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$\Delta I = \frac{1}{2}$ rule in the *PC*-conserving nonleptonic weak interactions

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The question of the $\Delta I = \frac{1}{2}$ selection rule in weak nonleptonic decays is studied. We assume that the weak amplitudes of the form $\langle \alpha | \mathcal{H}_w | 0 \rangle$ obey unsubtracted dispersion relations in the momentum-transfer variable $s = p_\alpha^2$ and that they obey Hölder's condition in s except at possible poles. Using unitarity, the determination of the weak amplitudes is transformed into solving a Hilbert problem. This in turn is transformed into a system of Fredholm's integral equations. The number of solutions is discussed, and the solution which satisfies our assumptions is found to be unique and can be expressed in terms of the strong interaction S-matrix elements. Weak amplitudes which possess a pole, namely the PC-conserving parity-violating $\Delta I = \frac{1}{2}$ interactions (poles K^0 and $\bar{K}^0$) and the parity-conserving $\Delta I = \frac{1}{2}$ interactions (poles κ^0 and $\bar{\kappa}^0$), are shown to be consistent with our assumptions. The other ΔI amplitudes either vanish or do not satisfy unsubtracted dispersion relations.

I. INTRODUCTION

It is well known[1,2] that the $\Delta I = \frac{1}{2}$ selection rule is in good agreement with experiment for the $\Delta S = \pm 1$ nonleptonic weak decays. The few percent deviation from this selection rule, namely in $K^+ \to \pi^+ \pi^0$ and $K \to 3\pi$ decays,[3] may be considered as an electromagnetic correction to the weak interactions.[4-6] The existence of a $\Delta I = \frac{3}{2}$ part in the fundamental weak Hamiltonian is another possibility.

From the theoretical point of view, the origin of this selection rule is an open question. In the usual current-current form of the weak interactions with currents belonging to an SU(3) octet. or in the charged intermediate vector-boson formulation of weak interactions, both the $\Delta I = \frac{1}{2}$ and the $\Delta I = \frac{3}{2}$ transitions (the octet and the 27-plet parts, respectively) may occur with the same order of magnitude. Thus, in order to explain this selection rule, two alternatives may be envisaged.[7] Either one must assume that the $\Delta I = \frac{1}{2}$ amplitude is enhanced and/or the $\Delta I = \frac{3}{2}$ amplitude is suppressed[8,9] (octet enhancement) or one must add neutral currents to the theory (or equivalently, the appropriate neutral intermediate vector bosons). There is no firm experimental evidence for the existence of neutral currents. The octet enhancement must be a dynamical effect of strong interactions.

Current algebra offered a hope of solving the problem of octet dominance by the current-current model of weak interactions using SU(3) $\times$ SU(3) current algebra and the soft-pion limit. This approach was only partially successful.[10] In particular, it failed to explain the $\Delta I = \frac{1}{2}$ rule for the P-wave baryonic decays and the smallness of $A(\Sigma_+^+)$ which is necessary in order to complete the proof of the $\Delta I = \frac{1}{2}$ for the S wave.

In this paper, we discuss this problem without assuming a specific model for the weak interactions. We decompose the weak Hamiltonian into a parity-conserving part $\mathcal{H}_w^{(+)}$ and a parity-violating part $\mathcal{H}_w^{(-)}$, each of which is in turn decomposed into $\Delta I = \frac{1}{2}, \frac{3}{2}$, etc., parts,

$$\mathcal{H}_w^{(\pm)} = \mathcal{H}_w^{(\pm)1/2} + \mathcal{H}_w^{(\pm)3/2} + \cdots , \tag{1}$$

and we make the following assumptions:

(a) We assume that all particles are spinless. This allows us to represent the weak amplitude of any process by a single function of the kinematical variables. We write

$$\langle \alpha | \mathcal{H}_w^{(\pm)I} | 0 \rangle = F_\alpha^{(\pm)I}(s, t_\alpha), \tag{2}$$

where $|\alpha\rangle$ stands for any state of particles, $s = (p_\alpha)^2$ is the square of the total energy in the center-of-mass system of the particles in the state $|\alpha\rangle$, and t_α stands for all the other kinematical variables needed to specify completely the state $|\alpha\rangle$. This assumption allows us to simplify the mathematical analysis. In the case of particles with spin, we must introduce the appropriate operators and spin-wave functions. The physical amplitude corresponds to $s = 0$.

