Broken-mass solutions to superconvergence relations

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We use an ansatz motivated by duality and the quark model to saturate superconvergence relations near $t=0$ for scattering of mesons in the vector and pseudoscalar nonets. The saturation is carried out with a few low-lying states, using the observed mass spectrum rather than the degenerate masses characteristic of a nonet—or SU(6)—symmetric model.

I. INTRODUCTION

Historically, rapid saturation schemes for meson-meson superconvergence relations (SCR's) have led to unpleasant consequences, namely degenerate masses for members of the vector-meson nonet.[1-3] In this paper we suggest a scheme which uses rapid saturation (that is, saturation with a small number of low-lying resonances), yet avoids the degenerate masses.

In the present section we first review the procedure for expressing SCR's in a helicity formalism; readers familiar with this formalism can skim Eqs. (1.1)–(1.10) for the notation. We then introduce and motivate the assumptions of the model. One of these assumptions is that s-channel couplings may be split up into an "st part" and an "su part." In Sec. II we show how to calculate the su part, given the st part. Then in Sec. III we solve some SCR's for meson-meson scattering, inserting only the lowest-lying resonances (those with quark orbital angular momentum $\mathcal{L}=0$), yet keeping the masses nondegenerate. In Sec. IV we ask whether the coupling scheme presented in Sec. III is unique; we find that there exist two further schemes which also produce rapid superconvergence. Section V contains a discussion of the W-spin and SU(3) properties of each coupling scheme (both our original scheme and the two new schemes introduced in Sec. IV). An appendix discusses some details of s-to-u-channel helicity crossing.

A helicity formalism seems best suited for bringing out the coupling structure which produces superconvergence. We therefore begin by reviewing briefly the usual procedure for constructing SCR's from helicity amplitudes.[4]

s-channel center-of-mass helicity amplitudes can be expanded as a sum of invariant M functions times factors depending on the center-of-mass scattering angle θ_s and three-momentum p. Consequently helicity amplitudes, considered as functions of s and t, possess kinematical singularities (coming from the square roots in $\cos\theta_s$, p, etc.) in addition to the dynamical singularities predicted by the Mandelstam representation.[5] Since SCR are essentially dispersion integrals in s, we must remove the kinematical singularities in s before we can disperse. Although we are interested in computing s-channel amplitudes $\langle\lambda_3'\lambda_4'\theta_s|H_s|\lambda_1'\lambda_2'\rangle$, it is somewhat easier to remove s kinematical singularities from a t-channel amplitude, so we first cross to the t channel by applying an s-to-t helicity crossing matrix,[6]

$$\langle\lambda_3\lambda_1\theta_t|H_t|\lambda_4\lambda_2\rangle=\prod_{i=1}^{4} d^{s_i}_{\lambda_i'\lambda_i}(x_i)\langle\lambda_3'\lambda_4'\theta_s|H_s|\lambda_1'\lambda_2'\rangle . \tag{1.1}$$

Wherever there is danger of confusing t- and s-channel helicities, we shall put primes on the latter. The s kinematical singularities of H_t are readily deduced from its partial-wave expansion:

$$\langle\lambda_3\lambda_1\theta_t|H_t|\lambda_4\lambda_2\rangle=\sum_J(2J+1)\,d^J_{\lambda\mu}(\theta_t)\times\langle\lambda_3\lambda_1|H_t^J|\lambda_4\lambda_2\rangle . \tag{1.2}$$

Equation (1.2) exhibits the s dependence of H_t explicitly in the d^J's, all of which have the form

$$d^J_{\lambda\mu}=(1-\cos\theta_t)^{|\lambda-\mu|/2}(1+\cos\theta_t)^{|\lambda+\mu|/2}\,\mathcal{P}_J(\cos\theta_t) . \tag{1.3}$$

λ and μ are the spin projections along the initial and final beam direction:

$$\lambda=\lambda_4-\lambda_2,\quad \mu=\lambda_3-\lambda_1 . \tag{1.4}$$

In Eq. (1.3), $\mathcal{P}_J$ is a power series in $\cos\theta_t$ and therefore a power series in s, since $\cos\theta_t$ is linear in s. The $1\pm\cos\theta_t$ terms contain various square-root, etc., kinematic singularities involving s; but these factors may be divided out of the sum (1.2), since each d^J contains the same $(1\pm\cos\theta_t)$ factors. What remains is a power series in s which can have singularities where it diverges, presumably only at dynamical singularities in s[7]:

$$\langle \lambda_3\lambda_1\theta_t|\bar{H}_t|\lambda_4\lambda_2\rangle \equiv \frac{\langle \lambda_3\lambda_1\theta_t|H_t|\lambda_4\lambda_2\rangle}{(\sin\theta_t/2)^{|\lambda-\mu|}(\cos\theta_t/2)^{|\lambda+\mu|}} . \tag{1.5}$$

We have written the $1\pm\cos\theta_t$ factors more compactly by using the half-angle identities $2\sin^2\theta_t/2 = 1-\cos\theta_t$, $2\cos^2\theta_t/2 = 1+\cos\theta_t$. In order to write a dispersion integral for $\bar{H}_t$, we need its asymptotic behavior for large s and small t. From Regge asymptotics for H_t plus the linearity of $\cos\theta_t$ in s, we get

$$\begin{aligned}&\langle \lambda_3\lambda_1|\bar{H}_t|\lambda_4\lambda_2\rangle \sim s^{\alpha(t)-Mx} ,\\ &Mx = \max(|\lambda|, |\mu|) ,\\ &\alpha(t) = \text{leading Regge trajectory} .\end{aligned} \tag{1.6}$$

Suppose $\alpha(t)-Mx$ is so negative that an unsubtracted dispersion integral for $s\bar{H}_t$ converges:

$$(1/\pi)\int_{\text{cuts}} ds' \operatorname{Im}[s'\langle\lambda_3\lambda_1|\bar{H}_t(s',t)|\lambda_4\lambda_2\rangle]/(s'-s) = s\langle\lambda_3\lambda_1|\bar{H}_t(s,t)|\lambda_4\lambda_2\rangle . \tag{1.7}$$

Then $\bar{H}_t$ is said to superconverge, and one gets a sum rule, or superconvergence relation, by taking $s=0$ in Eq. (1.7):

$$\int ds' \operatorname{Im}\langle\lambda_3\lambda_1|\bar{H}_t(s',t)|\lambda_4\lambda_2\rangle = 0 . \tag{1.8}$$

[Actually Eq. (1.8) is an infinite set of sum rules, one for each t.]

We say that an SCR is of the "spin" type if the t channel is nonexotic [$\alpha(t)\simeq\frac{1}{2}$], and $\alpha(t)-Mx$ in Eq. (1.6) is small because Mx is large ($Mx>\frac{3}{2}$). We say that an SCR is of the "SU(3)" type if the t channel is exotic and $\alpha(t)-Mx$ is small because $\alpha(t)$ is small.

We shall work within the usual sharp-resonance approximation, which assumes that the imaginary part on the right- (left-) hand cut is well approximated by a sum of s- (u-) channel Breit-Wigner resonances, with the imaginary part of the Breit-Wigner denominator approximated by a delta function:

$$\operatorname{Im}\langle\lambda_3\lambda_1|H_t|\lambda_4\lambda_2\rangle \simeq \prod_i d_{\lambda_i'\lambda_i}(\chi_i)\sum_J \pi\delta(s-s_J)\langle\lambda_1'\lambda_2'|H|\lambda'J\rangle\langle\lambda_3'\lambda_4'|H|\mu'J\rangle h(12\to J)h(34\to J)\,d^J_{\lambda'\mu'}(\theta_s) \quad (s>0) . \tag{1.9}$$

The integral in Eq. (1.8) is then trivial, and Eq. (1.8) reduces to an algebraic equation for the scalar couplings h, having the general form

$$\sum_J h(12\to J)h(34\to J)f_J(s_J, m_i^{\,2}, t) + \sum_J h(1\bar{4}\to J)h(3\bar{2}\to J)f_J'(u_J, m_i^{\,2}, t) = 0 . \tag{1.10}$$

The second sum comes from the crossed cut.

The functions f_J in Eq. (1.10) involve the $d^J(\theta_s)$, the helicity crossing matrices, and the $(1\pm\cos\theta_t)$ kinematic-singularity-removing factors. The f's are complicated functions, therefore, and it is not a good idea to try to solve Eq. (1.10) by brute-force methods. Some hints or guidance from models is essential.

For hints as to how to saturate spin SCR's, let us first review the usual procedure for satisfying SU(3) SCR's. Veneziano's original $\pi\pi\to\pi\omega$ dual amplitude (for example) may be extended to $\Pi\Pi\to\Pi V$ (Π and V members of the 0^- and 1^- nonets) by multiplying each beta function $B(ij)$ by the boxlike SU(3) tensors M_{ij} diagrammed in Fig. 1:[8-11]

$$A(\Pi\Pi\to\Pi V) = (M_{st}-\bar{M}_{st})B(s,t) + (\bar{M}_{su}-M_{su})B(s,u) + (\bar{M}_{tu}-M_{tu})B(t,u) .$$

Each corner of each box in Fig. 1 is an SU(3) Clebsch-Gordan coefficient linking quark-antiquark to the external meson. $\bar{M}_{ij}$ is identical to M_{ij} except that the quark loop is anticlockwise; in what follows we can ignore the distinction between M and $\bar{M}$. The contribution to Eq. (1.8) from each pole in $B(s,t)$ at $t=0$ will not spoil SU(3) superconvergence because the M_{st} box cannot contribute to exotic t channels. (When sliced down the middle vertically, M_{st} contains a quark-antiquark state in the t channel.) Each pole in $B(s,t)$ is multiplied by M_{st}, so that each level in the mass spectrum

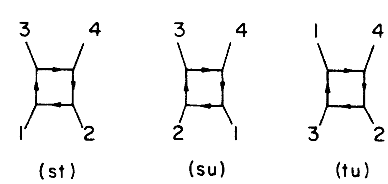

FIG. 1. Structure of M_{st}, M_{su}, and M_{tu}. All internal lines represent SU(3) quarks. Each trilinear vertex represents an SU(3) Clebsch-Gordan coefficient linking a quark-antiquark pair to the external meson.

superconverges by itself, and we have a theorist's dream, single-level superconvergence, the most rapid possible, similarly for the $B(t,u)$ term. The $B(s,u)$ term, however, is multiplied by an M_{su} which does contribute to exotic t channels. SU(3) superconvergence is preserved not by M_{su}, but by the asymptotic behavior of $B(t,u)$, which is $s^{\alpha(u)}$, negligible because u and $\alpha(u) \ll 0$. Many poles in $B(s,u)$ must contribute with alternating signs to produce this small resultant, so that presumably the superconvergence of $B(s,u)$ at $t=0$ is many-level and very slow.

