $\xi^2 = 1.4 \pm 0.2$, which is somewhat higher[42] than the KSRF value.

The parameter δ_{K^*} introduced by Fenster and Hussain[21] in their treatment of the $K^*K\pi$ vertex functions is taken to be -1.0, which is the preferred value these authors obtain in calculating $\Gamma(K^* \to K\pi)$, although they conclude that the value of $-\delta_{K^*}$ could be as small as -0.6.

For F_π we use the value of 92 MeV. We allow the ratio F_K/F_π to range between 1.0 and 1.30 and find that the most consistent value is 1.22. This is somewhat higher than that found by Gerstein and Schnitzer[23] (1.09) using the GMOR[10] symmetry-breaking scheme and somewhat lower than that of McKisic[26] (1.28) and Pagnamenta and Renner[27] (~1.25).

There appears to be some controversy regarding the mass and width of the κ meson. Our analysis favors the high mass solution of 1200 MeV,[41] and a value for F_κ^2/F_π^2 of 0.20. The latter value is lower than the 0.38 obtained by Gerstein and Schnitzer[23] and higher than that of Pond[28] (0.14) and corresponds to a value for $\Gamma(\kappa \to K\pi)$ of 500 MeV, substantially higher than the low[43] of 90 and close to a high[44] of ~450 MeV. The decay width is related to the appropriate vertex function $\overline{\Gamma}(q,p)$ (and thus to F_κ, see Appendix A) by the formula

$$\Gamma(\kappa \to K\pi) = \frac{q_c}{8\pi m_\kappa^2} \times \tfrac{3}{4} \times |\,\overline{\Gamma}(q,p)|^2 , \quad (51)$$

where q_c is the center-of-mass momentum of the K or π.

From the analysis given in Appendix A it follows that the K_{l3} form factor $f_+(t)$ is given at $t=0$, by[45]

$$f_+(0) = \frac{1}{2F_\pi F_K}(F_\pi^2 + F_K^2 - F_\kappa^2) ; \quad (52)$$

we find $f_+(0) = 0.94$. This, in turn, implies

$$\frac{F_K}{F_\pi f_+(0)} = 1.30 . \quad (53)$$

The phenomenological value of this last quantity is quoted[46] as

$$\frac{F_K}{F_\pi f_+(0)} = 1.27 \pm 0.03 . \quad (54)$$

The achievement of residue equality of the K^* and κ poles in the $K\pi$ scattering amplitude and correction terms depends on all of the parameters

TABLE I. Particle masses and widths used in the calculations.

Particle	Mass (MeV)	Width (MeV)
π	138	...
K	496	...
K^*	891	50
A_1	1100	...
K_A	1320	...
K_N	1420	55 (into $K\pi$)

TABLE II. The ratio of the residue of the K^* pole in the correction term to that in the $K\pi$ scattering amplitude for the reduced-K case with $F_K/F_\pi=1.22$, $F_\kappa/F_\pi=0.45$, and $1/\rho^2=1.1$.

δ_{K^*} \ ξ^2	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
0.00	0.366	0.368	0.371	0.375	0.380	0.386	0.393	0.400	0.408	0.416	0.424
−0.10	0.390	0.397	0.405	0.415	0.425	0.435	0.446	0.457	0.469	0.481	0.493
−0.20	0.416	0.428	0.442	0.456	0.471	0.487	0.502	0.519	0.535	0.552	0.568
−0.30	0.441	0.460	0.480	0.500	0.520	0.541	0.562	0.584	0.605	0.627	0.649
−0.40	0.468	0.493	0.519	0.545	0.572	0.598	0.625	0.652	0.680	0.707	0.735
−0.50	0.495	0.527	0.560	0.592	0.625	0.659	0.692	0.725	0.759	0.792	0.826
−0.60	0.524	0.563	0.602	0.642	0.682	0.722	0.762	0.802	0.842	0.882	0.922
−0.70	0.553	0.599	0.646	0.693	0.740	0.788	0.835	0.882	0.929	0.977	1.024
−0.80	0.582	0.637	0.692	0.747	0.802	0.857	0.911	0.966	1.021	1.076	1.131
−0.90	0.613	0.676	0.739	0.802	0.865	0.928	0.991	1.054	1.118	1.181	1.244
−1.00	0.644	0.716	0.788	0.860	0.931	1.003	1.075	1.146	1.218	1.290	1.362
−1.10	0.676	0.757	0.838	0.919	1.000	1.081	1.161	1.242	1.323	1.404	1.485
−1.20	0.709	0.800	0.890	0.980	1.071	1.161	1.251	1.342	1.432	1.523	1.613
−1.30	0.743	0.843	0.943	1.044	1.144	1.244	1.345	1.445	1.546	1.647	1.747
−1.40	0.777	0.888	0.998	1.109	1.220	1.331	1.442	1.553	1.664	1.775	1.887
−1.50	0.813	0.934	1.055	1.176	1.298	1.420	1.542	1.664	1.786	1.909	2.031

just mentioned, but perhaps most sensitively on ξ^2 and δ_{K^*}. In Table II we list for the reduced-K case the ratio of the residue of the K^* pole in the correction term to that in the scattering amplitude as a function of ξ^2 and δ_{K^*} for $F_K/F_\pi=1.22$. Consistency is represented by the curve in the $\xi^2-\delta_{K^*}$ plane for which the ratio is unity. The corresponding curve for the K^* pole in the reduced-π case is found to be nearly coincident.

$a^{1/2}-a^{3/2}$

For this combination of the scattering lengths, we have outlined five methods of evaluation, namely, by the threshold and $\nu=0$ sum rules in both the reduced-π and reduced-K approaches, and by use of the dispersion relation Eq. (31). We treat the reduced-π calculation first.

In the reduced-π approach, the "uncorrected" value of $a^{1/2}-a^{3/2}$ is $0.210m_\pi^{-1}$. For $\nu=m_\pi$, the K^* contribution to the correction term reduces this to $0.208m_\pi^{-1}$. Inclusion of the κ term raises this value to $0.218m_\pi^{-1}$. The result is fairly insensitive to changes in δ_{K^*} and ξ^2, remaining constant to within $\pm0.002m_\pi^{-1}$ for the ranges $1.1<\xi^2<1.7$ and $-1.5<\delta_{K^*}<-0.3$. It is also insensitive to values of the κ mass and width for which consistent solutions could be found. While a very broad, low-mass κ could appreciably increase the reduced-π prediction, its contribution to Eq. (31) would (assuming that the $I=\frac{1}{2}$ $K\pi$ cross section dominates that with $I=\frac{3}{2}$) make the dispersive value of $a^{1/2}-a^{3/2}$ much larger than the reduced-π value could ever become without enormous, higher-mass contributions. If we use Eq. (48) then we obtain $0.21m_\pi^{-1}$ and this remains constant for the same range of δ_{K^*} and ξ^2 as above.