(2) The weak amplitudes $F_\alpha^{(\pm)I}$ are analytic functions of s for fixed values of t_α. Their singularities are poles corresponding to scalar particles of the same parity and isospin as $\mathcal{H}_w^{(\pm)I}$ and a cut along the real-s axis starting at the threshold of the multiparticle states. The discontinuity across the cut, $\text{Abs}\, F_\alpha^{(\pm)I}$, is given by the unitarity relation:

$$\mathrm{Abs}\,F_\alpha^{(\pm)I}(s,t_\alpha)=\tfrac{1}{2}(2\pi)^4\sum_j \delta^4(p_\alpha-p_j)\times\langle\alpha|T^+|j\rangle\langle j|\mathcal{H}_w^{(\pm)I}|0\rangle$$
$$=\tfrac{1}{2}(2\pi)^4\sum_j \delta^4(p_\alpha-p_j)\times T_{\alpha j}^{(\pm)I}(s,t_{\alpha j})F_j^{(\pm)I}(s,t_j),\qquad(3)$$

where $T_{\alpha j}^{(\pm)I}$ is the strong-interaction transition amplitude for $j\to\alpha$ in the even- (odd-) parity state and isospin state I. We assume that this is a known function of the variable s and the other auxiliary kinematical variables $t_{\alpha j}$ needed to specify the states α and j. In our notation, particle states labeled by a Latin letter (j) represent multiparticle states, while Greek letters (α) represent multiparticle states as well as the single-particle state which we label by a if the particle is on-mass-shell and $\underline{a}$ if it is off-mass-shell.

(3) The weak amplitudes $F_\alpha^{(\pm)I}$ and the strong-interaction transition amplitudes $T_{\alpha\beta}^{(\pm)I}$ are assumed to satisfy Hölder's condition in s except at the poles. This means that some positive constants M and m exist such that

$$|F_\alpha(s,t_\alpha)-F_\alpha(s',t_\alpha)|<M|s-s'|^m,$$
$$|T_{\alpha\beta}(s,t_{\alpha\beta})-F_{\alpha\beta}(s',t_{\alpha\beta})|<M|s-s'|^m\qquad(4)$$

for any pair of variables s and s' except at the poles. This implies that F_α and $T_{\alpha\beta}$ are continuous in s and bounded by a polynomial of degree less than m at infinity. If $m\geq 1$, the derivatives of F_α and $T_{\alpha\beta}$ with respect to s are constant, and this does not correspond to the physical situation of functions with a cut. Thus, we assume that $m<1$. Then F_α satisfies a dispersion relation with at most one subtraction. We write

$$F_\alpha(s,t_\alpha)=\tilde{F}_\alpha(s,t_\alpha)+\frac{1}{\pi}\int\frac{ds'}{s'-s-i\epsilon}\,\mathrm{Ab}\,F_\alpha(s',t_\alpha)\qquad(5)$$

in the case of unsubtracted dispersion relations and

$$F_\alpha(s,t_\alpha)=F_\alpha(0,t_\alpha)+\tilde{F}_\alpha(s,t_\alpha)-\tilde{F}_\alpha(0,t_\alpha)+\frac{s}{\pi}\int\frac{ds'}{s'-s-i\epsilon}\frac{1}{s'}\,\mathrm{Ab}\,F_\alpha(s',t_\alpha)\qquad(6)$$

in the case of one subtraction made at the physical point $s=0$. $\tilde{F}_\alpha$ is the possible pole term.

The main result of our analysis can be stated as follows. If the weak amplitude $F_\alpha^{(\pm)I}$ has no pole and obeys an unsubtracted dispersion relation in s, it must vanish. In the case of the CP-conserving parity-violating $\Delta I=\frac{1}{2}$ interaction, the K^0 (or $\bar{K}^0$) is the pole, and in the case of parity-conserving $\Delta I=\frac{1}{2}$ interaction, the κ^0 (or $\bar{\kappa}^0$) resonance may be a pole. In these two cases, the weak amplitudes $F_\alpha^{(\pm)1/2}$ are found to be related linearly to the pole term. In the cases $\Delta I=\frac{3}{2},\frac{5}{2}$, etc., no scalar particles or resonance are known to exist. We deduce that the $\Delta I\geq\frac{3}{2}$ weak amplitudes either do not exist, or, at least some of them, must obey subtracted dispersion relations. In such a case, the unsubtracted amplitudes may be related to the subtraction constants and may thus be arbitrary.