The foregoing information about the SU(3) dependence of the amplitude suggests our first two assumptions about the spin dependence of the amplitude:

(1) s-channel couplings may be split into an st and an su part (similarly for crossed t- and u-channel couplings).

(2) st and su parts superconverge independently, and only the st part superconverges rapidly at $t \cong 0$ (similarly for the crossed couplings).

Assumptions (1) and (2) imply that we must construct s-channel couplings which separate naturally into st and su parts and then insert only the st part into the SCR. (In practice the st and su parts are closely related by the requirements of charge-conjugation invariance and factorization; therefore, determining the st part determines the total coupling.)

In making these assumptions, we in effect are saying that the spin superconvergence properties of the st and su parts are similar to their SU(3) superconvergence properties, at least near $t=0$. We can give an argument which suggests that rapid superconvergence, if it occurs at all, occurs nearer $t=0$ than $t \ll 0$. For $t \ll 0$ the resonances contribute with alternating signs, and the superconvergence of the st part is slow (just as the superconvergence of the su part, discussed above, was slow for $u \ll 0$). As we move toward $t=0$, the Legendre polynomials P_{Lm} associated with each resonance become more coherent, and it becomes less likely that a nonsuperconvergent contribution from one resonance will be canceled by a corresponding contribution from another resonance. It becomes more likely that cancellations will occur between a small number of resonances having the same value of orbital angular momentum L, but contributing to the SCR with different signs because they have different values of total spin S or/and total angular momentum J.

We need more information than is supplied by the above assumptions (1) and (2). Among the meson-meson reactions with two or more external V's, there are eleven independent helicity configurations which satisfy $Mx=2$ and therefore spin superconverge. Which of the eleven are the ones that single-level superconverge? Also, which value of t in Eq. (2) is the rapid superconvergence value?

To answer these questions, we construct a specific model for the couplings, one motivated by the diagrams of Fig. 1. Consider the butterfly-shaped diagram of Fig. 2, which is meant to represent the spin dependence of the st part of the amplitude. (For simplicity, we postpone discussion of the su part and isospin dependence until Sec. II.) Each trilinear vertex in the diagram represents a Clebsch-Gordan coefficient, just as in Fig. 1, but now the Clebsch-Gordan coefficient couples spin rather than SU(3) multiplets. The body of the butterfly-shaped diagram is made up of quark and antiquark coupling in some manner with L (L'), the orbital angular momentum S_1S_2 (S_3S_4), to form angular momentum $\mathcal{L}$. If the diagram is cut in half horizontally, the intermediate values obtained are those in $\frac{1}{2}\otimes\frac{1}{2}\otimes\mathcal{L} \equiv \mathcal{S}\otimes\mathcal{L}$. These J values are just those predicted by the quark model, if we identify $\mathcal{S}$ ($=0$ or 1) and $\mathcal{L}$ with quark spin and quark orbital angular momentum, respectively. If the diagram is cut vertically through the two horizontal quark lines coming from S_1 and S_3 we find that possible values for intermediate quark total spin are ≤ 1; therefore, $|\lambda_3' - \lambda_1'| \leq 1$. This fact suggests spin superconvergence in the crossed t channel. [Of course, we want t-channel helicities $|\lambda_3 - \lambda_1| \leq 1$, not s-channel helicities $|\lambda_3' - \lambda_1'| \leq 1$, and s- and t-channel helicity amplitudes differ by quite a complex transformation, Eq. (1.1). Nevertheless, when we do cross to the t channel in Sec. III, we do find a zero in the $|\lambda_3 - \lambda_1| = 2$ amplitude.]

To see in more detail how the intermediate J values arise when the diagram is sliced horizontally, let us consider the simple case $\mathcal{L}=0$. Then spin-parity conservation requires $L=L'=1$, and the body of Fig. 2 reduces, as shown in Fig. 3(a). This diagram contains six Clebsch-Gordan coefficients, but we focus on the two forming the left

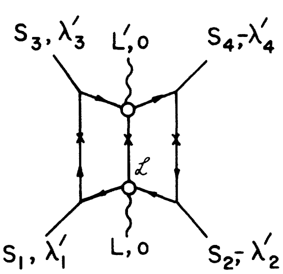

FIG. 2. st part of the meson-meson helicity amplitude. L (L') is the orbital angular momentum in the initial (final) state; $\mathcal{L}$ is quark orbital angular momentum. Lines with arrowheads are spin-$\frac{1}{2}$ quarks. x's denote rotation matrices.

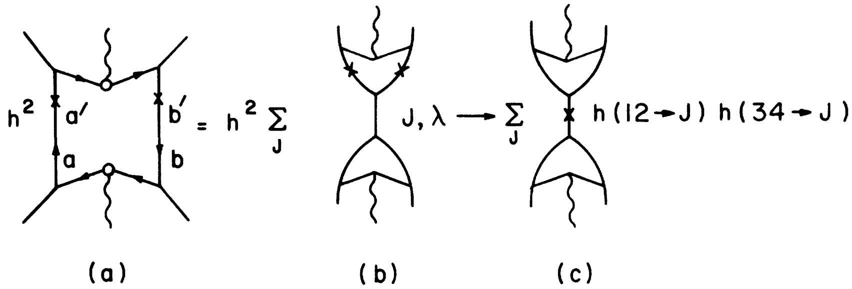

FIG. 3. (a) Diagram of Fig. 2 for the case $\mathcal{L}=0$; $L=L'=1$. (b) Diagram (a) with contributions of definite J projected out using Eq. (1.12). The couplings h are still degenerate. (c) Diagram (b) with each J assigned a different coupling strength.

edge of Fig. 3(a), and for clarity suppress all quark indices except those associated with quark line aa' (we use a bra-ket notation $\langle jm|j_1m_1j_2m_2\rangle$ for Clebsch-Gordan coefficients):

$$\cdots\langle S_1\lambda_1|\tfrac{1}{2}a\cdots\rangle\,\delta_{aa'}\,d^{1/2}_{a'a''}(\theta_s)\,\langle S_3\lambda_3|\tfrac{1}{2}a''\cdots\rangle\cdots. \tag{1.11}$$

The x on line aa' stands for the $d(\theta_s)$ rotation matrix in Eq. (1.11); the reason for inserting this matrix will become apparent shortly. We have anticipated a subsequent step by inserting a Kronecker delta $\delta_{aa'}$. We insert a similar Kronecker delta $\delta_{bb'}$ on the right-hand antiquark line. Then we replace the Kronecker delta's by using the unitarity of the Clebsch-Gordan coefficients:

$$\delta_{aa'}\,\delta_{bb'}=\sum_J\langle J\lambda'|\tfrac{1}{2}a\,\tfrac{1}{2}b\rangle\langle J\lambda'|\tfrac{1}{2}a'\,\tfrac{1}{2}b'\rangle. \tag{1.12}$$

The sum is over a complete set of intermediate states, i.e., $J=\frac{1}{2}\otimes\frac{1}{2}=1$ or 0. Equation (1.12) exhibits explicitly the contribution to the helicity amplitude from the intermediate pseudoscalar ($J=0$) and vector ($J=1$) mesons making up the $\mathcal{L}=0$, $\mathcal{S}=J=1$ or 0 level of the quark model. In diagrammatic language, Eq. (1.11) changes Fig. 3(a) into Fig. 3(b).

The upper J vertex in Fig. 3(b) has a rotation matrix on each of its quark legs. Using the rotation property of Clebsch-Gordan coefficients,

$$\langle J\lambda|\tfrac{1}{2}m\,\tfrac{1}{2}n\rangle\,d^{1/2}_{mm'}(\theta_s)\,d^{1/2}_{nn'}(\theta_s)=d^J_{\lambda\mu}(\theta_s)\langle J\mu|\tfrac{1}{2}m'\,\tfrac{1}{2}n'\rangle. \tag{1.13}$$

We can move these matrices through to the J line. We get just the $d^J_{\lambda\mu}$ characteristic of a helicity amplitude for a spin-J intermediate particle. This result explains why rotation matrices were inserted on all internal lines running vertically in Figs. 2 and 3.

Next we explain the azimuthal quantum numbers $\lambda_1, 0, -\lambda_2$ indicated for angular momenta S_1, L, S_2 in Fig. 2 (and subsequent figures). The vertex making up the lower half of Fig. 3(b) is one way of coupling the three angular momenta L, S_1, S_2 to form J. By the theory of angular momentum recoupling, this vertex is therefore some weighted linear combination of the Clebsch-Gordan coefficients $\langle L0S\lambda|J\lambda\rangle\times\langle S_1\lambda_1S_2-\lambda_2|S\lambda\rangle$, which arise when the angular momenta are recoupled in familiar "spin-orbit" order: $S_1\otimes S_2=$ total spin S; $S\otimes L=J$. From Eq. (B.5) of Jacob and Wick, helicity amplitudes can be expanded in a series of spin-orbit coupling amplitudes[12]:

$$\begin{aligned}\langle\lambda_3\lambda_4\theta_s|H_s|\lambda_1\lambda_2\rangle=\sum[(2L+1)(2L'+1)]^{1/2}\langle L0S\lambda|J\lambda\rangle\langle S_1\lambda_1S_2-\lambda_2|S\lambda\rangle\,d^J_{\lambda\mu}(\theta_s)\\ \times\langle L'S'|H^J|LS\rangle\langle L'0S'\mu|J\mu\rangle\langle S_3\lambda_3S_4-\lambda_4|S'\mu\rangle\,.\end{aligned} \tag{1.14}$$

Therefore, the top and bottom vertices diagrammed in Fig. 3(b) yield (after recoupling) a specific set of reduced matrix elements $\langle L'S'|H^J|LS\rangle$ in expansion (1.13). To get the Clebsch-Gordan coefficients right, we must choose the azimuthal indices in Figs. 2 and 3 to match the azimuthal indices in expansion (1.13), QED.

Since expansion (1.13) is valid relativistically, so is the coupling procedure of Fig. 3(b), even though the latter superficially appears nonrelativistic. Indeed, the nonrelativistic "feel" to helicity amplitudes coupled in a quark-loop or spin-orbit formalism may go a long way toward explaining the nonrelativistic feel of the quark model itself.