In the reduced-K approach, on the other hand, the uncorrected (soft-kaon) value of $a^{1/2}-a^{3/2}$ is $0.141m_\pi^{-1}$, considerably different from the reduced-π case due to the F_K^{-2} factor. The correction terms are necessarily quite large, with the K^* term reducing the uncorrected value to $0.120m_\pi^{-1}$ when evaluated at threshold. The κ term restores this value to $0.177m_\pi^{-1}$. The result is quite sensitive to variation in any parameter. The $\nu=0$ sum rule also gives $a^{1/2}-a^{3/2}\approx0.18m_\pi^{-1}$, when the second term on the right-hand side of Eq. (33) is taken into account.

In Table III the combination $a^{1/2}-a^{3/2}$, as determined in the reduced-K case, is shown as a function of ξ^2 and δ_{K^*} for $F_K/F_\pi=1.22$. Comparing Table III with Table II one sees that $a^{1/2}-a^{3/2}$ varies by $\lesssim15\%$ along the consistency curve in the $\xi^2-\delta_{K^*}$ plane.

Increasing F_K/F_π above 1.22 (or decreasing it) makes it more difficult to satisfy all the consistency conditions. In fact, it is probable that F_K/F_π cannot exceed 1.3. For $F_K/F_\pi=1.3$ the satisfaction of all consistency conditions requires $|F_\kappa/F_\pi|\cong0.6$. Then, using Eq. (52), we find $F_K/[F_\pi f_+(0)]\cong1.45$, which is much higher than the result (54) deduced from experiment.

We will now calculate $a^{1/2}-a^{3/2}$ using the fixed-t dispersion relation in Eq. (31). This integral is evaluated by means of the narrow-resonance approximation

$$\sigma_R(s)=4\pi^2(2l_R+1)\frac{m_R\Gamma(R\to K\pi)}{q_c^{\,2}}\delta(s-m_R^{\,2}),\qquad(55)$$

where m_R and $\Gamma(R\to K\pi)$ are the mass and partial

TABLE III. Predictions for the S-wave, $I=\frac{1}{2}$ and $\frac{3}{2}$ $K\pi$ scattering length combination $a^{1/2}-a^{3/2}$ (in units of m_π^{-1}) in the **reduced-K** case with $F_K/F_\pi=1.22$, $F_\kappa/F_\pi=0.45$, and $1/\rho^2=1.1$. The soft-kaon value is $a^{1/2}-a^{3/2}=0.141\ m_\pi^{-1}$.

δ_{K^*} \ ξ^2	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
0.0	0.206	0.202	0.198	0.194	0.190	0.186	0.182	0.178	0.175	0.171	0.168
−0.10	0.205	0.201	0.197	0.192	0.188	0.184	0.180	0.177	0.173	0.169	0.165
−0.20	0.205	0.200	0.196	0.191	0.187	0.183	0.179	0.175	0.171	0.167	0.163
−0.30	0.204	0.199	0.195	0.190	0.186	0.181	0.177	0.173	0.169	0.165	0.161
−0.40	0.203	0.198	0.194	0.189	0.184	0.180	0.176	0.171	0.167	0.163	0.158
−0.50	0.203	0.198	0.193	0.188	0.183	0.179	0.174	0.169	0.165	0.160	0.156
−0.60	0.202	0.197	0.192	0.187	0.182	0.177	0.172	0.168	0.163	0.158	0.154
−0.70	0.202	0.196	0.191	0.186	0.181	0.176	0.171	0.166	0.161	0.156	0.151
−0.80	0.201	0.195	0.190	0.185	0.179	0.174	0.169	0.164	0.159	0.154	0.149
−0.90	0.200	0.195	0.189	0.184	0.178	0.173	0.167	0.162	0.157	0.152	0.147
−1.00	0.200	0.194	0.188	0.182	0.177	0.171	0.166	0.161	0.155	0.150	0.145
−1.10	0.199	0.193	0.187	0.181	0.176	0.170	0.164	0.159	0.153	0.148	0.142
−1.20	0.199	0.192	0.186	0.180	0.174	0.169	0.163	0.157	0.151	0.146	0.140
−1.30	0.198	0.192	0.185	0.179	0.173	0.167	0.161	0.155	0.150	0.144	0.138
−1.40	0.197	0.191	0.185	0.178	0.172	0.166	0.160	0.154	0.148	0.142	0.136
−1.50	0.197	0.190	0.184	0.177	0.171	0.164	0.158	0.152	0.146	0.140	0.134

width into $K\pi$ of the resonant particle, R, and where q_c is the center-of-mass momentum of K or π. Including the K^*, κ, and K_N resonances, we obtain

$$a^{1/2}-a^{3/2}=0.18m_\pi^{-1} \tag{56}$$

($0.093m_\pi^{-1}$ from the K^*, $0.065m_\pi^{-1}$ from the κ, $0.018m_\pi^{-1}$ from the K_N).

To bring this value up to agreement with the result obtained from the reduced-π analysis, the width of the κ would have to be raised to about 800 MeV. While we could get a consistent solution for this width, it would completely saturate the dispersion relation without appreciably raising the reduced-K prediction for $a^{1/2}-a^{3/2}$. Higher mass contributions required to increase the latter value would oversaturate the dispersion relation (assuming the $I=\frac{1}{2}$ cross section is dominant).

An alternative is to use the low mass solution for the κ, taking a mass of about 900 MeV. Recent experiments rule out an "intermediate" width particle, implying either a very broad, hidden resonance or a very narrow ($\Gamma<7$ MeV) one.[47] The narrow resonance would not saturate the sum rule, a width of 150 MeV being required. It seems extremely unlikely that a κ meson with this width (or one between 7 and 150 MeV) is sufficiently broad to be hidden, as the ϵ meson is in the π-π case.[48] Moreover, a width close to 150 MeV would lead to the same problems discussed above in connection with a κ with a width of 800 MeV.

It is probable that exact evaluation[3] (as opposed to narrow-resonance approximation) of Eq. (31) as well as inclusion of higher-energy contributions would eliminate the present discrepancy between (56) and the reduced-π result.

$$a^{1/2}+2a^{3/2}$$

From the expressions for the even amplitude, Eqs. (25) and (47), the value of $a^{1/2}+2a^{3/2}$ may be obtained, if $\sigma(0)$ and $\sigma'(0)$ are known. Although these terms have not been rigorously calculated, they may be approximated by assuming the GMOR symmetry-breaking scheme,[10] which leads to

$$\sigma(0)=-\tfrac{1}{2}m_K^{2},\qquad \sigma'(0)=-\tfrac{1}{2}m_\pi^{2}. \tag{57}$$

Substituting these results, one finds that $a^{1/2}+2a^{3/2}=0.006m_\pi^{-1}$ for the reduced-π case. This is close to zero, as expected from earlier theory. Similarly, in the reduced-K case, the same combination is calculated to be $0.005m_\pi^{-1}$, which is in good agreement with the reduced-π case. This result is somewhat surprising due to possible subtractions neglected in the dispersive contributions to $R^{(+)}$, or to extrapolation effects arising from taking the K off the mass shell in the σ-term calculations leading to (57).