In Sec. II we reduce the problem of determining the weak amplitudes to that of solving a Hilbert boundary-value problem. This Hilbert problem is studied in detail in Sec. III by solving first the corresponding homogeneous problem and then the inhomogeneous one. The solution is found to be unique if it satisfies our assumptions.

Our result bears some similarity to an earlier work of Nishijima.[11] In that work, an eigenvalue problem for strong interactions is obtained for each set of the weak amplitudes satisfying the selection rules $\Delta I=\frac{1}{2},\frac{1}{3}$, etc., when assuming unsubtracted dispersion relations for the weak amplitudes combined with the unitarity and charge independence. The existence of the $\Delta I=\frac{1}{2}$ amplitudes gives some relations among the strong-interaction coupling constants and leads to the vanishing of the $\Delta I=\frac{3}{2},\frac{5}{2}$, etc., amplitudes. Other works must be mentioned in this context: Riazuddin[12] and, independently, Katz and Tatur[13] have used the unsubtracted dispersion relations for the weak amplitudes in a technique similar to that of Li and Pagels[14] in order to derive some conditions on the strong interactions and the octet dominance of nonleptonic decays. Our approach is completely different from those cited above although we arrive at a similar conclusion regarding the weak amplitudes for the various values of ΔI.

II. THE HILBERT PROBLEM FOR THE WEAK AMPLITUDES

The unitarity relation (3) relates only the weak amplitudes of the same parity and isospin. This allows us to study the problem of all the amplitudes of the same parity and isospin separately. In the following, the parity and the isospin indices and the explicit dependence of the amplitudes on the auxiliary variables t_α are omitted. The unitarity relation (3) can be written in the form

$$\mathrm{Abs}\,F_\alpha(s)=\tfrac{1}{2}\sum_j M_{\alpha j}(s)F_j(s)\;,\qquad(7)$$

where we have defined the strong-interaction S-matrix elements by

$$S_{\alpha\beta}=\delta_{\alpha\beta}-iM_{\alpha\beta}\,,\qquad M_{\alpha\beta}(s)=(2\pi)^4\delta^4(p_\alpha-p_\beta)T_{\alpha\beta}(s)\,. \tag{8}$$

The summation which includes the phase-space integration over the states j makes both sides of (7) and all equations where state summation is involved depend on the same kinematical variables s and t. The dispersion relation (5) may be written in the form

$$F_\alpha(s)=\tilde{F}_\alpha(s)+\frac{1}{2\pi}\sum_j\int_{s_j}^{\infty}\frac{ds'}{s'-s-i\epsilon}M_{\alpha j}(s')F_j(s')\,, \tag{9}$$

where $s_j=(\sum m_j)^2$ is the threshold of the state $|j\rangle$ and the pole term $\tilde{F}_\alpha(s)$ is of the form

$$\tilde{F}_\alpha(s)=g_\alpha/(\mu^2-s)\,, \tag{10}$$

where g_α depends on the auxiliary variables t_α. We assume that the contribution of the particle a to the amplitude of the same quantum numbers is of the form (10) whether the particle is stable (μ real) or unstable (μ complex with a negative imaginary part $-i\Gamma$, where Γ is the width of the resonance). The resonance is treated, in this context, as a state different from the multiparticle states to which it may decay.

The meaning of the residue g_α becomes evident if we consider the contribution of the one-particle state to Abs $F_\alpha(s)$. We find

$$\pi\delta(s-\mu^2)T_{\alpha a}(s=\mu^2)F_a\quad(p_\alpha=p_a)\,, \tag{11}$$

where F_a is the one-particle weak amplitude

$$\langle a(p)|\mathcal{H}_w|0\rangle=F_a \tag{12}$$

and the contribution of (11) to the dispersion integral is given by (10) with

$$g_\alpha=F_a T_{\alpha a}(s=\mu^2)\quad(p_\alpha=p_a)\,. \tag{13}$$

Let us define the sectionally analytic function

$$\phi_\alpha(z)=\frac{1}{2\pi}\sum_j\int_{-\infty}^{\infty}\frac{ds}{s-z}M_{\alpha j}(s)F_j(s)\theta(s-s_j)\,, \tag{14}$$

where the integration has been extended formally to the whole real axis as the absorptive part due to the multiparticle state $|j\rangle$ vanishes for $s<s_j$. This fact is made explicit by the step function defined as

$$\theta(s-s_j)=0\ \text{ if } s\leq s_j\,,\qquad =1\ \text{ if } s>s_j\,. \tag{15}$$