In Fig. 3(a) or Fig. 3(b), each J couples with the same strength h (as indicated by the over-all factor h^2 multiplying these figures). In Sec. III, where we solve the SCR's in detail, we start out initially with split couplings as well as split masses, as indicated in Fig. 3(c). To satisfy the SCR's, however, we find we must define reduced couplings

$$\bar{h}(12\to J)\equiv\frac{h(12\to J)}{[s_J-(m_1-m_2)^2]^{1/2}[s_J-(m_1+m_2)^2]^{1/2}}\,, \tag{1.15}$$

where s_J is the (mass)^2 of resonance J, and then we must set the $\bar{h}$'s equal:

$$\bar{h}(12\to J=0)\,\bar{h}(34\to J=0)=\bar{h}(12\to J=1)\,\bar{h}(34\to J=1)\ . \tag{1.16}$$

Thus the dynamics picks just that solution where the (reduced) couplings are not split. The dynamics therefore allows us to factor out the couplings $\bar{h}$ and recover the original butterfly-shaped diagram, Fig. 3(a).

The detailed investigation of our model in Sec. III suggests the following answer to a question raised earlier. [Which value of t in Eq. (1.8) gives the most rapid superconvergence?]

(3) When $Mx=|\lambda_1-\lambda_3|=2$, Eqs. (1.8) superconverge most rapidly at the point $t=(m_1-m_3)^2$.

We list this statement as one of our assumptions, rather than a consequence of our model. It is possible to insert resonances having only a single value of $\mathcal{L}$, pick (say) $t=0$ in Eq. (1.8), and wind up with equations which are internally consistent. Thus SCR's do not force us to pick $t=(m_1-m_3)^2$. However, Eqs. (1.9) assume the simple form (1.15) at this point [a remarkably simple form, considering the complexity of the original equations (1.9)]. Furthermore, $t=(m_1-m_3)^2$ is the only point we have been able to find where the masses can be left nondegenerate.

Our model also tells us which of the eleven helicity amplitudes superconverge rapidly—i.e., superconverge even when we insert only $\mathcal{L}=0$ resonances. We recall that the amplitudes (1.5) are free of kinematic s singularities, but may still possess kinematic t singularities. In particular, they may possess singularities of the form $[t-(m_1-m_3)^2]^{n/2}$, $n=$integer, since $t=(m_1-m_3)^2$ is a pseudothreshold in the t channel. From the discussion of kinematic t singularities given in the references, six out of the eleven meson-meson helicity amplitudes normally are constant at pseudothreshold ($n=0$), while the remaining five vanish as the square root ($n=1$). It is the first six which superconverge even when saturated with only one $\mathcal{L}$ value: With the choice of couplings suggested by Fig. 3(c) and Eq. (1.15), Eq. (1.10) for each of these SCR's adds up to a function of t which vanishes linearly at pseudothreshold. The f_J and f_J' in the remaining five SCR's each contain a factor $[t-(m_1-m_3)^2]^{1/2}$, but of course this is a kinematic singularity and must be divided out. The sum that remains does not vanish linearly, and presumably more than one $\mathcal{L}$ is needed to saturate these five SCR's. Thus the six helicity amplitudes which are "closest" to pseudothreshold [by one factor of $(t-(m_1-m_3)^2)^{1/2}$] are the ones which superconverge most rapidly.

The foregoing rule continues to work even for elastic reactions $m_1=m_3$ when the pseudothreshold singularities coalesce with the $t=0$ singularities: Amplitudes which are normally finite at $t=0$ superconverge rapidly; those which vanish as $t^{1/2}$ do not. The former amplitudes have $\sum\lambda_i=$even and are the "Class I" amplitudes of Gilman and Harari; the latter have $\sum\lambda_i=$odd and are their "Class II" amplitudes.[3]

Away from $t=0$, the rule "$\sum\lambda_i=$even" no longer characterizes the helicity amplitudes which superconverge rapidly. When helicity amplitudes are expanded in terms of M functions, the expansion coefficients have singularities in masses as well as in s and t; therefore, the expansion which works for $m_1\neq m_3$ breaks down for $m_1=m_3$, and vice versa. This circumstance explains why Hara *et al.* find one set of kinematical singularities for equal masses, and another set for unequal masses, with no smooth extrapolation connecting the two sets.[5] It is not particularly surprising, therefore, that the rule $\sum\lambda_i=$even for $m_1=m_3$ does not extrapolate to the $m_1\neq m_3$ case. It is more surprising that any simple rule extrapolates between the two cases. Of course, the equations (1.10) for the couplings also extrapolate smoothly in the masses, as they should because they are derived using kinematic singularity-free helicity amplitudes, rather than the helicity amplitudes themselves.

Equation (1.15) leads to a generalization of the Gilman-Kugler-Meshkov prescription for the mass dependence of meson decay rates.[13] Given the couplings defined in Eq. (1.9), we may calculate decay rates from Fermi's golden rule:

$$\Gamma(J\to 12)=\text{const}\times|\langle\lambda_1'\lambda_2'|H|\lambda'J\rangle\,h(12\to J)|^2P_{12}/s_J\ . \tag{1.17}$$

P_{12} is the three-momentum in the final state. From the results of Sec. V, the reduced couplings defined in Eq. (1.15) preserve W spin:

$$\langle\lambda_1'\lambda_2'|H|\lambda'J\rangle\,\bar{h}(12\to J)=\text{const}\times\langle W_1\,\lambda_1'\,W_2-\lambda_2'|\,W_J\lambda'\rangle\ . \tag{1.18}$$

Combining Eqs. (1.15), (1.17), and (1.18), we find

$$\Gamma(J\to 12)=\text{const}\times[s_J-(m_1+m_2)^2]\,[s_J-(m_1-m_2)^2]\,|\langle W_1\lambda_1' W_2-\lambda_2'|W_J\lambda'\rangle|^2 P_{12}/s_J\,. \tag{1.19}$$

In the limit that $m_2=m_\pi\ll m_1$, Eq. (1.19) becomes

$$\Gamma(J\to 12)\simeq \text{const} \times(s_J-m_1^2)^2\,|\langle W_1\lambda_1' W_2-\lambda_2'|W_J\lambda'\rangle|^2 P_{12}/s_J\,. \tag{1.20}$$

Equation (1.20) is just the prescription derived for pion decays by Gilman, Kugler, and Meshkov using the Melosh transformation, current algebra, and partial conservation of axial-vector current (PCAC). Equation (1.19) generalized their result to the cases where m_2 is not small.

In their original paper on SCR's, de Alfaro, Fubini, Furlan, and Rossetti (AFFR) derive the relation[1]

$$g_{\omega\rho\pi}{}^2 m_\rho{}^2-4g_{\rho\pi\pi}{}^2=0 \tag{1.21}$$

(in their notation). We split our couplings into an st plus su part, whereas AFFR do not. Nevertheless, Eq. (1.21) holds also in the present formalism because the su isospin box does not contribute to the $I_t=1$ amplitude used to derive relation (1.21).

In the present section we have described only the st part of our couplings; in the next section we indicate how to construct the su part given the st part.

II. CONSTRUCTING THE *su* BOX

(In this section we will drop the primes on s-channel helicities. All the helicities in this section will be s-channel.)

Factorization and charge conjugation may be used to generate the su box, given the st box. By factorization, the st plus su amplitude must be a product of a $12\to J$ coupling

$$\langle J\lambda m_J|H^J|S_1\lambda_1 m_1 S_2\lambda_2 m_2\rangle$$

times a corresponding $J\to 34$ coupling [the m's are SU(3) indices]:

$$\langle J\lambda m_J|H^J|S_1\lambda_1 m_1 S_2\lambda_2 m_2\rangle=G_C H_C+G_A H_A\,. \tag{2.1}$$

The right-hand side of this equation is diagrammed in Fig. 4. $G_C=G_C(m_1m_2m_J)$ is a clockwise SU(3) quark loop, while G_A is an anticlockwise quark loop. If G_C gives the amplitude for $S_1S_2\to J$, then G_A gives the amplitude for the antiparticle reaction $\bar S_1\bar S_2\to\bar J$, since replacing clockwise quark by anticlockwise quark amounts to replacing quark by antiquark. By C invariance, then, H_A and H_C must be identical, except for a phase, since C does not affect spin dependence,

$$H_A=\eta H_C\,, \tag{2.2}$$

where η is the product of the charge-conjugation parities of S_1S_2J. In the quark model, C parity $=(-1)^{\mathcal{L}+S}$, so for ground states ($\mathcal{L}=0$),

$$\eta=\eta_1\eta_2\eta_J=(-1)^{S_1+S_2+J}\,. \tag{2.3}$$

In Fig. 4 we have deliberately omitted any clockwise or anticlockwise arrowheads on the quark lines internal to the factors giving the helicity dependence, in order to emphasize that the same helicity tensor is used in both halves of the diagram. Whatever order one chooses for the spin-$\frac{1}{2}$ quark indices in the kets of the Clebsch-Gordan coefficients making up this helicity tensor, one should choose the same order and use the same helicity tensor for both halves of Fig. 4. Because of C invariance, the effect of switching from clockwise to anticlockwise spin-$\frac{1}{2}$ quarks is entirely accounted for by the factor η. Now we construct a similar clockwise and anticlockwise loop for the final-state coupling $34\to J$, "star" it (which changes G_A into G_C), multiply it into the $12\to J$ coupling, include the usual $d^J_{\lambda\mu}$, and sum over J, m_J. If we take degenerate internal masses so as to open the sums into the butterfly form, we find

$$\langle\lambda_3 m_3\lambda_4 m_4\theta|H|\lambda_1 m_1\lambda_2 m_2\rangle=(M_{st}+\eta_1\eta_2\eta_3\eta_4\bar M_{st})\langle\lambda_3\lambda_4\theta|H_{st}|\lambda_1\lambda_2\rangle + (\eta_1\eta_2 M_{su}+\eta_3\eta_4\bar M_{su})\sum_J(-1)^J\langle J\lambda\|H_C\|S_1\lambda_1S_2\lambda_2\rangle d^J_{\lambda\mu}(\theta)\langle J_\mu\|H_C\|S_3\lambda_3S_4\lambda_4\rangle\,. \tag{2.4}$$

M_{ij} is the clockwise SU(3) loop of Fig. 1, $\bar M_{ij}$ is the corresponding anticlockwise loop, H_{st} is the butterfly diagram of Fig. 3(a), and H_C is the lower half of this butterfly, as in Eq. (2.1) and Fig. 4.