IV. SUMMARY AND DISCUSSION

From the above study of on-mass-shell current-algebra sum rules for $K\pi$ scattering several results should be emphasized.

(1) It appears certain that the on-mass-shell current-algebra prediction for the combination $a^{1/2}-a^{3/2}$ of $I=\frac{1}{2}$ and $\frac{3}{2}$ S-wave $K\pi$ scattering lengths is very close to its soft-pion value[12-14] of $\approx 0.21m_\pi^{-1}$.

(2) For the value of $F_K/F_\pi = 1.22$, favored by the present analysis, result (1) implies that the soft-K prediction for $a^{1/2} - a^{3/2}$ of $\sim 0.14 m_\pi^{-1}$ is in error by ~30%. This provides a yardstick by which one can estimate kaon mass extrapolation effects in $K\pi$ scattering.

(3) The K^* and κ contributions to the "correction terms" can raise the soft-K value of $a^{1/2} - a^{3/2}$ to about $\sim 0.18 m_\pi^{-1}$.

(4) The present investigation implies that $F_K/F_\pi \lesssim 1.30$; not only does the soft-K value of $a^{1/2} - a^{3/2}$ ($\sim 0.12 m_\pi^{-1}$ for $F_K/F_\pi = 1.3$) decrease for increasing F_K/F_π, but so also do the K^* and κ corrections, which are of opposite sign.

It is worthwhile to expand briefly on some of these conclusions.

The smallness of the corrections to the soft-π prediction, upon which result (1) depends, should hold for any κ mass $\gtrsim 1.0$ GeV. A large correction could only be obtained, if, as in the $\pi\pi$ case,[48] a very broad, low-lying κ were found. A κ width large enough to cause an appreciable change in the reduced-π prediction for $a^{1/2} - a^{3/2}$ would greatly oversaturate the dispersion relation for this combination of scattering lengths (unless the $I = \frac{3}{2}$ $K\pi$ cross section is surprisingly large), by making the dispersive value much larger than the reduced-π value could become.

The discrepancy between the predicted reduced-K value of $a^{1/2} - a^{3/2} = 0.177 m_\pi^{-1}$ and the "correct" value of $0.21 m_\pi^{-1}$ [result (3)] could be erased by taking $F_K/F_\pi \approx 1.0$. However, for values of $F_K/F_\pi < 1.10$ it becomes increasingly difficult to satisfy the consistency requirements for the K^* and κ residues and widths, while still maintaining reasonable values for other parameters. In addition, we know from the fixed-t dispersion relation for $a^{1/2} - a^{3/2}$ that there may be important contributions to the reduced-K value that have not been taken into account. Indeed, by approximating the dispersion integral (in the narrow approximation) with the $K^*(890)$, a κ of mass 1200 MeV and width 500 MeV, and the $K_N(1420)$ the dispersion relation predicts $a^{1/2} - a^{3/2} = 0.18 m_\pi^{-1}$. κ parameters, such as a mass and width of 900 and 150 MeV, respectively, which could raise this to $\sim 0.21 m_\pi^{-1}$, are ruled out experimentally[47] and are not favored by the present model. A more realistic form for the resonances, however, would probably result in a small increase.[3]

APPENDIX A: VERTEX FUNCTIONS

As has been previously noted, Fenster and Hussain[21] have calculated the vertex functions involving $K^*K\pi$. However, $\partial_\mu V_i^\mu$ $(i = 4, \ldots, 7)$ is set equal to zero in their calculations when clearly it can be used as an interpolating field for the κ meson. In order to include the κ in our calculations then, it is obviously necessary to expand the previous calculations. This is done by adding four new vertex functions $\overline{\Gamma}$ which involve $\partial_\mu V_4^\mu$, etc., to the four functions Γ which arise from the K^*. This gives a pole dependence (note here that K^* has f-type coupling while κ has d-type) for the three-point functions of the form[49]

$$\int d^4x\, d^4y\, e^{-iqx} e^{ipy} \langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = if_{abc} \frac{F_\pi F_K m_\pi^2 m_K^2}{g_{K^*}(q^2+m_\pi^2)(p^2+m_K^2)} \Delta_{K^*}^{\lambda\eta}(k)\Gamma_\eta(q,p) + id_{abc} \frac{F_\pi F_K m_\pi^2 m_K^2 F_\kappa k^\lambda}{(q^2+m_\pi^2)(p^2+m_K^2)(k^2+m_\kappa^2)} \overline{\Gamma}(q,p), \tag{A1}$$

$$\int d^4x\, d^4y\, e^{-iqx} e^{ipy} \langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 = d_{abc} \frac{F_\pi F_K F_\kappa m_\pi^2 m_K^2 m_\kappa^2}{(q^2+m_\pi^2)(p^2+m_K^2)(k^2+m_\kappa^2)} \overline{\Gamma}(q,p), \tag{A2}$$

$$\begin{aligned}\int d^4x\, d^4y\, e^{-iqx} e^{ipy} \langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = {}& f_{abc} \frac{F_\pi m_\pi^2}{g_{K^*} g_{K_A}(q^2+m_\pi^2)} \Delta_{K_A}^{\nu\sigma}(p)\Delta_{K^*}^{\lambda\eta}(k)\Gamma_{\sigma\eta}(p,q) \\ &+ f_{abc} \frac{F_\pi F_K m_\pi^2}{g_{K^*}(q^2+m_\pi^2)(p^2+m_K^2)} p^\nu \Delta_{K^*}^{\lambda\eta}(k)\Gamma_\eta(q,p) \\ &+ d_{abc} \frac{F_\pi m_\pi^2 F_\kappa k^\lambda}{g_{K_A}(q^2+m_\pi^2)(k^2+m_\kappa^2)} \Delta_{K_A}^{\nu\eta}(p)\overline{\Gamma}_\eta(p,q) \\ &+ d_{abc} \frac{F_\pi m_\pi^2 F_K F_\kappa k^\lambda p^\nu}{(q^2+m_\pi^2)(k^2+m_\kappa^2)(p^2+m_K^2)} \overline{\Gamma}(q,p),\end{aligned} \tag{A3}$$

$$\begin{aligned}\int d^4x\, d^4y\, e^{-iqx} e^{ipy} \langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 = {}& -i \frac{d_{abc} F_\pi m_\pi^2 F_\kappa m_\kappa^2}{g_{K_A}(q^2+m_\pi^2)(k^2+m_\kappa^2)} \Delta_{K_A}^{\nu\eta}(p)\overline{\Gamma}_\eta(p,q) \\ &- id_{abc} \frac{F_\pi m_\pi^2 F_\kappa m_\kappa^2 F_K p^\nu}{(q^2+m_\pi^2)(p^2+m_K^2)(k^2+m_\kappa^2)} \overline{\Gamma}(q,p),\end{aligned} \tag{A4}$$