$\phi_\alpha(z)$ has a cut along the real axis starting at the lowest threshold s_j up to ∞. Let $\phi_\alpha^{(+)}(s)$ [or $\phi_\alpha^{(-)}(s)$] be the limits of $\phi_\alpha(z)$ as z approaches a point s on the real axis from the upper (or the lower) half of the complex z plane, respectively. The weak amplitudes $F_\alpha(s)$ are related to $\phi_\alpha^{(+)}(s)$ by Eq. (10) as

$$F_\alpha(s)=\tilde{F}_\alpha(s)+\phi_\alpha^{(+)}(s) \tag{16}$$

and the discontinuity of $\phi_\alpha(z)$ on the real axis is given by

$$\phi_\alpha^{(+)}(s)-\phi_\alpha^{(-)}(s)=i\sum_j M_{\alpha j}(s)\phi_j^{(+)}(s)\theta(s-s_j)\,, \tag{17}$$

or, using Eq. (16),

$$\phi_\alpha^{(+)}(s)-\phi_\alpha^{(-)}(s)=f_\alpha(s)+i\sum_j M_{\alpha j}(s)\phi_j^{(+)}(s)\theta(s-s_j)\,, \tag{18}$$

where we have defined

$$f_\alpha(s)=i\sum_j M_{\alpha j}(s)\tilde{F}_j(s)\theta(s-s_j)=\frac{i}{\mu^2-s}\sum_j M_{\alpha j}(s)g_j\theta(s-s_j)\,, \tag{19}$$

which are known functions in terms of g_j and the strong-interaction transition amplitudes.

In the case of the one-particle state, the weak amplitude (12) is generalized to off-mass-shell particle $\underline{a}$ in a way similar to the pion decay form factor,[15] for instance, to become

$$\langle\underline{a}(p)|\mathcal{H}_w|0\rangle=F_{\underline{a}}(p)\,. \tag{20}$$

Then (18) gives the discontinuity of $\phi_a(s)$ along the cut.

We define the G matrix by

$$\begin{aligned}G_{ij}(s)&=\delta_{ij}-iM_{ij}(s)\theta(s-s_j)\,,\\ G_{i\underline{a}}(s)&=0\,,\\ G_{\underline{a}i}(s)&=-iM_{\underline{a}j}(s)\theta(s-s_j)\,,\\ G_{\underline{a}\underline{a}'}(s)&=\delta_{\underline{a}\underline{a}'}\,.\end{aligned} \tag{21}$$

This allows us to write Eq. (18) in the form

$$G(s)\phi^{(+)}(s)-\phi^{(-)}(s)=f(s)\,. \tag{22}$$

The problem of the determination of the weak amplitudes $F_\alpha(s)$ is equivalent to finding $\phi_\alpha(z)$ which are analytic in the cut z plane and which satisfy the inhomogeneous Hilbert boundary-value problem (22) on the real axis with the supplementary condition at infinity

$$\phi(z)\underset{z\to\infty}{\longrightarrow}0\,. \tag{23}$$

The matrix $G(s)$ as well as the functions $f(s)$ satisfy Hölder's conditions except, perhaps, at $s=\mu^2$. However, if the particle a is stable, we have necessarily $\mu^2<s_j$ for any multiparticle state

$|j\rangle$, and $f_\alpha(s)$ as given by Eq. (19) vanishes below s_j. Similarly, $G_{\alpha\beta}$ becomes identical to $\delta_{\alpha\beta}$ for $s< s_j$. Thus the point $s=\mu^2$ is not a point of discontinuity for f_α or $G_{\alpha\beta}$. In the case of an unstable particle a, the corresponding pole lies off the real axis. We conclude that in both cases, $G_{\alpha\beta}$ and f_α satisfy Hölder's condition on the real axis by virtue of our assumption (3). Consequently, the classical methods of the study of the Hilbert problem with continuous coefficients can be applied.

It is easy to verify that

$$G_{\alpha\beta}(s)=\delta_{\alpha\beta}-iM_{\alpha\beta}(s)\theta(s-s_\beta). \tag{24}$$

In the following, the matrix elements $G_{\alpha\beta}$ appear only acting on functions in the form $G_{\alpha\beta}\phi_\beta$. In such a sum over intermediate states, only those with $s>s_\beta$ contribute. This makes $G_{\alpha\beta}$ completely identical to the S-matrix elements. Hence we are able to generalize it to complex values of s.