We may identify the H_{st} term in Eq. (2.4) with the st part. We now need to show that the remaining terms in Eq. (2.4) have the crucial property expected of an su part, namely that they superconverge rapidly at $u\cong 0$ if the st part superconverges rapidly at $t\cong 0$. To establish this property, we first show that the remaining terms in Eq. (2.4) add up to give the H_{st} term except for a relabeling

$3 \leftrightarrow 4$ everywhere, i.e., except for $t \leftrightarrow u$. Let us introduce a new angular variable:

$$\bar{\theta}_s \equiv \pi - \theta_s \ . \qquad (2.5)$$

$\bar{\theta}_s$ is the scattering angle if we take $u=0$ rather than $t=0$ as the "forward" direction: $\cos\theta_s = \hat{P}_1 \cdot \hat{P}_3$, while $\cos\bar{\theta}_s = \hat{P}_1 \cdot \hat{P}_4$. Also, if $\theta_s = f(s, t, m_3, m_4)$, then $\bar{\theta}_s = f(s, u, m_4, m_3)$. We switch to this new variable via the identity

$$(-1)^J d^J_{\lambda\mu}(\theta_s) = (-1)^{-\lambda} d^J_{\lambda-\mu}(\bar{\theta}_s) \ . \qquad (2.6)$$

To complete the interchange of $3 \leftrightarrow 4$, we change the signs of the azimuthal quantum numbers in each of the four Clebsch-Gordan coefficients making up H_C in Fig. 4, using the identity

$$\langle j_1 m_1 j_2 m_2 | j m \rangle = (-1)^{j_1 + j_2 - j} \langle j_1 - m_1 j_2 - m_2 | j - m \rangle \qquad (2.7)$$

repeatedly. This multiplies H_C by an over-all phase $(-1)^{S_3 + S_4 + L' + J}$ and ensures that helicity λ_3, rather than λ_4, now carries the minus sign. We next reverse the coupling order at the quark plus antiquark $\rightarrow J$ vertex in H_C, resulting in a further phase change:

$$\langle \tfrac{1}{2} m_3 \tfrac{1}{2} m_4 | J - \mu \rangle = (-1)^{1-J} \langle \tfrac{1}{2} m_4 \tfrac{1}{2} m_3 | J - \mu \rangle \ . \qquad (2.8)$$

This coupling-order reversal ensures that, when the J sum in Eq. (2.4) is carried out and the couplings are opened up into a butterfly shape, the vertical quark lines will crisscross the diagram (rather than run perpendicularly upward); compare the M_{su} box in Fig. 1 (after the bottom half of the diagram has been twisted through 180° so as to put particle 1 on the left). Now we relabel the dummy index $-\mu \rightarrow \mu$, collect all the phases [keeping in mind $(-1)^L = -1$], and carry out the sum over J in Eq. (2.4). We find that the M_{su} and M_{tu} terms in Eq. (2.4) add up to give the H_{st} butterfly-shaped diagram except for an interchange $3 \leftrightarrow 4$:

$$su\,\text{part} = (\bar{M}_{su} + \eta_1\eta_2\eta_3\eta_4 M_{su}) \langle \lambda_4 \lambda_3 \bar{\theta}_s | H_{st} | \lambda_1 \lambda_2 \rangle (-1)^{-\lambda} \ . \qquad (2.9)$$

The factor of $(-1)^{-\lambda}$ might make one uncomfortable at first, but in fact it is required for two reasons: (a) It ensures that the sum of st plus su parts obeys Bose symmetry when $S_3 \equiv S_4$. (b) It ensures that the su part superconverges at $u=0$. The s-to-u helicity crossing matrix differs from the s-to-t one by a phase $(-1)^{\lambda}$ and a relabeling $3 \leftrightarrow 4$. (For details, see the Appendix.) Hence the $(-1)^{-\lambda}$

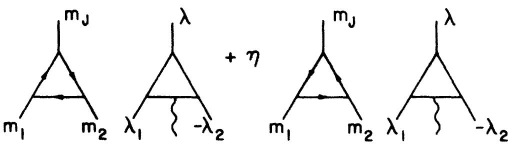

FIG. 4. Right-hand side of Eq. (2.1). The phase η is explained in the text.

in Eq. (2.9) just cancels a phase in the s-to-u crossing matrix, leaving one with a u-channel $u=0$ amplitude which is the t-channel $t=0$ amplitude except for a relabeling $3 \leftrightarrow 4$ everywhere. Hence the su part superconverges rapidly at $u \cong 0$ if the st part superconverges rapidly at $t \cong 0$, Q.E.D. In conclusion, expression (2.9) has all the desirable properties required of the su part. We remark that result (2.9) holds even when the internal resonances have $\mathcal{L} > 0$.

III. SUPERCONVERGENCE AT $t = (m_1 - m_3)^2$

In this section we verify rapid superconvergence in detail, using nondegenerate masses and the couplings of Fig. 3(c).

We first write out the $S_1 S_2 \rightarrow J$ vertices [lower half of Fig. 3(c)] in Eqs. (3.4)–(3.7). Next we multiply the $S_1 S_2 \rightarrow J$ vertex by an $S_3 S_4 \rightarrow J$ vertex (obtained from the $S_1 S_2 \rightarrow J$ vertex by a simple relabeling), include a $d^J(\theta_s)$, and cross to the t channel using Eq. (1.1). We specialize to the superconverging case $\lambda_1 = +1$, $\lambda_3 = -1$ while keeping λ_2 and λ_4 arbitrary. (There is another helicity choice which yields $|\lambda_3 - \lambda_1| = Mx = 2$, namely $\lambda_1 = -1$, $\lambda_3 = +1$, but it gives no new information because of parity conservation.) We obtain thereby the contribution of each direct-channel J to the crossed channel amplitudes $\langle \lambda_3 = +1\, \lambda_1 = -1\, \theta_t | H_t | \lambda_4 \lambda_2 \rangle$ [see Eqs. (3.8)–(3.12)]. We evaluate these contributions for the six choices of $\lambda_1 \cdots \lambda_4$ which superconverge rapidly and remove kinematic singularities (Secs. III A–III F). We then find in each case that Eqs. (1.15) and (1.16) hold at $t = (m_1 - m_3)^2$.

In order to obtain spin superconvergence, we must have $Mx = 2$; therefore, we set $S_1 = S_3 = 1$ and let (S_2, S_4) take on the values (0, 0), (0, 1), and (1, 1) in turn. SCR's for $(S_2, S_4) = (1, 0)$ are identical to those for $(S_2, S_4) = (0, 1)$ because of time-reversal invariance in the direct channel; similarly, SCR's for $S_2 = S_4 = 1$ and (S_1, S_3) arbitrary yield no new information because of time-reversal invariance in the crossed channel.

We define factorized couplings $\langle S_3 S_4 | H | J \rangle$ in the sharp-resonance limit,

$$\mathrm{Im}\langle \lambda_3' \lambda_4' \theta_s | H_s | \lambda_1' \lambda_2' \rangle = \sum_J \pi \delta(s - s_J)\, d^J_{\lambda'\mu'}(\theta_s) \langle S_3 \lambda_3' S_4 \lambda_4' | H | J \mu' \rangle h(34 \rightarrow J) \langle S_1 \lambda_1' S_2 \lambda_2' | H | J \lambda' \rangle h(12 \rightarrow J) \ , \qquad (3.1)$$

where the h's are reduced matrix elements, and $\langle S_1S_2|H|J\rangle$ (we suppress the helicity dependence for simplicity) is the SU(2) tensor given by the lower half of Fig. 3(c):

$$\langle S_1S_2|H|J\rangle = \sum_{m_i,\, m_i'} \langle S_1\lambda_1'|\tfrac{1}{2}m_1\,\tfrac{1}{2}m_1'\rangle\langle S_2-\lambda_2'|\tfrac{1}{2}m_2\,\tfrac{1}{2}m_2'\rangle\langle J\lambda|\tfrac{1}{2}m_1\tfrac{1}{2}m_2\rangle\langle L0|\tfrac{1}{2}m_1'\,\tfrac{1}{2}m_2'\rangle\,. \tag{3.2}$$

By the preceding discussion, we can take $S_1=1$ without loss of generality. For $S_2=1$ and $J=0$ Eq. (3.2) becomes

$$\langle 11|H|0\rangle = \sum\left[\delta(\lambda_1',1)\,\delta(m_1,m_1')\,\delta(m_1,\tfrac{1}{2})+\delta(\lambda_1',-1)\,\delta(m_1,m_1')\,\delta(m_1,-\tfrac{1}{2})+\delta(\lambda_1',0)\,\delta(m_1,-m_1')/\sqrt{2}\,\right]$$
$$\times[m_1\to m_2,\, m_1'\to m_2',\, \lambda_1'\to-\lambda_2'][\delta(m_1,-m_2)(-1)^{m_1-1/2}/\sqrt{2}\,][\delta(m_1',-m_2')/\sqrt{2}\,]\;. \tag{3.3}$$

We use the notation $\delta(m,m')$ for a Kronecker delta implying $m=m'$. Carrying out the sums in Eq. (3.3), we get

$$2\langle 11|H|0\rangle=\delta(\lambda_1',1)\,\delta(-\lambda_2',-1)-\delta(\lambda_1',-1)\,\delta(-\lambda_2',1)\;. \tag{3.4}$$

Calculations of the remaining vertices proceed similarly:

$$2\langle 11|H|1\rangle=[\delta(\lambda_1',1)\,\delta(-\lambda_2',-1)+\delta(\lambda_1',-1)\,\delta(-\lambda_2',1)+\delta(\lambda_1',0)\,\delta(\lambda_2',0)]\;\delta(\lambda',0)$$
$$+[\delta(\lambda_1',1)\,\delta(\lambda_2',0)+\delta(\lambda_1',0)\,\delta(-\lambda_2',1)]\,\delta(\lambda',1)$$
$$+[\delta(\lambda_1',-1)\,\delta(\lambda_2',0)+\delta(\lambda_1',0)\,\delta(-\lambda_2',-1)]\,\delta(\lambda',-1)\;, \tag{3.5}$$

$$2\langle 10|H|0\rangle=-\delta(\lambda_1',0)\;, \tag{3.6}$$

$$2\langle 10|H|1\rangle=[\delta(\lambda_1',1)-\delta(\lambda_1',-1)]\,\delta(\lambda_1',\lambda')\;. \tag{3.7}$$