$$\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0$$
$$= if_{abc}\frac{1}{g_{K_A}g_{A_1}g_{K^*}}\Delta_{A_1}^{\mu\tau}(q)\Delta_{K_A}^{\nu\sigma}(p)\Delta_{K^*}^{\lambda\eta}(k)\Gamma_{\tau\sigma\eta}(q,p) - if_{abc}\frac{F_K p^\nu}{g_{K^*}g_{A_1}(p^2+m_K^2)}\Delta_{A_1}^{\mu\tau}(q)\Delta_{K^*}^{\lambda\eta}(k)\Gamma'_{\tau\eta}(q,p)$$
$$+ if_{abc}\frac{F_\pi q^\mu}{g_{K_A}g_{K^*}(q^2+m_\pi^2)}\Delta_{K_A}^{\nu\sigma}(p)\Delta_{K^*}^{\lambda\eta}(k)\Gamma_{\sigma\eta}(p,q) + if_{abc}\frac{F_\pi F_K p^\nu q^\mu}{g_{K^*}(q^2+m_\pi^2)(p^2+m_K^2)}\Delta_{K^*}^{\lambda\eta}(k)\Gamma_\eta(q,p)$$
$$+ id_{abc}\frac{F_\kappa k^\lambda}{(k^2+m_\kappa^2)g_{K_A}g_{A_1}}\Delta_{A_1}^{\mu\tau}(q)\Delta_{K_A}^{\nu\sigma}(p)\overline{\Gamma}_{\tau\sigma}(q,p) - id_{abc}\frac{F_\kappa F_K p^\nu k^\lambda}{g_{A_1}(p^2+m_K^2)(k^2+m_\kappa^2)}\Delta_{A_1}^{\mu\tau}(q)\overline{\Gamma}'_\tau(q,p)$$
$$+ id_{abc}\frac{F_\kappa F_\pi q^\mu k^\lambda}{g_{K_A}(q^2+m_\pi^2)(k^2+m_\kappa^2)}\Delta_{K_A}^{\nu\sigma}(p)\overline{\Gamma}_\sigma(p,q) + id_{abc}\frac{F_\kappa F_K F_\pi q^\mu p^\nu k^\lambda}{(q^2+m_\pi^2)(p^2+m_K^2)(k^2+m_\kappa^2)}\overline{\Gamma}(q,p), \quad \text{(A5)}$$

and

$$\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0$$
$$= d_{abc}\frac{F_\kappa m_\kappa^2}{g_{K_A}g_{A_1}(k^2+m_\kappa^2)}\Delta_{A_1}^{\mu\tau}(q)\Delta_{K_A}^{\nu\sigma}(p)\overline{\Gamma}_{\tau\sigma}(q,p) - d_{abc}\frac{F_\kappa m_\kappa^2 F_K p^\nu}{g_{A_1}(p^2+m_K^2)(k^2+m_\kappa^2)}\Delta_{A_1}^{\mu\tau}(q)\overline{\Gamma}'_\tau(q,p)$$
$$+ d_{abc}\frac{F_\kappa F_\pi m_\kappa^2 q^\mu}{g_{K_A}(q^2+m_\pi^2)(k^2+m_\kappa^2)}\Delta_{K_A}^{\nu\sigma}(p)\overline{\Gamma}_\sigma(p,q) + d_{abc}\frac{F_\kappa F_K F_\pi m_\kappa^2 q^\mu p^\nu}{(q^2+m_\pi^2)(p^2+m_K^2)(k^2+m_\kappa^2)}\overline{\Gamma}(q,p). \quad \text{(A6)}$$

We can substitute these relations into the usual Ward-Takahashi identities to obtain the expressions for the desired vertex functions. However, in order to evaluate the Ward-Takahashi identities, we need to know the current divergences so that all commutators will be calculable. Using the GMOR model[10] we obtain[50]

$$[A_a^0(x), A_b^\mu(y)]\delta(x^0-y^0) = if_{abc}V_c^\mu(x)\delta(x-y),$$
$$[V_a^0(x), V_b^\mu(y)]\delta(x^0-y^0) = if_{abc}V_c^\mu(x)\delta(x-y),$$
$$[V_a^0(x), A_b^\mu(y)]\delta(x^0-y^0) = if_{abc}A_c^\mu(x)\delta(x-y),$$
$$[A_a^0(x), V_b^\mu(y)]\delta(x^0-y^0) = if_{abc}A_c^\mu(x)\delta(x-y),$$
$$[V_a^0(x), \partial_\mu A_b^\mu(y)]\delta(x^0-y^0) = if_{abc}\partial_\mu A_c^\mu(x)\delta(x-y),$$
$$[A_a^0(x), \partial_\mu V_b^\mu(y)]\delta(x^0-y^0) = \frac{ic}{\omega_d}f_{b8e}d_{aed}\partial_\mu A_d^\mu(y)\delta(x-y),$$

and (A7)

$$[A_a^0(x), \partial_\mu A_b^\mu(y)]\delta(x^0-y^0) = \frac{4i}{3c}\omega_b d_{abe}f_{e8d}\partial_\mu V_d^\mu(y)\delta(x-y),$$

where we take[10] $c=-1.25$. Here,

$$\omega_a = \begin{cases} \dfrac{\sqrt{2}+c}{\sqrt{3}}, & a=1,2,3 \\ \dfrac{\sqrt{2}-\frac{1}{2}c}{\sqrt{3}}, & a=4,5,6,7 \\ \dfrac{\sqrt{2}-c}{\sqrt{3}}, & a=8. \end{cases} \quad \text{(A8)}$$

We have seven independent Ward-Takahashi identities, assuming $\partial_\lambda V_c^\lambda \neq 0$. They are

$$(p-q)_\lambda \int d^4x\, d^4y\, e^{-iqx}e^{-ipy}\langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = -i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0$$
$$+ f_{adc}\int d^4y\, e^{ipy}\langle T\{\partial_\mu A_d^\mu(0), \partial_\nu A_b^\nu(y)\}\rangle_0$$
$$+ f_{bdc}\int d^4x\, e^{-iqx}\langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_d^\nu(0)\}\rangle_0, \quad \text{(A9)}$$

$$(p-q)_\lambda \int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = -i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0$$
$$+ f_{adc}\int d^4y\, e^{ipy}\langle T\{\partial_\mu A_d^\mu(0), A_b^\nu(y)\}\rangle_0$$
$$+ f_{bdc}\int d^4x\, e^{-iqx}\langle T\{\partial_\mu A_a^\mu(x), A_d^\nu(0)\}\rangle_0, \quad \text{(A10)}$$

$$(p-q)_\lambda \int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = -i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\rangle_0 + f_{adc}\int d^4y\, e^{ipy}\langle T\{A_d^\mu(0), A_b^\nu(y)\}\rangle_0 + f_{bdc}\int d^4x\, e^{-iqx}\langle T\{A_a^\mu(x), A_d^\nu(0)\}\rangle_0\,, \tag{A11}$$