The inverse matrix of G exists and it is given by

$$G_{\alpha\beta}{}^{-1}(s)=\delta_{\alpha\beta}+iM_{\beta\alpha}\theta(s-s_\beta). \tag{25}$$

This shows that $G^-(s)$ is identical to the inverse of the strong-interaction S matrix [Eq. (8)]. In the following we shall identify completely the G matrix with the S matrix.

III. STUDY OF THE HILBERT PROBLEM

The Hilbert problem for several unknown functions has been studied in the case of a contour which bounds a finite and connected region in the complex z plane.[16,17] The case of an open contour may be reduced to the previous case by completing the open contour by a curve on which $G_{\alpha\beta}\equiv\delta_{\alpha\beta}$ and $f_\alpha\equiv 0$.

Let L be the part of the real axis situated between $-R$ and $+R$, where R has a large but finite value. We close L by the semicircle C of radius R in the upper half of the complex z plane. Let Γ be the union of C and L, D^+ be the region bounded by Γ, and D^- be the complement of $D^+ + L$ in the entire z plane (Fig. 1). It is evident that the solution of the Hilbert problem (22) is the limit of the solution of the following Hilbert problem:

$$\hat{G}(s)\phi^{(+)}(s)-\phi^{(-)}(s)=\hat{f}(s),\quad \text{on } \Gamma \tag{26}$$

where

$$\begin{aligned}&\hat{G}=G,\ \hat{f}=f \ \text{ if } s\in L\\ &\hat{G}=1,\ \hat{f}=0 \ \text{ if } s\in C \text{ or } |s|>R\,.\end{aligned} \tag{27}$$

Furthermore, the matching at z on the circle C may be made smoothly in such a way that $\hat{G}_{\alpha\beta}$ and $\hat{f}_\alpha$ always obey Hölder's conditions in the neighborhood of $|z|=R$.

In order to solve the inhomogeneous Hilbert problem (26), we solve first the homogeneous problem obtained by taking $\hat{f}=0$,

$$\hat{G}(s)\hat{\psi}^{(+)}(s)-\hat{\psi}^{(-)}(s)=0 \quad \text{on } \Gamma, \tag{28}$$

with the general condition at infinity

$$\hat{\psi}_\alpha(z)\to\gamma_\alpha(z), \tag{29}$$

where $\gamma_\alpha(z)$ are polynomials in z. For this, we transform the Hilbert problem (28) into two integral equations which may be solved. Once the general solution of the homogeneous Hilbert problem is known, the solution of the inhomogeneous problem is easy to obtain.

A. Transformation of the Hilbert problem to a system of integral equations

Using Cauchy's theorem, we may write

$$\begin{aligned}&\hat{\psi}(z)=\frac{1}{2i\pi}\int_\Gamma\frac{dz'}{z'-z}\,\hat{\psi}^{(+)}(z'),\quad z\in D^+\\ &0=\frac{1}{2i\pi}\int_\Gamma\frac{dz'}{z'-z}\,\hat{\psi}^{(+)}(z'),\quad z\in D^-\\ &\hat{\psi}(z)=\gamma(z)-\frac{1}{2i\pi}\int_\Gamma\frac{dz'}{z'-z}\,\hat{\psi}^{(-)}(z'),\quad z\in D^-\\ &0=\gamma(z)-\frac{1}{2i\pi}\int_\Gamma\frac{dz'}{z'-z}\,\hat{\psi}^{(-)}(z'),\quad z\in D^+\,.\end{aligned} \tag{30}$$

In particular, for s real, we obtain

$$\hat{\psi}^{(+)}(s)=\frac{P}{i\pi}\int_\Gamma\frac{dz}{z-s}\,\hat{\psi}^{(+)}(z), \tag{31}$$

$$\hat{\psi}^{(-)}(s)=2\gamma(s)-\frac{P}{i\pi}\int_\Gamma\frac{dz}{z-s}\,\hat{\psi}^{(-)}(z). \tag{32}$$

Using Eq. (28), we may write Eq. (32) in the form

$$\hat{G}(s)\hat{\psi}^{(+)}(s)=2\gamma(s)-\frac{P}{i\pi}\int_\Gamma\frac{dz}{z-s}\,\hat{G}(z)\hat{\psi}^{(+)}(z) \tag{33}$$