The $\langle S_3S_4|H|J\rangle$ vertices in Eq. (3.1) follow from Eqs. (3.4)–(3.7) by a relabeling $1\to3$, $2\to4$, $\lambda'\to\mu'$. Now suppose we multiply together the quantities occurring in the Jth term of Eq. (3.1), drop the delta function [because it goes away anyway at Eq. (1.8)], and cross to the t channel, as in Eq. (1.1). What we get are the following quantities:

$$\langle\lambda_3\lambda_1|(J=1\text{ or }0)|S_4\lambda_4S_2\lambda_2\rangle\equiv\left[\prod_{i=1}^{4}d_{\lambda_i'\lambda_i}(\chi_i)\,d^J_{\lambda'\mu'}(\theta_s)\,\langle S_3\lambda_3'S_4\lambda_4'|H|J\mu'\rangle\,h(34\to J)\,\langle S_1\lambda_1'S_2\lambda_2'|H|J\lambda'\rangle\,h(12\to J)\right]_{s=s_J}\,. \tag{3.8}$$

Equation (3.8) is just the contribution from direct-channel resonance $J=1$ or 0 to the SCR with the indicated t-channel helicity values (except that we have not removed some kinematical singularities yet). In detail, the quantities (3.8) are

$$8\langle+1,-1|(J=0)|1\lambda_41\lambda_2\rangle=h(12\to\Pi)\,h(34\to\Pi)\,[(1+\cos\chi_1)\,d_{1\lambda_2}(\chi_2)-(1-\cos\chi_1)\,d_{-1\lambda_2}(\chi_2)]$$
$$\times[(1-\cos\chi_3)\,d_{1\lambda_4}(\chi_4)-(1+\cos\chi_3)\,d_{-1\lambda_4}(\chi_4)]\;. \tag{3.9}$$

A superscript S_1 is understood on all d's involving χ_i. Also all $(J=0)$ contributions are evaluated at $s=s_0$, and all $(J=1)$ contributions are evaluated at $s=s_1$:

$$4\langle+1,-1|(J=1)|1\lambda_41\lambda_2\rangle=h(12\to V)\,h(34\to V)$$
$$\times\{\delta(\lambda',0)\,d_{\lambda_21}(\chi_1-\chi_2)+\delta(\lambda',1)\,[\tfrac{1}{2}((1+\cos\chi_1)\,d_{0\lambda_2}(\chi_2)+\sin\chi_1\,d_{-1\lambda_2}(\chi_2)/\sqrt{2}\,]$$
$$+\delta(\lambda',-1)\,[\tfrac{1}{2}(1-\cos\chi_1)\,d_{0\lambda_2}(\chi_2)+\sin\chi_1\,d_{1\lambda_2}(\chi_2)/\sqrt{2}\,]\}\,d^1_{\lambda'\mu'}(\theta_s)$$
$$\times\{\delta(\mu',0)\,d_{\lambda_4-1}(\chi_3-\chi_4)+\delta(\mu',1)\,[\tfrac{1}{2}(1-\cos\chi_3)\,d_{0\lambda_4}(\chi_4)+(-\sin\chi_3/\sqrt{2}\,)\,d_{-1\lambda_4}(\chi_4)]$$
$$+\delta(\mu',-1)\,[\tfrac{1}{2}(1+\cos\chi_3)\,d_{0\lambda_4}(\chi_4)+(-\sin\chi_3/\sqrt{2}\,)d_{1\lambda_4}(\chi_4)]\}\;, \tag{3.10}$$

$$8\langle+1,-1|(J=0)|1\lambda_400\rangle=h(12\to\Pi)\,h(34\to\Pi)\,(-\sin\chi_1/\sqrt{2}\,)\,[(1-\cos\chi_3)\,d_{1\lambda_4}(\chi_4)-(1+\cos\chi_3)\,d_{-1\lambda_4}(\chi_4)]\,, \tag{3.11}$$

$$8\langle+1,-1|(J=1)|1\lambda_400\rangle=h(12\to V)\,h(34\to V)\,[(H\cos\chi_1)\,\delta(\lambda',1)-(1-\cos\chi_1)\,\delta(\lambda',-1)]\,d^1_{\lambda'\mu'}(\theta_s)$$
$$\times\{\delta(\mu',0)\,d_{\lambda_4-1}(\chi_3-\chi_4)+\delta(\mu',1)\,[\tfrac{1}{2}(1-\cos\chi_3)\,d_{0\lambda_4}(\chi_4)+(-\sin\chi_3/\sqrt{2}\,)\,d_{-1\lambda_4}(\chi_4)]$$
$$+\delta(\mu',-1)\,[\tfrac{1}{2}(1-\cos\chi_3)\,d_{0\lambda_4}(\chi_4)+(-\sin\chi_3/\sqrt{2}\,)\,d_{1\lambda_4}(\chi_4)]\}\;, \tag{3.12}$$

$$8\langle +1,-1|(J=0)|0000\rangle = h(12-\Pi)h(34-\Pi)(-\sin\chi_1\sin\chi_3)\,, \tag{3.13}$$

$$16\langle +1,-1|(J=1)|0000\rangle = h(12-V)h(34-V)[(1+\cos\chi_1)\delta(\lambda',1)-(1-\cos\chi_1)\delta(\lambda',-1)]\,d'_{\lambda'\mu'}(\theta_s)$$
$$\times[(1-\cos\chi_3)\delta(\mu',1)-(1+\cos\chi_3)\delta(\mu',-1)]\,. \tag{3.14}$$

We have used the rotation matrix

$$d'(\chi)=\begin{bmatrix} \tfrac{1}{2}(1+\cos\chi) & -(\sin\chi)/\sqrt{2} & \tfrac{1}{2}(1-\cos\chi) \\ (\sin\chi)/\sqrt{2} & \cos\chi & -(\sin\chi)/\sqrt{2} \\ \tfrac{1}{2}(1-\cos\chi) & (\sin\chi)/\sqrt{2} & \tfrac{1}{2}(1+\cos\chi) \end{bmatrix}\,. \tag{3.15}$$

We need expressions for the sines and cosines of the crossing and scattering angles as functions of s and t. We use the same angles as Wang (except $\sin\chi_i \to -\sin\chi_i$, because we are crossing s to t whereas she is crossing t to s):[5]

$$\cos\chi_i = [\mp(s+m_i^2-m_j^2)(t+m_i^2-m_k^2)-2m_i^2(m_3^2-m_1^2+m_2^2-m_4^2)]\,(\mathcal{S}_{ij}\mathcal{T}_{ik})^{-1}\,, \tag{3.16}$$

$$\sin\chi_i = -2m_i\,\phi^{1/2}/\mathcal{S}_{ij}\mathcal{T}_{ik}\,, \tag{3.17}$$

$$\cos\theta_s = [2st+s^2-s\textstyle\sum m_i^2+(m_1^2-m_2^2)(m_3^2-m_4^2)]/\mathcal{S}_{12}\mathcal{S}_{34}\,, \tag{3.18}$$

$$\sin\theta_s = 2(s\phi)^{1/2}/\mathcal{S}_{12}\mathcal{S}_{34}\,, \tag{3.19}$$

$$\cos\theta_t = [2st+t^2-t\textstyle\sum m_i^2+(m_4^2-m_2^2)(m_3^2-m_1^2)]/\mathcal{T}_{13}\mathcal{T}_{24}\,. \tag{3.20}$$

The upper sign in Eq. (3.16) goes with the particles which cross ($i=1$ and 4). $\mathcal{S}_{12}$ and $\mathcal{S}_{34}$ contain the threshold singularities in the initial and final states of the s channel, and similarly for $\mathcal{T}_{13}$ and $\mathcal{T}_{24}$; e.g.,

$$\mathcal{S}_{12}^2=[s-(m_1+m_2)^2]\,[s-(m_1-m_2)^2]\equiv\mathcal{S}_{12}^+\mathcal{S}_{12}^-\,,$$
$$\mathcal{T}_{13}^2=[t-(m_1+m_3)^2]\,[t-(m_1-m_3)^2]\equiv\mathcal{T}_{13}^+\mathcal{T}_{13}^-\,,\ \text{etc.} \tag{3.21}$$

Therefore, the indices j,k in Eqs. (3.16) and (3.17) are determined unambiguously by the index i. ϕ is Kibble's polynomial

$$\phi = st(\textstyle\sum m_i^2-s-t)-s(m_2^2-m_4^2)(m_1^2-m_3^2)-t(m_1^2-m_2^2)(m_3^2-m_4^2)$$
$$-(m_1^2m_4^2-m_3^2m_2^2)(m_1^2+m_4^2-m_3^2-m_2^2)\,. \tag{3.22}$$

It is convenient to introduce some abbreviations for the polynomials occurring in the numerators of the cosines, Eqs. (3.16)–(3.20):

$$P_i\equiv\mathcal{S}_{ij}\mathcal{T}_{ik}\cos\chi_i\,,\quad i=1,\ldots,4$$
$$P_s\equiv\mathcal{S}_{12}\mathcal{S}_{34}\cos\theta_s\,, \tag{3.23}$$
$$P_t\equiv\mathcal{T}_{13}\mathcal{T}_{24}\cos\theta_t\,.$$

We now consider each of the six rapidly superconvergent cases in turn (parts A through F immediately following). In each case we will not only verify superconvergence, we will also verify explicitly that each $J=0$ and 1 contribution is free of kinematic singularities in t, as well as mixed s and t kinematic singularities coming from nonintegral or negative powers of the Kibble polynomial ϕ. Sometimes our final expression for the $J=0$ and 1 contributions will have apparent kinematic s singularities (factors of $\mathcal{S}_{ij}$ or $\sqrt{s}$); but these are canceled by corresponding singularities implicit in the reduced s-channel helicity couplings $\bar{h}$.

Where extensive algebra is required to get from one equation to the next, we have grouped terms so as to clarify which terms in the first equation lead to which terms in the next equation.