$$p_\nu \int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 - \frac{4}{3c}\omega_a d_{bae}f_{e8d}\int d^4y\, e^{iky}\langle T\{\partial_\mu V_d^\mu(y), V_c^\lambda(0)\}\rangle_0 - f_{bcd}\int d^4x\, e^{-iqx}\langle T\{\partial_\mu A_a^\mu(x), A_d^\lambda(0)\}\rangle_0\,, \tag{A12}$$

$$p_\nu \int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 = i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 - \frac{4}{3c}\omega_a d_{bae}f_{e8d}\int d^4y\, e^{iky}\langle T\{\partial_\mu V_d^\mu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 - \frac{c}{\omega_d}f_{c8e}d_{bed}\int d^4x\, e^{-iqx}\langle T\{\partial_\mu A_a^\mu(x), \partial_\lambda A_d^\lambda(0)\}\rangle_0\,, \tag{A13}$$

$$q_\mu \int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 = -i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), V_c^\lambda(0)\}\rangle_0 + f_{abd}\int d^4y\, e^{iky}\langle T\{V_d^\mu(y), V_c^\lambda(0)\}\rangle_0 + f_{acd}\int d^4y\, e^{ipy}\langle T\{A_b^\nu(y), A_d^\lambda(0)\}\rangle_0\,, \tag{A14}$$

and

$$q_\mu \int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 = -i\int d^4x\, d^4y\, e^{-iqx}e^{ipy}\langle T\{\partial_\mu A_a^\mu(x), A_b^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 + f_{abd}\int d^4y\, e^{iky}\langle T\{V_d^\nu(y), \partial_\lambda V_c^\lambda(0)\}\rangle_0 + \frac{c}{\omega_d}f_{c8e}d_{aed}\int d^4y\, e^{ipy}\langle T\{A_b^\nu(y), \partial_\lambda A_c^\lambda(0)\}\rangle_0\,, \tag{A15}$$

where $k = p - q$.

As it turns out, the Γ's and $\overline{\Gamma}$'s separate into two sets of equations, and the Γ's retain the same form as was calculated by Fenster and Hussain for K^*. Since we have not made such strong assumptions as they have (e.g., $F_\pi = F_K$), then the functional forms of our vertex functions will be somewhat different, as shown below. Additionally, once a two-point function for the vector current and its divergence has been evaluated (analogous to the expression in Fenster and Hussain for the axial-vector currents) the $\overline{\Gamma}$'s can be solved. The values for those vertex functions which will be of use in our calculations, are given below. Thus, after considerable algebra, we have

$$\Gamma_{\mu\nu\lambda}(q,p) = \frac{1}{2}\frac{m_{K^*}^2}{g_{K^*}}\left(\frac{g_{K_A}}{g_{A_1}} + \frac{g_{A_1}}{g_{K_A}}\right)[g_{\mu\nu}(p+q)_\lambda + (2+\delta_{K^*})(g_{\mu\lambda}k_\nu - g_{\nu\lambda}k_\mu) - g_{\mu\lambda}p_\nu - g_{\nu\lambda}q_\mu]\,, \tag{A16}$$

$$\Gamma_{\sigma\eta}(p,q) = \frac{d}{2}\{q_\sigma(p+q)_\eta + (2+\delta_{K^*})[q_\eta k_\sigma - (q\cdot k)g_{\sigma\eta}] - q_\eta p_\sigma - q^2 g_{\sigma\eta}\} + \frac{e}{2}[(p^2 + m_{K_A}^2)g_{\sigma\eta} - p_\sigma p_\eta]\,, \tag{A17}$$

where

$$d = -\frac{m_{K^*}}{F_\pi m_{K_A}^2 m_{A_1}^2 (C_{K^*}C_{K_A})^{1/2}}(m_{K_A}^2 C_{K_A} + m_{A_1}^2 C_{A_1})\,, \tag{A18}$$

and

$$e=-\frac{2m_{K^*}}{F_\pi m_{K_A}}\left(\frac{C_{K^*}}{C_{K_A}}\right)^{1/2}, \tag{A19}$$

with

$$C_a=\frac{g_a}{m_a^{\,2}}, \tag{A20}$$

and

$$\overline{\Gamma}_{\tau\sigma}(q,p)=\frac{g_{A_1}}{2\beta F_\kappa g_{K_A}}[(p^2+2m_{K_A}{}^2+q^2)g_{\tau\sigma}-p_\tau p_\sigma-q_\tau q_\sigma]-\frac{g_{K_A}}{2\beta F_\kappa g_{A_1}}[(p^2+q^2+2m_{A_1}{}^2)g_{\tau\sigma}-p_\tau p_\sigma-q_\tau q_\sigma], \tag{A21}$$

where $\beta=\pm i=(f_{abc}/d_{abc})_{K^+\kappa^+\pi^0,\,\text{etc.}}$; obviously $|\beta|^2=1$ so it will disappear from the calculations. Also

$$\overline{\Gamma}_\sigma(p,q)=d(p,q)q_\sigma+e(p,q)p_\sigma, \tag{A22}$$

where

$$d(p,q)=\frac{g_{K_A}}{2F_\pi F_\kappa C_{K_A}}\left[2C_{K_A}-2C_{A_1}+p^2\left(\frac{C_{K_A}}{m_{A_1}{}^2}-\frac{C_{A_1}}{m_{K_A}{}^2}\right)+\frac{2F_\kappa{}^2}{m_{K_A}{}^2}p^2+2F_\kappa{}^2\right], \tag{A23}$$

$$e(p,q)=\frac{g_{K_A}}{2F_\pi F_\kappa C_{K_A}}\left[-(p\cdot q)\left(\frac{C_{K_A}}{m_{A_1}{}^2}-\frac{C_{A_1}}{m_{K_A}{}^2}\right)+2X_K F_\kappa{}^2-2F_\kappa{}^2-2F_\kappa{}^2\frac{(p\cdot q)}{m_{K_A}{}^2}\right], \tag{A24}$$

$$X_K=-\frac{2}{c\sqrt{3}}\omega_K\frac{m_\kappa{}^2}{m_K{}^2}=6.369, \tag{A25}$$

and

$$X_\pi=\frac{2}{c\sqrt{3}}\omega_\pi\frac{m_\kappa{}^2}{m_\pi{}^2}=-6.690. \tag{A26}$$

We can obtain $\overline{\Gamma}'_\sigma(q,p)$ from the above by substituting $\pi\leftrightarrow K$, $A_1\leftrightarrow K_A$, $p\leftrightarrow -q$. Finally,

$$\overline{\Gamma}(q,p)=\frac{1}{2F_\pi F_K F_\kappa}\{2(C_{A_1}-C_{K_A})(p\cdot q)-2F_\kappa{}^2[(p\cdot q)+p^2X_K-p^2]\}. \tag{A27}$$

To go from our results to the corresponding ones of Fenster and Hussain one must multiply by 2 because of different normalizations.