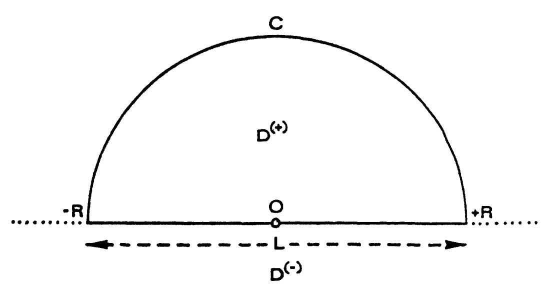

FIG. 1. Transformation of the Hilbert problem on the real axis to the Hilbert problem on $\Gamma=C+L$. On L, we have $\hat{G}_{\alpha\beta}=G_{\alpha\beta}$ and $\hat{f}_\alpha=f_\alpha$. On C and for s real and such that $|s|>R$, we take $\hat{G}_{\alpha\beta}=\delta_{\alpha\beta}$ and $\hat{f}_\alpha=0$.

This may be combined with (31) to obtain the integral equation

$$\hat{\psi}^{(+)}(s) = \hat{G}^{-1}(s)\gamma(s) + \frac{P}{i\pi}\int_\Gamma \frac{dz}{z-s}[1-\hat{G}^{-1}(s)\hat{G}(z)]\hat{\psi}^{(+)}(z). \tag{34}$$

Similarly, we find the equation for $\hat{\psi}^{(-)}$ to be

$$\hat{\psi}^{(-)}(s) = \gamma(s) + \frac{P}{2i\pi}\int_\Gamma \frac{dz}{z-s}[\hat{G}(s)\hat{G}^{-1}(z)-1]\hat{\psi}^{(-)}(z). \tag{35}$$

We have shown that every solution of Hilbert's problem (28) satisfies the integral equations (34) and (35). Conversely, if $\hat{\psi}^{(+)}$ and $\hat{\psi}^{(-)}$ are solutions of (34) and (35), respectively, we have

$$\hat{G}(s)\hat{\psi}^{(+)}(s) - \hat{\psi}^{(-)}(s) = \frac{P}{2i\pi}\int_\Gamma \frac{dz}{z-s}[\hat{G}(s)\hat{G}^{-1}(z)-1] \times [G(z)\psi^{(+)}(z) - \psi^{(-)}(z)] . \tag{36}$$

This implies that $\hat{\psi}(z)$ is a solution of the Hilbert problem if the integral equation

$$\phi(s) = \frac{P}{2i\pi}\int_\Gamma \frac{dz}{z-s}[\hat{G}(s)\hat{G}^{-1}(z)-1]\phi(z) \tag{37}$$

has only the trivial solution $\phi = 0$. This will be discussed later.

B. Study of the integral equations

Let us study the integral equation (35). It is of the weakly singular type of the form

$$\hat{\psi}^{(-)}(s) = \gamma(s) + \lambda\int_\Gamma dz\, N(s,z)\psi^{(-)}(z), \tag{38}$$

where the matrix kernel $N(s,z)$ is given by

$$N(s,z) = \frac{1}{2i\pi}\frac{1}{z-s}[\hat{G}(s)\hat{G}^{-1}(z)-1] = \frac{n(s,z)}{(z-s)^r}, \tag{39}$$

where $r < 1$ and the constant λ is equal to 1. This equation may be iterated to become less singular. The $(p-1)$-fold iterated equation is

$$\hat{\psi}^{(-)}(s) = \gamma^{(p)}(s) + \lambda^p\int_\Gamma dz\, N^{(p)}(s,z)\psi^{(-)}(z), \tag{40}$$

where we have defined

$$N^{(p)}(s,z) = \int_\Gamma N^{(p-1)}(s,u)N(u,z)du = \int\cdots\int_\Gamma du_1\cdots du_{p-1}N(s_1u_1) \times N(u_1,u_2)\cdots N(u_{p-1},z), \tag{41}$$

$$\gamma^{(p)}(s) = \gamma(s) + \sum_{q=1}^{p-1}\lambda^q\int_\Gamma dz\, N^{(q)}(s,z)\gamma(z). \tag{42}$$

If, for instance, $n(s,z)$ is bounded on Γ, the $(p-1)$-fold iterated kernel $N^{(p)}(s,z)$ is bounded by $c/(z-s)^{pr+1-p}$. This shows that if $p > 1/(1-r)$, $N^{(p)}(s,z)$ is regular.