A. $S_2=S_4=0$

The $J=0$ and 1 contributions to this SCR follow from Eqs. (3.13) and (3.14). We can suppress the bra and ket enclosing the $(J=0)$ and $(J=1)$ in those equations, and we absorb some threshold factors into the h's as in Eq. (1.15):

$$(J=0)=\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)(-2m_1m_3)(\phi/4\mathcal{T}_{13}^2)\,, \tag{3.24}$$

$$\begin{aligned}(J=1)&=\bar{h}(12\to V)\bar{h}(34\to V)[P_s-P_1P_3/\mathcal{T}_{13}^2]/8\\&=\bar{h}(12\to V)\bar{h}(34\to V)(m_1^2+m_3^2-t)(\phi/4\mathcal{T}_{13}^2)\,.\end{aligned} \tag{3.25}$$

The final parentheses in Eqs. (3.24) and (3.25) cancel out when the kinematic singularities are removed; the $J=0$ and 1 contributions then sum to give a quantity which vanishes at $t=(m_1-m_3)^2$, provided we assume Eq. (1.16), Q. E. D.

B. $S_2=0, S_4=1, \lambda_4=\pm1$

From Eqs. (3.11) and (3.12),

$$\langle+1,-1|\,(J=0)\,|1+100\rangle=h(12\to\Pi)h(34\to\Pi)(\cos\chi_3-\cos\chi_4)\sin\chi_1/8\sqrt{2}\,, \tag{3.26}$$

$$\begin{aligned}\langle+1,-1|\,(J=1)\,|1+100\rangle&=h(12\to V)h(34\to V)\\&\times\{-\sin\theta_s[1-\cos(\chi_3-\chi_4)]+\tfrac{1}{2}[\cos\chi_1+\cos\theta_s][\sin\chi_4-\sin\chi_3-\sin(\chi_4-\chi_3)]\\&+\tfrac{1}{2}[\cos\chi_1-\cos\theta_s][\sin\chi_4-\sin\chi_3+\sin(\chi_4-\chi_3)]\}\,.\end{aligned} \tag{3.27}$$

The $J=0$ and 1 contributions for $\lambda_4=-1$ are identical to the above, except that we must replace $\cos\chi_4$ and $\sin\chi_4$ by their negatives everywhere.

Now consider the linear combinations

$$\begin{aligned}&\langle+1,-1|\,(J)\,|1+100\rangle/(\sin\tfrac{1}{2}\theta_t)^3(\cos\tfrac{1}{2}\theta_t)\pm\langle1,-1|\,(J)\,|1-100\rangle/(\sin\tfrac{1}{2}\theta_t)(\cos\tfrac{1}{2}\theta_t)^3\\&\quad=[\langle1,-1|\,(J)\,|1+100\rangle\tfrac{1}{2}(1+\cos\theta_t)\pm\langle1,-1|\,(J)\,|1-100\rangle\tfrac{1}{2}(1-\cos\theta_t)]/(\tfrac{1}{2}\sin\theta_t)^3\,.\end{aligned} \tag{3.28}$$

The half-angles in Eq. (3.28) are just right to remove the kinematical singularities in s, but kinematical singularities in t have not been removed yet. The linear combination with the upper sign in Eq. (3.28) vanishes as the square root at pseudothreshold, hence does not superconverge rapidly, by the rule given in the Introduction. The $J=0$ and 1 contributions to the other linear combination are

$$\begin{aligned}(J=0)&=\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)(2m_1\sqrt{\phi}/\sqrt{2}\,\mathcal{T}_{13}\mathcal{T}_{24}\sin^3\theta_t)[P_4-P_tP_3/\mathcal{T}_{13}^2]\\&=\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)m_1(t+m_3^2-m_1^2)(4\phi^{3/2}/\sqrt{2}\,\mathcal{T}_{13}^3\mathcal{T}_{24}\sin^3\theta_t)\,,\end{aligned} \tag{3.29}$$

$$\begin{aligned}(J=1)&=\bar{h}(12\to V)\bar{h}(34\to V)\\&\times\{\sin\theta_s[\cos(\chi_3-\chi_4)-\cos\theta_t]+\cos\chi_1\sin\chi_4-\cos\theta_s\sin(\chi_4-\chi_3)-\cos\theta_t\sin\chi_3\cos\chi_1\}/(\sqrt{2}\sin^3\theta_t)\\&=\bar{h}(12\to V)\bar{h}(34\to V)\\&\times\{\sin\theta_s[s-(m_3-m_4)^2]2\phi/\mathcal{S}_{34}^2\mathcal{T}_{13}\mathcal{T}_{24}\\&+(\phi^{1/2}/\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})[-2m_4P_1+(2m_4/\mathcal{S}_{34}^2)P_sP_3-(2m_3/\mathcal{S}_{34}^2)P_sP_4+(2m_3/\mathcal{T}_{13}^2)P_1P_t]\}/(\sqrt{2}\sin^3\theta_t)\\&=\bar{h}(12\to V)\bar{h}(34\to V)\\&\times\{\sin\theta_s\,\text{term}+(\phi^{1/2}/\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})(-2m_4P_1+(2m_4/\mathcal{S}_{34}^2)[\mathcal{S}_{34}^2P_1-2\phi(s+m_3^2-m_4^2)]\\&-(2m_3/\mathcal{S}_{34}^2)[\mathcal{S}_{34}^2P_2+2\phi(s+m_4^2-m_3^2)]+(2m_3/\mathcal{T}_{13}^2)[\mathcal{T}_{13}^2P_2+2\phi(t+m_1^2-m_3^2)])\}/(\sqrt{2}\sin^3\theta_t)\\&=\bar{h}(12\to V)\bar{h}(34\to V)\\&\times\{\sqrt{s}\,\mathcal{T}_{13}^2[s-(m_3-m_4)^2]/\mathcal{S}_{34}^2+\mathcal{T}_{13}^2m_3-\mathcal{T}_{13}^2[m_3(s+m_4^2-m_3^2)+m_4(s+m_3^2-m_4^2)]/\mathcal{S}_{34}^2\\&+2m_1m_3(m_1-m_3)\}(4\phi^{3/2}/\sqrt{2}\,\mathcal{T}_{13}^3\mathcal{T}_{24}\sin^3\theta_t)\,.\end{aligned} \tag{3.30}$$

The final parentheses in Eqs. (3.29) and (3.30) cancel out when kinematic t singularities are removed. Every term in the curly bracket, Eq. (3.30), is proportional to $\mathcal{T}_{13}^2$ and therefore vanishes at the superconvergence point, except the last term. This term is just right to cancel the $(J=0)$ contribution, Q. E. D.

Note that the equations for $\bar{h}$ would have been far more complex had we chosen $t=0$ rather than $t=(m_1-m_3)^2$ as the superconvergence point.

The $(s)^{1/2}$ term in Eq. (3.30) comes entirely from the $\sin\theta_s$ contribution to $(J=1)$. $\sin\theta_s$ terms do not contribute at the superconvergence point, even though in the unequal mass case, $t=(m_1-m_3)^2$ no longer coincides with $\sin\theta_s=0$. In fact in all six of the SCR's we will investigate here, the terms proportional to $\sin\theta_s$ vanish at $t=(m_1-m_3)^2$. We will need this result in the next section.

C. $S_2 = S_4 = 1, \lambda_2 = \lambda_4 = 0$

From Eqs. (3.9) and (3.10),

$$(J=0) = h(12 \to \Pi) h(34 \to \Pi) \sin\chi_2 \sin\chi_4 / 8$$
$$= \bar{h}(12 \to \Pi)\bar{h}(34 \to \Pi)(2m_2 m_4 \mathcal{T}_{13}{}^2)(\phi/4\mathcal{T}_{24}{}^2\mathcal{T}_{13}{}^2) . \qquad (3.31)$$

This contribution superconverges by itself: The final parenthesis cancels out when kinematical s and t singularities are removed, leaving a quantity which vanishes at $t=(m_1-m_3)^2$. Similarly, the $(J=1)$ contribution will superconverge by itself:

$$(J=1) = h(12 \to V) h(34 \to V)$$
$$\times\{-\cos\theta_s[\sin(\chi_1-\chi_2)\sin(\chi_3-\chi_4)+\cos(\chi_1-\chi_2)\cos(\chi_3-\chi_4)-1]+[\cos\chi_2\cos\chi_4-\cos\theta_s]$$
$$+\sin\theta_s[-\sin(\chi_1-\chi_2)\cos(\chi_4-\chi_3)+\sin(\chi_3-\chi_4)\cos(\chi_1-\chi_2)]\}/8$$
$$=\bar{h}(12 \to V)\bar{h}(34 \to V)$$
$$\times\{P_s[-(-2m_1P_2+2m_2P_1)(-2m_3P_4+2m_4P_3)\phi(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2$$
$$-(P_1P_2+4m_1m_2\phi)(P_3P_4+4m_3m_4\phi)/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2+1]$$
$$+[-P_s+P_2P_4/\mathcal{T}_{24}{}^2]+2\sqrt{s}\,\phi[(2m_1P_2-2m_2P_1)(P_3P_4+4m_3m_4\phi)$$
$$-(2m_3P_4-2m_4P_3)(P_1P_2+4m_1m_2\phi)]/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2\}/8 . \qquad (3.32)$$

Let us denote the first, second, and third square brackets in Eq. (3.52) by B_1, B_2, and B_3, respectively:

$$(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2B_1=[-4m_2m_4P_1P_3-4m_1m_3P_2P_4+4(m_3P_1-m_1P_3)(m_2P_4-m_4P_2)]\phi-16m_1m_2m_3m_4\phi^2$$
$$-(P_1P_3)(P_2P_4)+(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2$$
$$=\{-4m_2m_4[\mathcal{T}_{13}{}^2P_s-2(m_1{}^2+m_3{}^2-t)\phi]-4m_1m_3[\mathcal{T}_{24}{}^2P_s-2(m_2{}^2+m_4{}^2-t)\phi]$$
$$+4\mathcal{T}_{13}^-[m_1(s+m_3{}^2-m_4{}^2)+m_3(s+m_1{}^2-m_2{}^2)]\mathcal{T}_{24}^-[m_4(s+m_2{}^2-m_1{}^2)+m_2(s-m_3{}^2+m_4{}^2)]\}\phi$$
$$-16m_1m_2m_3m_4\phi^2-[\mathcal{T}_{13}{}^2P_s-2(m_3{}^2+m_1{}^2-t)\phi][\mathcal{T}_{24}{}^2P_s-2(m_2{}^2+m_4{}^2-t)\phi]$$
$$+(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2$$
$$=\{-\mathcal{T}_{13}{}^2P_s\mathcal{T}_{24}^--\mathcal{T}_{24}{}^2P_s\mathcal{T}_{13}^-+4\mathcal{T}_{13}^-\mathcal{T}_{24}^-[m_1(s+m_3{}^2-m_4{}^2)+m_3(s+m_1{}^2-m_2{}^2)]$$
$$\times[m_4(s+m_2{}^2-m_1{}^2)+m_2(s-m_3{}^2+m_4{}^2)]\}\phi-4\mathcal{T}_{13}^-\mathcal{T}_{24}^-\phi^2+(\mathcal{T}_{13}\mathcal{T}_{24})^2(\mathcal{S}_{12}{}^2\mathcal{S}_{34}{}^2-P_s{}^2) .$$