APPENDIX B: CALCULATIONS FOR REDUCED K AND REDUCED π

The K^* and κ contributions to the correction terms will be evaluated using the results of Appendix A. For the reduced-K case the matrix elements $\langle\pi|\tilde{A}|K^*\rangle$ and $\langle\pi|\tilde{A}|\kappa\rangle$ can be obtained from Eq. (A3) in terms of the vertex functions $\Gamma_{\mu\nu}$ and $\overline{\Gamma}_\mu$, respectively. These vertex functions have been determined by means of Ward-Takahashi identities and are given by Eqs. (A17)–(A20) and (A22)–(A26). The corresponding reduced-π matrix elements can be obtained from these expressions by making the changes $K\leftrightarrow\pi$, $p\leftrightarrow -q$, $K\leftrightarrow A_1$, etc.

We note that the second and fourth terms of Eq. (A3) do not contribute to the matrix elements of $\tilde{A}$, since the latter contain no K pole. Hence, multiplying both sides of Eq. (A3) by $(q^2+m_\pi{}^2)(k^2+m_{K^*}{}^2)$ and taking the limit $q^2\to -m_\pi{}^2, k^2\to -m_{K^*}{}^2$ leads to

$$(2\pi)^3(2q^0 2k^0)^{1/2}\langle\pi_a(q)|\tilde{A}_b^\nu(0)|K_c^*(k)\rangle =f_{abc}\frac{1}{g_{K_A}}\epsilon_{K^*}^\eta(k)\Delta_{K_A}^{\nu\sigma}(p)\Gamma_{\sigma\eta}(p,q),$$

where $\epsilon_{K^*}^\eta$ is the K^* polarization vector and $p=k-q$. Similarly, simultaneous projection of the π and κ poles gives

$$(2\pi)^3(2q^0 2k^0)^{1/2}\langle\pi_a(q)|\tilde{A}_b^\nu(0)|\kappa_c(k)\rangle =-id_{abc}\frac{1}{g_{K_A}}\Delta_{K_A}^{\nu\eta}(p)\overline{\Gamma}_\eta(p,q).$$

Substituting back into $r^{\mu\nu}$ [Eq. (21)] we have the following contributions.

For K^*:

$$r^{\mu\nu}_{bd,ac} = -\pi\left[f_{dbe}f_{cae}\sum_{\text{pol}}\frac{1}{g_{K_A}{}^2}\epsilon^{\eta}_{K^*}(k)\Delta^{\mu\sigma}_{K_A}(p)\Gamma_{\sigma\eta}(p,-p_\beta)\epsilon^{\rho}_{K^*}(k)\Delta^{\nu\tau}_{K_A}(q)\Gamma_{\tau\rho}(q,-p_\alpha)\delta(s+k^2)\right]$$
$$+\pi\left[f_{dae}f_{cbe}\frac{1}{g_{K_A}{}^2}\sum_{\text{pol}}\epsilon^{\eta}_{K^*}(-k')\Delta^{\nu\sigma}_{K_A}(q)\Gamma_{\sigma\eta}(q,p_\beta)\epsilon^{\rho}_{K^*}(-k')\Delta^{\mu\tau}_{K_A}(p)\Gamma_{\tau\rho}(p,p_\alpha)\delta(u+k'^2)\right],$$

where $k=q+p_\alpha=p+p_\beta$ and $k'=-q+p_\beta$.

For κ:

$$r^{\mu\nu}_{bd,ac} = -\pi\left[d_{dbe}d_{cae}\frac{1}{g_{K_A}{}^2}\Delta^{\mu\eta}_{K_A}(p)\overline{\Gamma}_\eta(p,-p_\beta)\Delta^{\nu\sigma}_{K_A}(q)\overline{\Gamma}_\sigma(q,-p_\alpha)\delta(s+k^2)\right]$$
$$+\pi\left[d_{dae}d_{cbe}\frac{1}{g_{K_A}{}^2}\Delta^{\nu\sigma}_{K_A}(q)\overline{\Gamma}_\sigma(q,p_\beta)\Delta^{\mu\eta}_{K_A}(p)\overline{\Gamma}_\eta(p,p_\alpha)\delta(u+k'^2)\right].$$

We can now evaluate the absorptive part in terms of p,q,p_α,p_β, by using Eqs. (A17)–(A20) and (A22)–(A26) and then change variables to P,Q,Δ. Integration over s and u in the dispersion relations will then yield $A,\ldots,C_4$. Defining $N^{\pm}_{K^*}$ and $N^{\pm}_{\kappa}$ as

$$N^{\pm}_{K^*}=\frac{1}{m_{K^*}{}^2-s}\pm\frac{1}{m_{K^*}{}^2-u},$$

$$N^{\pm}_{\kappa}=\frac{1}{m_{\kappa}{}^2-s}\pm\frac{1}{m_{\kappa}{}^2-u},$$

we have for the K^* contribution

$$A^{(\pm)}=\frac{L}{16}J_1N^{\pm}_{K^*},$$

$$B_1^{(\pm)}=\frac{L}{16}J_2N^{\mp}_{K^*},$$

$$B_2^{(\pm)}=\frac{L}{32}J_3N^{\mp}_{K^*},$$

$$C_1^{(\pm)}=\frac{L}{16}J_4N^{\pm}_{K^*},$$

$$C_2^{(\pm)}=\frac{L}{32}J_5N^{\pm}_{K^*},$$

$$C_3^{(\pm)}=\frac{L}{64}J_6N^{\pm}_{K^*},$$

and

$$C_4^{(\pm)}=\frac{L}{16}J_7N^{\pm}_{K^*},$$

where

$$L=\frac{-g_{K_A}{}^2}{(m_{K_A}{}^2-m_{K}{}^2)^2},$$

and where

$$J_1=H_5+H_6+H_9,$$

$$J_2=H_2+H_3,$$

$$J_3=H_4-H_5+H_6+H_8-H_9,$$

$$J_4=H_1+H_7+H_{10},$$

$$J_5=H_1-H_2+H_3-H_7+H_{10},$$

$$J_6=-H_1+H_2-H_3+H_4-H_5+H_6+H_7-H_8+H_9+H_{10},$$

and

$$J_7=H_{11}.$$

If we define

$$\theta_1=-d-d\delta_{K^*}+\frac{dm_K{}^2}{m_{K_A}{}^2}-\frac{d\delta_{K^*}}{m_{K_A}{}^2}(p\cdot k),$$