The solution of Eq. (40) is well known.[18] It is of the form

$$\psi^{(-)}(s) = \gamma^{(p)}(s) + \frac{\lambda^p}{D(\lambda^p)}\int_\Gamma \Delta(s,z,\lambda^p)\gamma^{(p)}(z)dz, \tag{43}$$

where $D(\lambda^p)$ is the Fredholm determinant and Δ is the first Fredholm minor. $D(\lambda^p)$ and $\Delta(t,z,\lambda^p)$ can be written as power series in $(\lambda^p)^n$ which are convergent for any λ. The solution (43) has a meaning only if $D(\lambda^p) \neq 0$, in which case λ^p is said to be a regular value of $N^{(p)}(t,z)$. In our case λ^p is equal to 1 and we assume that this is a regular value. This condition is sufficient for the integral equation (35) to have a solution and this solution is unique (Fredholm's first theorem).

On the other hand, if $\lambda^p = 1$ is a regular value of $N^{(p)}(s,z)$, Fredholm's second theorem states that the homogeneous integral equation (37) has only the trivial solution $\phi = 0$. This completes the proof of the equivalence of the Hilbert problem (28) with the asymptotic behavior (29) and the integral equations (34) and (35).

C. General solution of the homogeneous Hilbert problem

In the case $\lambda^p = 1$, using (42), the solution (43) may be written in the form

$$\hat{\psi}^{(-)}(s) = \gamma(s) + \int_\Gamma dz\, \mathfrak{N}(s,z)\gamma(z), \tag{44}$$

where we have defined

$$\mathfrak{N}(s,z) = \sum_{q=1}^{p-1}N^{(q)}(s,z) + \frac{1}{D(1)}\Delta(s,z,1) + \frac{1}{D(1)}\sum_{q=1}^{p-1}\int_\Gamma du\,\Delta(s,u,1)N(u,z). \tag{45}$$

We may look for solutions of the Hilbert problem (28) of three types:

(a) Solutions which remain constant as $z \to \infty$.

These correspond to γ_α equal to constants in (44). Assuming that the number of weak amplitudes is finite n and taking for γ_α the special value given by

$$\gamma_\alpha^{(\beta)} = (0, 0, 0, \ldots, 0, 1, 0, \ldots, 0) = \delta_\alpha^\beta, \tag{46}$$

which corresponds to the solution $\psi^{(\beta)}$ satisfying $\psi_\beta^{(\beta)} \to 1$ and $\psi_\alpha^{(\beta)} \to 0$ if $\alpha \neq \beta$, we get for such a solution

$$\psi_\alpha^{(\beta)}(s) = \delta_\alpha^\beta + \int_\Gamma dz\, \mathfrak{N}_{\alpha\beta}(s, z)\,. \tag{47}$$

The general solution of the Hilbert problem (28) which remains bounded at infinity is a linear combination of such solutions; i.e.,

$$\hat{\psi}_\alpha^{(-)} = \sum_\beta C_\beta \psi_\alpha^{(\beta)} = C_\alpha + C_\beta \int_\Gamma dz\, \mathfrak{N}_{\alpha\beta}(s, z), \tag{48}$$

where C_α are arbitrary coefficients.

(b) Solutions which vanish at infinity. Let us consider the solutions which behave like

$$\hat{\psi}_\alpha(z) \to z^{-k}, \quad k > 0\,. \tag{49}$$

If the homogeneous Hilbert problem (28) has such a solution, then any function of the form $P(z)\hat{\psi}_\alpha(z)$, where $P(z)$ is an arbitrary polynomial is also a solution. Particularly, the functions ψ_α, $z\psi_\alpha, \ldots, z^{k-1}\psi_\alpha$ are solutions which vanish at infinity. We deduce that the functions $\psi_\alpha^{(-)}(s)$, $s\psi_\alpha^{(-)}(s), \ldots, s^{k-1}\psi_\alpha^{(-)}(s)$ are solutions of the integral equation (35), which is equivalent to the Hilbert problem, but with $\gamma_\alpha = 0$. However, this becomes identical to the integral equation (37) for which we have no solution [if $\lambda^p = 1$ is a regular value of $N^{(p)}(s, z)$]. We deduce that the homogeneous Hilbert problem (28) has no solutions which vanish at infinity.

(c) Solutions which have a polynomial as principal value at infinity. Such solutions are given by (44) with $\gamma_\alpha(s)$ equal to the principal value of $\hat{\psi}_\alpha$ at infinity.