[$\mathcal{T}_{13}^{\pm}\equiv t-(m_1\pm m_3)^2$, as at Eq. (3.21).] The last parenthesis equals $4s\phi$, so that we can take out a factor $\mathcal{T}_{13}^-\phi$ from each term and define

$$B_1\equiv 2\mathcal{T}_{13}^-\phi f_1(s,t)/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2 , \qquad (3.33)$$

where f_1 is a kinematic-singularity-free, finite polynomial at pseudothreshold. As for B_2 and B_3, we have

$$B_2=2(t-m_2{}^2-m_4{}^2)\phi/\mathcal{T}_{24}{}^2 , \qquad (3.34)$$
$$B_3=[(2m_1P_3-2m_3P_1)(P_2P_4+4m_2m_4\phi)+(2m_4P_2-2m_2P_4)(P_1P_3+4m_1m_3\phi)]$$
$$=[(2m_1P_3-2m_3P_1)(\mathcal{T}_{24}{}^2P_s+2\mathcal{T}_{24}^-\phi)]+(2m_4P_2-2m_2P_4)(\mathcal{T}_{13}{}^2P_s+2\mathcal{T}_{13}^-\phi)]$$
$$=\mathcal{T}_{13}^-\mathcal{T}_{24}^-\{[m_1(s+m_3{}^2-m_4{}^2)+m_3(s+m_1{}^2-m_2{}^2)][\mathcal{T}_{24}^+P_s+2\phi]$$
$$+[m_4(s+m_2{}^2-m_1{}^2)+m_1(s+m_4{}^2-m_3{}^2)][\mathcal{T}_{13}^+P_s+2\phi]\}$$
$$\equiv\mathcal{T}_{13}^-f_2(s,t) . \qquad (3.35)$$

Inserting Eqs. (3.33)–(3.35) into the $(J=1)$ contribution, we get

$$(J=1)=\bar{h}(12\to V)\bar{h}(34\to V)$$
$$\times\{\mathcal{T}_{13}^-P_sf_1(s,t)/(\mathcal{S}_{12}\mathcal{S}_{34})^2+\mathcal{T}_{13}{}^2(t-m_2{}^2-m_4{}^2)+\mathcal{T}_{13}^-s^{1/2}f_2(s,t)/(\mathcal{S}_{12},\mathcal{S}_{34})^2\}(\phi/4\mathcal{T}_{13}{}^2\mathcal{T}_{24}{}^2), \qquad (3.36)$$

where the f's are finite polynomials at pseudothreshold. The final parenthesis in Eq. (3.36) cancels out when the kinematical singularities are removed, leaving a quantity which vanishes at the superconvergence point, Q. E. D.

This particular SCR puts no constraints on the h's, since the $J=0$ and 1 contributions vanish independently.

D. $S_2 = S_4 = 1, \lambda_2 = \lambda_4 = \pm 1$

Consider the quantities

$$\langle +1, -1 | (J) | 1+1\, 1+1 \rangle \pm \langle +1, -1 | (J) | 1-1\, 1-1 \rangle. \tag{3.37}$$

The linear combination with the lower sign in Eq. (3.37) vanishes as the square root at pseudothreshold, hence does not superconverge, by the rule given in the Introduction. The $J=0$ and 1 contributions to the other linear combination are

$$\begin{aligned}(J=0) &= h(12\to\Pi)h(34\to\Pi)(-\cos\chi_1\cos\chi_3 + \cos\chi_2\cos\chi_4)/8 \\ &= \bar{h}(12\to\Pi)\bar{h}(34\to\Pi)[(m_1^2+m_3^2-t)\mathcal{T}_{24}^2 - (m_2^2+m_4^2-t)\mathcal{T}_{13}^2](\phi/4\mathcal{T}_{13}^2\mathcal{T}_{24}^2),\end{aligned} \tag{3.38}$$

$$\begin{aligned}(J=1) = h(12\to V)h(34\to V) &\\ \times\{-\cos\theta_s[&\sin(\chi_1-\chi_2)\sin(\chi_3-\chi_4) + \cos(\chi_1-\chi_2)\cos(\chi_3-\chi_4) - 1] \\ &+ \sin\theta_s[-\sin(\chi_1-\chi_2)\cos(\chi_4-\chi_3) + \sin(\chi_3-\chi_4)\cos(\chi_1-\chi_2)] \\ &+ \sin\chi_2\sin\chi_4 - \sin\chi_1\sin\chi_3\}/8\,.\end{aligned} \tag{3.39}$$

The two square brackets in Eq. (3.39) are identical to the first and third square brackets in Eq. (3.32). Hence we can eliminate the square brackets immediately, using results (3.33) and (3.35) of part C:

$$\begin{aligned}(J=1) &= \bar{h}(12\to V)\bar{h}(34\to V) \\ &\times\{\mathcal{T}_{13}^- P_s f_1(s,t)/(\mathcal{S}_{12}\mathcal{S}_{34})^2 + \mathcal{T}_{13}^- s^{1/2} f_2(s,t)/(\mathcal{S}_{12}\mathcal{S}_{34})^2 + (2m_2m_4\mathcal{T}_{13}^2 - 2m_1m_3\mathcal{T}_{24}^2)\}(\phi/4\mathcal{T}_{13}^2\mathcal{T}_{24}^2).\end{aligned} \tag{3.40}$$

The final parentheses in Eqs. (3.38) and (3.40) cancel out when kinematical singularities are removed. Since the f's are well behaved, finite polynomials, every term in the $(J=1)$ contribution, Eq. (3.40), vanishes at pseudothreshold, except a term which just cancels the $(J=0)$ contribution, Eq. (3.38), Q. E. D.

E. $S_2 = S_4 = 1, \lambda_2 = -\lambda_4 = \pm 1$

The linear combinations

$$\begin{aligned}&\langle +1, -1 | (J) | 1+1\, 1-1 \rangle/(\cos\theta_t/2)^4 \pm \langle +1, -1 | (J) | 1-1\, 1+1 \rangle/(\sin\theta_t/2)^4 \\ &\quad = 4[\langle +1, -1 | (J) | 1+1\, 1-1 \rangle (1-\cos\theta_t)^2 \pm \langle +1, -1 | (J) | 1-1\, 1+1 \rangle (1+\cos\theta_t)^2]/(\sin\theta_t)^4\end{aligned} \tag{3.41}$$

are free of kinematic s singularities. The linear combination with the lower sign in Eq. (3.41) has a square root singularity at $t=(m_1-m_3)^2$, hence does not superconverge rapidly there. The $J=0$ contribution to the other linear combination is

$$\begin{aligned}(J=0) &= -h(12\to\Pi)h(34\to\Pi) \\ &\quad\times[(\cos\chi_1\cos\chi_3 + \cos\chi_2\cos\chi_4)(1+\cos^2\theta_t) - 2\cos\theta_t(\cos\chi_2\cos\chi_3 + \cos\chi_1\cos\chi_4)]/2(\sin\theta_t)^4 \\ &= -\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)[(P_1P_3/\mathcal{T}_{13}^2 + P_2P_4/\mathcal{T}_{24}^2)(2\mathcal{T}_{13}^2\mathcal{T}_{24}^2 - 4t\phi) - 2P_t(P_2P_3 + P_1P_4)](\mathcal{T}_{13}^2\mathcal{T}_{24}^2/32t^2\phi^2).\end{aligned}$$

We rewrite the $2\mathcal{T}_{13}^2\mathcal{T}_{24}^2$ terms inside the square bracket in a number of ways and leave the other terms alone for the moment:

$$(J=0) = -\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)$$
$$\times\{[P_2P_t + 2\phi(t+m_2^2-m_4^2)]P_3 + [P_4P_t - 2\phi(t+m_4^2-m_2^2)]P_1$$
$$+[P_3P_t + 2\phi(t+m_3^2-m_1^2)]P_2 + [P_1P_t - 2\phi(t+m_1^2-m_3^2)]P_4$$
$$-2P_t(P_2P_3+P_1P_4) + [P_1P_3/\mathcal{T}_{13}^2 + P_2P_4/\mathcal{T}_{24}^2](-4t\phi)\}(\mathcal{T}_{13}^2\mathcal{T}_{24}^2/32t^2\phi^2)$$
$$= -\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)$$
$$\times\{(t+m_2^2-m_4^2)P_3 - (t+m_4^2-m_2^2)P_1 + (t+m_3^2-m_1^2)P_2 - (t+m_1^2-m_3^2)P_4$$
$$+[P_s - (m_3^2+m_1^2-t)\phi/\mathcal{T}_{13}^2 - (m_4^2+m_2^2-t)\phi/\mathcal{T}_{24}^2](-4t)\}(\mathcal{T}_{13}^2\mathcal{T}_{24}^2/16t^2\phi)$$
$$= -\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)$$
$$\times\{4+[-(m_3^2+m_1^2-t)/\mathcal{T}_{13}^2 - (m_4^2+m_2^2-t)/\mathcal{T}_{24}^2](-4t)\}\mathcal{T}_{13}^2\mathcal{T}_{24}^2/16t^2. \tag{3.42}$$

The $J=1$ contribution to the linear combination in Eq. (3.41) with the upper sign is

$$(J=1) = h(12\to V)h(34\to V)$$
$$\times\{(1+\cos^2\theta_t)[\cos\theta_s(1+\cos(\chi_1-\chi_2)\cos(\chi_3-\chi_4)+\sin(\chi_1-\chi_2)\sin(\chi_3-\chi_4)) - \sin\chi_1\sin\chi_3 - \sin\chi_2\sin\chi_4]$$
$$-\cos\theta_t[\cos\theta_s(\cos(\chi_1-\chi_2)+\cos(\chi_3-\chi_4)) - \sin\chi_2\sin\chi_3 - \sin\chi_1\sin\chi_4]$$
$$+\sin\theta_s[(1+\cos^2\theta_t)(\cos(\chi_1-\chi_2)\sin(\chi_4-\chi_3) - \cos(\chi_3-\chi_4)\sin(\chi_2-\chi_1))$$
$$-2\cos\theta_t(\sin(\chi_4-\chi_3) - \sin(\chi_2-\chi_1))]\}/2\sin^4\theta_t. \tag{3.43}$$