$$\theta_2=e(m_{K_A}{}^2-m_K{}^2)+d\lambda,$$

and

$$\lambda=2p_\beta\cdot k+\delta_{K^*}(p_\beta\cdot k)+m_\pi{}^2,$$

where d and e are given in Eqs. (A18) and (A19), then

$$H_1=\theta_1{}^2(p_\alpha\cdot p_\beta)+2\frac{\theta_1 d\lambda}{m_{K_A}{}^2}(p\cdot p_\alpha)+\frac{d^2\lambda^2}{m_{K_A}{}^2}(p\cdot q)$$
$$+\frac{\theta_1{}^2}{m_{K^*}{}^2}(k\cdot p_\alpha)^2+\frac{2\theta_1 d\lambda}{m_{K_A}{}^2m_{K^*}{}^2}(k\cdot p_\alpha)(k\cdot q)$$
$$+\frac{2\theta_1\theta_2}{m_{K^*}{}^2}(k\cdot p_\alpha)+\frac{d^2\lambda^2}{m_{K^*}{}^2m_{K_A}{}^4}(k\cdot p)^2$$
$$+\frac{d\lambda\theta_2}{m_{K_A}{}^2m_{K^*}{}^2}(k\cdot p)+\frac{\theta_2{}^2}{m_{K^*}{}^2},$$

$$H_2=-\theta_1 d\delta_{K^*}(p_\alpha\cdot p_\beta)-\frac{d^2\delta_{K^*}\lambda}{m_{K_A}{}^2}(p\cdot p_\alpha)$$
$$-\frac{\theta_1 d\delta_{K^*}}{m_{K^*}{}^2}(k\cdot p_\alpha)^2+\frac{\theta_1\theta_2}{m_{K^*}{}^2}(k\cdot p_\alpha)$$
$$-\frac{d^2\delta_{K^*}\lambda}{m_{K_A}{}^2m_{K^*}{}^2}(k\cdot p)(k\cdot p_\alpha)+\frac{d\lambda\theta_2}{m_{K_A}{}^2m_{K^*}{}^2}(k\cdot p)$$
$$-\frac{d\delta_{K^*}\theta_2}{m_{K^*}{}^2}(k\cdot p_\alpha)+\frac{\theta_2{}^2}{m_{K^*}{}^2},$$

$$H_3 = \theta_1 \theta_2 ,$$

$$H_4 = -d\delta_{K^*}\theta_1(p_\alpha \cdot p_\beta) - \frac{d^2\delta_{K^*}\lambda}{m_{K_A}^2}(p_\beta \cdot q) - \frac{d\delta_{K^*}\theta_1}{m_{K^*}^2}(k \cdot p_\alpha)^2 - \frac{d^2\delta_{K^*}\lambda}{m_{K_A}^2 m_{K^*}^2}(k\cdot p_\alpha)(k \cdot q) - \frac{d\delta_{K^*}\theta_2}{m_{K^*}^2}(k\cdot p_\alpha) + \frac{\theta_1\theta_2}{m_{K^*}^2}(k\cdot p_\alpha) + \frac{\theta_2^2}{m_{K^*}^2},$$

$$H_5 = d^2\delta_{K^*}^2(p_\alpha \cdot p_\beta) + \frac{d^2\delta_{K^*}^2}{m_{K^*}^2}(k\cdot p_\alpha)^2 - \frac{2d\delta_{K^*}\theta_2}{m_{K^*}^2}(k \cdot p_\alpha) + \frac{\theta_2^2}{m_{K^*}^2},$$

$$H_6 = -d\delta_{K^*}\theta_2 , \quad H_7 = \frac{d\lambda\theta_2}{m_{K_A}^2},$$

$$H_8 = \theta_1\theta_2 + \frac{\theta_2 d\lambda}{m_{K_A}^2 m_{K^*}^2}(k\cdot q) , \quad H_9 = -d\delta_{K^*}\theta_2 ,$$

$$H_{10} = \frac{\theta_2 d\lambda}{m_{K_A}^2} + \frac{\theta_2 d\lambda}{m_{K_A}^2 m_{K^*}^2}(k\cdot q) , \text{ and } H_{11} = \theta_2^2 .$$

For the κ contribution

$$A^{(\pm)} = \frac{L}{4} d_\kappa^2 N_\kappa^{\pm},$$

$$B_1^{(\pm)} = \frac{L}{4} d_\kappa h_\kappa N_\kappa^{\mp},$$

$$B_2^{(\pm)} = \frac{L}{8}(d_\kappa h_\kappa - d_\kappa^2) N_\kappa^{\mp},$$

$$C_1^{(\pm)} = \frac{L}{4} h_\kappa^2 N_\kappa^{\pm},$$

$$C_2^{(\pm)} = \frac{L}{8}(h_\kappa^2 - d_\kappa h_\kappa) N_\kappa^{\pm},$$

$$C_3^{(\pm)} = -\frac{L}{16}(d_\kappa - h_\kappa)^2 N_\kappa^{\pm},$$

$$C_4^{(\pm)} = 0 ,$$

where $d_\kappa = d(q, -p_\alpha)$,

$$e_\kappa = e(q, -p_\alpha) ,$$

and

$$h_\kappa = \frac{(q\cdot p_\alpha)}{m_{K_A}^2} d_\kappa + \left(\frac{m_K^2}{m_{K_A}^2} - 1\right) e_\kappa .$$

$d(q, -p_\alpha)$ and $e(q, -p_\alpha)$ may be obtained from Eqs. (A23) and (A24), respectively.

The corresponding correction terms for the reduced-π case can be obtained from the above expressions by interchanging the labels $K \leftrightarrow \pi$ and $K_A \leftrightarrow A_1$.

---

*Work supported in part by the National Research Council of Canada.

[1]For a review of current-algebra applications the reader is referred to S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968); B. Renner, *Current Algebras and Their Applications* (Pergamon, London, 1968).

[2]The kinematic factors used by Adler (Ref. 3) are typical of this approach.

[3]S. L. Adler, Phys. Rev. 140, B736 (1965).

[4]These approaches cover a rather broad spectrum. A representative sample includes W. I. Weisberger, Phys. Rev. 143, 1302 (1966); F. T. Meiere and M. Sugawara, *ibid.* 153, 1709 (1967); S. Fubini and G. Furlan, Ann. Phys. (N.Y.) 48, 322 (1968); H. D. Dahmen and K. D. Rothe, Nucl. Phys. B15, 45 (1970).

[5]L. S. Brown, W. J. Pardee, and R. D. Peccei, Phys. Rev. D 4, 2801 (1971).

[6]S. L. Adler, Phys. Rev. Lett. 14, 1051 (1965); W. I. Weisberger, *ibid.* 14, 1047 (1965).

[7]T. P. Cheng and R. F. Dashen, Phys. Rev. Lett. 26, 594 (1971).

[8]R. H. Graham and B. Weisman, Phys. Rev. D 10, 315 (1974).

[9]I. J. Muzinich and S. Nussinov, Phys. Lett. 19, 248 (1965); K. Kawarabayashi, W. D. McGlinn, and W. W. Wada, Phys. Rev. Lett. 15, 897 (1965).

[10]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968). See also S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[11]Previous current-algebra approaches to $K\pi$ scattering, which make use of soft-meson techniques, include the study of Adler-Weisberger type relations by the authors of Ref. 9 and the derivation of scattering-length sum rules in Refs. 12–14.