We deduce that, within our assumptions, the most general solution of the homogeneous Hilbert problem (28) is given by Eq. (44) with $\gamma_\alpha(s)$ equal to an arbitrary polynomial. The solutions which have the lowest principal values at infinity correspond to γ_α equal to constants. The n independent solutions (47) form the fundamental system of solutions. We define the fundamental matrix X by

$$X_{\alpha\beta}(s) = \psi_\alpha^{(\beta)}(s)\,. \tag{50}$$

This obeys the Hilbert equation

$$G(s)X^+(s) = X^-(0) \quad \text{on } \Gamma\,. \tag{51}$$

Assuming that the inverse matrix X^{-1} exists, we may write

$$G(s) = X^{(-)}(s)[X^{(+)}]^{-1}\,. \tag{52}$$

D. Solution of the inhomogeneous Hilbert problem

Using Eq. (52), we may write the inhomogeneous Hilbert equation (27) in the form

$$[X^{-1}\hat{\phi}]^{(+)} - [X^{-1}\phi]^{(-)} = [X^{-1}]^{(-)} f \quad \text{on } \Gamma\,. \tag{53}$$

This has the general solution

$$X^{-1}(z)\hat{\phi}(z) = \frac{1}{2i\pi}\int_\Gamma \frac{dz'}{z'-z}[X^{(-)}(z')]^{-1}\hat{f}(z') + P(z), \tag{54}$$

where $P(z)$ is an arbitrary polynomial. We may write also

$$\hat{\phi}(z) = \frac{1}{2i\pi}X(z)\int_\Gamma \frac{dz'}{z'-z}[X^{(-)}(z')]^{-1}\hat{f}(z') + X(z)P(z)\,. \tag{55}$$

Taking the limit $R \to \infty$, the Hilbert problems (27) and (22) become identical and the solution of (22) is the limit of (55) as $R \to \infty$; i.e.,

$$\phi(z) = \frac{1}{2i\pi}X(z)\int_\Gamma \frac{dz'}{z'-z}[X^{(-)}(z')]^{-1}f(z') + X(z)P(z)\,. \tag{56}$$

As $z \to \infty$, $X_{\alpha\beta} \to \delta_{\alpha\beta}$ and the contribution of the semicircle C to the integral in (56) vanishes if $f_\alpha(z)$, i.e., the strong-interaction transition amplitudes, vanish at infinity. The supplementary condition $\phi(z) \to 0$ as $z \to \infty$ is satisfied only if the arbitrary polynomials $P(z)$ in (56) are taken to be identically zero. Thus the solution of the Hilbert problem (22) which vanishes at infinity is given by

$$\phi(z) = \frac{1}{2i\pi}X(z)\int_{-\infty}^{\infty} \frac{dz'}{z'-z}[X^{(-)}(z')]^{-1}f(z') \tag{57}$$

and the weak amplitudes (16) become

$$F(s) = \tilde{F}(s) + \tfrac{1}{2}G^{-1}(s)f(s) + \frac{1}{2i\pi}X^{(+)}(s)\,\mathrm{P}\int_{-\infty}^{\infty}\frac{dz}{z-s}[X^{(-)}(z)]^{-1}f(z), \tag{58}$$

or, using Eq. (19), we get for the physical amplitude $(s=0)$

$$F_\alpha(0) = \tilde{F}_\alpha(0) + \frac{1}{2\pi}X_{\alpha\beta}^{(+)}(0)\,\mathrm{P}\int_{s_j}^{\infty}\frac{ds}{s}[X^{(-)}(s)]_{\beta\tau}{}^{-1}M_{\tau j}(s)\tilde{F}_j(s)\,.$$

IV. CONCLUSION

We have shown that the solution of the Hilbert problem (22) is unique and is given by Eq. (59) in terms of the pole term $\bar{F}_\alpha$ if such a term exists. If there is no pole term, the Hilbert problem (22) becomes homogeneous and our discussion of Sec. III shows that this problem has no solution which vanishes at infinity, and the weak amplitudes either do not exist or they must obey subtracted dispersion relations. Thus, only the weak amplitudes which have a pole term are consistent with the assumption of unsubtracted dispersion relations. This is the case of the $\Delta I = \frac{1}{2}$ parity-violating amplitudes (K^0 and $\bar{K}^0$ poles) and the $\Delta I = \frac{1}{2}$ parity-conserving amplitude (κ^0 and $\bar{\kappa}^0$ poles). The other isospin transition amplitudes either do not exist or they must obey subtracted dispersion relations.

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