[12]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[13]Y. Tomozawa, Nuovo Cimento 46A, 707 (1966).

[14]R. W. Griffith, Phys. Rev. 176, 1705 (1968).

[15]L. S. Brown *et al.* (Ref. 5). See also S. Fubini and G. Furlan (Ref. 4), and G. Altarelli, N. Cabibbo, and L. Maiani, Nucl. Phys. B34, 621 (1971).

[16]See, e.g., E. Reya, Phys. Rev. D 6, 200 (1972); 7, 3472 (1973); J.-Y. Harnois, Phys. Lett. 56B, 379 (1975).

[17]By this we mean the equation which originates from the standard Lehmann-Symanzik-Zimmermann reduction of the pions from the $K\pi$ scattering amplitude.

[18]Unsubtracted dispersion relations have a theoretical justification only in the case of the $a^{1/2} - a^{3/2}$ sum rule; however, their broader use does have several pre-

cedents. The analysis of Brown *et al.* (Ref. 5) is based on assumptions which are equivalent to the use of unsubtracted dispersion relations. Also, our use of them in Ref. 8 led to results which are not patently unreasonable.

[19]The earliest hard-meson analyses (carried out on the $A_1\rho\pi$ system) were by H. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967); S. G. Brown and G. B. West, Phys. Rev. Lett. 19, 812 (1967); R. Arnowitt, M. H. Friedman, and P. Nath, *ibid.* 19, 1085 (1967). SU(3) generalizations have been carried out by the authors of Refs. 20–23.

[20]C. S. Lai and B.-L. Young, Phys. Rev. 169, 1241 (1968).

[21]S. Fenster and F. Hussain, Phys. Rev. 169, 1314 (1968).

[22]Y. Ueda, Phys. Rev. 174, 2082 (1968).

[23]I. S. Gerstein, H. J. Schnitzer, and S. Weinberg, Phys. Rev. 175, 1873 (1968); I. S. Gerstein and H. J. Schnitzer, *ibid.* 175, 1876 (1968).

[24]In Appendix A we extend their analysis to the case in which the strangeness-changing vector current is not conserved; its divergence is assumed to be dominated by the κ.

[25]What we consider to be the physical range of the κ parameters will be described below.

[26]J. M. McKisic, Phys. Rev. D 2, 531 (1970).

[27]A. Pagnamenta and B. Renner, Phys. Rev. 172, 1761 (1968); R. Rockmore, Phys. Rev. D 2, 2693 (1970).

[28]P. Pond, Phys. Rev. D 3, 2210 (1971).

[29]J. S. Kang, Phys. Rev. D 7, 2637 (1973).

[30]The four-momenta of the particles are indicated in parentheses. The subscripts $a,\ldots,d$ are SU(3) indices which run from 1 to 3 (4 to 7) for the pions (kaons).

[31]Here $A_a^\nu(x)$ is the strangeness-changing axial-vector current and $\phi_a(x)$ is the interpolating field for the kaons. m_K and F_K are the kaon mass and decay constant, respectively.

[32]The derivation of this is a straightforward SU(3) generalization of that given for $\pi\pi$ scattering by S. Weinberg, in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T., Cambridge, Mass., 1971), Vol. 1. We follow Weinberg's normalization of amplitudes and metric $(+,+,+,-)$, but our vector and axial-vector currents are normalized to one half of his.

[33]See, e.g., L. S. Brown *et al.* (Ref. 5).

[34]The argument is given in Ref. 8. It should have been pointed out there that the presence of fixed poles in the angular momentum plane does not affect the conclusions regarding subtractions. For a discussion of fixed poles in the context of current-algebra applications see, e.g., D. J. Gross, in *Lectures on Current Algebras and Its Applications,* edited by S. B. Treiman, R. Jackiw, and D. J. Gross (Princeton Univ. Press, Princeton, New Jersey, 1972).

[35]A numerical calculation using the narrow-resonance approximation for the cross sections shows that the contribution of the first term relative to the second is of the order of 25 : 1.

[36]S. Weinberg, Phys. Rev. Lett. 18, 507 (1967); S. L. Glashow, H. J. Schnitzer, and S. Weinberg, *ibid.* 19, 139 (1967).

[37]Note that the *exact* sum rules are still assumed, since they are necessary for the cancellation of Schwinger terms in the Ward-Takahashi identities [see, e.g., Schnitzer and Weinberg (Ref. 19)].

[38]The relevant definitions are

$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_a^\mu(0)|\kappa_b(k)\rangle = i\,\delta_{ab}F_\kappa k^\mu,\qquad a,b=4,\ldots,7;$$

$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|V_a^\mu(0)|\rho_b(k,\epsilon)\rangle = \delta_{ab}g_\rho\epsilon^\mu,\qquad a,b=1,\ldots,3,$$

where ϵ^μ is the polarization of the ρ;

$$(2\pi)^{3/2}(2k^0)^{1/2}\langle 0|A_a^\mu(0)|A_{1b}(k,\epsilon)\rangle = \delta_{ab}g_{A_1}\epsilon^\mu,\qquad a,b=1,\ldots,3;$$

and similarly for g_{K^*} and g_{K_A}.

[39]We have adopted the point of view that g_{K_A} and m_{K_A} characterize a two-parameter model of the $Y=1$, $J=1$ axial-vector spectral function; hence, $g_{K_A}{}^2/m_{K_A}{}^2+F_K{}^2$ should be greater than $g_\rho{}^2/m_\rho{}^2$, since the ρ is not expected to fully dominate the $I=1$ vector spectral function. When the calculations were completed, it was indeed found that $\rho^2=0.9$ resulted in the best matching of the K^* and κ poles in R with their counterparts in M. It was not found to be necessary to modify the saturated SU(2) × SU(2) sum rule in the present work, although this was explored in Ref. 8.

[40]K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).

[41]Particle Data Group, Phys. Lett. 50B, 1 (1974).

[42]The $\rho\to\mu^+\mu^-$ rate (see Ref. 41) leads to the even higher value of $\xi^2\approx 2.0$.

[43]D. J. Crennell *et al.*, Phys. Rev. Lett. 22, 487 (1969).

[44]T. G. Trippe *et al.*, Phys. Lett. 28B, 203 (1968).

[45]S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[46]L.-M. Chounet, J.-M. Gaillard, and M. K. Gaillard, Phys. Rep. 4C, 199 (1972).

[47]M. J. Matison *et al.*, Phys. Rev. D 9, 1872 (1974).

[48]S. D. Protopopescu *et al.*, Phys. Rev. D 7, 1279 (1973).

[49]The notation follows that of Fenster and Hussain (Ref. 21), with the exception that our currents are normalized to one half theirs. F_κ is defined analogously to F_π and F_K (see Ref. 38).

[50]The Schwinger terms (which do not contribute to the Ward-Takahashi identities) are omitted.