We analyze Eq. (3.43) in two steps. First we show that the $\sin\theta_s$ terms in Eq. (3.40) vanish at pseudothresholds; then we show that the remaining terms in Eq. (3.43) cancel the $(J=0)$ contribution, Eq. (3.42):

$$\sin\theta_s \text{ terms} = h(12\to V)h(34\to V)\sin\theta_s$$
$$\times\{\mathcal{T}_{13}^{-2}\mathcal{T}_{24}^{-2}(2\mathcal{T}_{13}^2\mathcal{T}_{24}^2 - 4t\phi)[(P_1P_2+4m_1m_2\phi)(-2m_4P_3+2m_3P_4)$$
$$-(P_3P_4+4m_3m_4\phi)(-2m_2P_1+2m_1P_2)]$$
$$-2P_t[\mathcal{S}_{12}^2(-2m_4P_3+2m_3P_4) - \mathcal{S}_{34}^2(-2m_2P_1+2m_1P_2)]\}$$
$$\times(\mathcal{T}_{13}\mathcal{T}_{24})^2/32t^2(\mathcal{S}_{12}\mathcal{S}_{34})^2\phi^{3/2}$$
$$= \bar{h}(12\to V)\bar{h}(34\to V)s^{1/2}$$
$$\times\{2[4m_1m_2\phi(-2m_4P_3+2m_3P_4) - 4m_3m_4\phi(-2m_2P_1+2m_1P_2)]$$
$$+2[(P_1P_2-\mathcal{S}_{12}^2P_t)(-2m_4P_3+2m_3P_4) - (P_3P_4-\mathcal{S}_{34}^2P_t)(-2m_2P_1+2m_1P_2)]$$
$$+\mathcal{T}_{13}^{-2}\mathcal{T}_{24}^{-2}(-4t\phi)[(P_1P_2+4m_1m_2\phi)(-2m_4P_3+2m_3P_4)$$
$$-(P_3P_4+4m_3m_4\phi)(-2m_2P_1+2m_1P_2)]\}$$
$$\times(\mathcal{T}_{13}\mathcal{T}_{24})^2/16t^2(\mathcal{S}_{12}^2\mathcal{S}_{34}^2)\phi \tag{3.44}$$
$$= \bar{h}(12\to V)\bar{h}(34\to V)s^{1/2}$$
$$\times\{2\phi[4m_1m_2(-2m_4P_3+2m_3P_4) - 4m_3m_4(-2m_2P_1+2m_1P_2)]$$
$$+2\phi[2(s-m_1^2-m_2^2)(-2m_4P_3+2m_3P_4) - (s-m_3^2-m_4^2)(-2m_2P_1+2m_1P_2)]$$
$$+\mathcal{T}_{13}^{-2}\mathcal{T}_{24}^{-2}(-4t\phi)[-\mathcal{T}_{13}^{-}f_2(s,t)]\}(\mathcal{T}_{13}\mathcal{T}_{24})^2/16t^2(\mathcal{S}_{12}\mathcal{S}_{34})^2\phi. \tag{3.45}$$

Here we compared the last square bracket in Eq. (3.44) with the corresponding bracket in Eq. (3.32), noticed they were equal except for a sign, and used Eq. (3.35). The ϕ singularity in the denominator now cancels out. The t^2 singularity cancels when we remove kinematical t singularities, but the over-all factor of $(\mathcal{T}_{13}\mathcal{T}_{24})^2$ remains. Hence the $\sin\theta_s$ terms, Eq. (3.45), do not contribute at $\mathcal{T}_{13}^{-}=0$. [In fact they do not contribute at either pseudothreshold. Since the SCR now under consideration has both $|\lambda_3-\lambda_1|=2$ and $|\lambda_2-\lambda_4|=2$, we expect it to superconverge both at $\mathcal{T}_{13}^{-}=0$ and at $\mathcal{T}_{24}^{-}=0$, and it will. Note that there is a factor of $\mathcal{T}_{24}^{-}$ implicit in $f_2(s,t)$.]

Now let us return to Eq. (3.43) and check out the non-$\sin\theta_s$ terms:

$$
\begin{aligned}
(J=1) &= \bar{h}(12\to V)\bar{h}(34\to V) \\
&\times\{\mathcal{T}_{13}^{-2}\mathcal{T}_{24}^{-2}(2\mathcal{T}_{13}^{2}\mathcal{T}_{24}^{2}-4t\phi)[P_s+P_s(P_1P_2+4m_1m_2\phi)(P_3P_4+4m_3m_4\phi)/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2 \\
&\qquad + P_s(-2m_2P_1+2m_1P_2)(-2m_4P_3+2m_3P_4)\phi/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2 \\
&\qquad -(4m_1m_3/\mathcal{T}_{13}^{2}+4m_2m_4/\mathcal{T}_{24}^{2})\phi] \\
&\quad -2P_t[P_s(P_1P_2+4m_1m_2\phi)/\mathcal{S}_{12}^{2}+P_s(P_3P_4+4m_3m_4\phi)/\mathcal{S}_{34}^{2}-(4m_2m_3+4m_1m_4)\phi/(\mathcal{T}_{13}\mathcal{T}_{24})^2]\} \\
&\times(\mathcal{T}_{13}\mathcal{T}_{24})^4/32t^2\phi^2+\sin\theta_s \text{ terms}. \qquad (3.46)
\end{aligned}
$$

Comparing the first square brackets in Eqs. (3.46) and (3.32), we see we can use Eq. (3.33) to simplify the $(-4t\phi)$ term in Eq. (3.46). We do not use Eq. (3.33) on the $2\mathcal{T}_{13}^{2}\mathcal{T}_{24}^{2}$ terms, but rather regroup the latter with the remaining terms in Eq. (3.46):

$$
\begin{aligned}
(J=1) &= \bar{h}(12\to V)\bar{h}(34\to V) \\
&\times\{(-4t\phi)[2P_s-2P_s\mathcal{T}_{13}^{-}f_1(s,t)\phi/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2-(4m_1m_3/\mathcal{T}_{13}^{2}+4m_2m_4/\mathcal{T}_{24}^{2})\phi](\mathcal{S}_{12}\mathcal{S}_{34})^2 \\
&\quad +2P_s[(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2+(P_1P_2)(P_3P_4)+(P_1P_2)(P_3P_4)-(P_1P_2)(P_3P_4)-P_t\mathcal{S}_{34}^{2}P_1P_2-P_t\mathcal{S}_{12}^{2}P_3P_4] \\
&\quad +2\phi P_s[4m_1m_2(P_3P_4-\mathcal{S}_{34}^{2}P_t)+4m_3m_4(P_1P_2-\mathcal{S}_{12}^{2}P_t)+16m_1m_2m_3m_4\phi] \\
&\quad +2\phi[4m_1m_3(P_sP_2P_4-\mathcal{T}_{24}^{2}\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2})+4m_2m_4(P_sP_1P_3-\mathcal{T}_{13}^{2}\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2}) \\
&\qquad -4m_2m_3(P_sP_1P_4-P_t\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2})-4m_1m_4(P_sP_2P_3-P_t\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2})]\} \\
&\times(\mathcal{T}_{13}\mathcal{T}_{24})^2/(\mathcal{S}_{12}\mathcal{S}_{34})^2 32t^2\phi^2+\sin\theta_s \text{ terms}. \qquad (3.47)
\end{aligned}
$$

The f_1 term in the first square bracket vanishes at pseudothreshold and is free of kinematic ϕ poles, while the last parenthesis in this bracket is just right to cancel the nonsuperconvergent part of the $(J=0)$ contribution, Eq. (3.42). For the moment, then, we can ignore these terms and the $\sin\theta_s$ terms, indicating them by dots $(\cdots)$ in what follows. We take the remaining $(2P_s)$ term in the first square bracket and put it into the second square bracket:

$$
\begin{aligned}
(J=1) &= \bar{h}(12\to V)\bar{h}(34\to V) \\
&\times\{\cdots+2P_s[-4t\phi\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2}+(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2+(P_1P_2)(\mathcal{S}_{34}^{2}P_t+2(s-m_3^2-m_4^2)\phi) \\
&\qquad +(\mathcal{S}_{12}^{2}P_t+2(s-m_1^2-m_2^2)\phi)(P_3P_4)-(\mathcal{S}_{12}^{2}P_t+2(s-m_1^2-m_2^2)\phi)(\mathcal{S}_{34}^{2}P_t+2(s-m_3^2-m_4^2)\phi) \\
&\qquad -P_t\mathcal{S}_{34}^{2}P_1P_2-P_t\mathcal{S}_{12}^{2}P_3P_4] \\
&\quad +2\phi P_s[4m_1m_2(s-m_3^2-m_4^2)(2\phi)+4m_3m_4(s-m_1^2-m_2^2)(2\phi)+16m_1m_2m_3m_4\phi] \\
&\quad +2\phi[4m_1m_3(2\phi(t-m_2^2-m_4^2)P_s+\mathcal{T}_{24}^{2}P_s^2-\mathcal{T}_{24}^{2}\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2})+4m_2m_4(2\phi(t-m_1^2-m_3^2)P_s+\mathcal{T}_{13}^{2}P_s^{2}-\mathcal{T}_{13}^{2}\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2}) \\
&\qquad -4m_2m_3(2(s+m_1^2-m_2^2)P_4+2(s-m_3^2-m_4^2)\mathcal{S}_{12}^{2})\phi+4m_1m_4(2(s+m_2^2)P_3-2(s-m_3^2-m_4^2)\mathcal{S}_{12}^{2})\phi]\} \\
&\times(\mathcal{T}_{13}\mathcal{T}_{24})^2/(\mathcal{S}_{12}\mathcal{S}_{34})^2 32t^2\phi^2 \\
&= \bar{h}(12\to V)\bar{h}(34\to V) \\
&\times\{\cdots+2P_s[\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2}(\mathcal{T}_{13}^{2}\mathcal{T}_{24}^{2}-4t\phi-P_t^{2})+2(s-m_1^2-m_2^2)\phi(P_3P_4-P_t\mathcal{S}_{34}^{2})+2(s-m_3^2-m_4^2)\phi(P_1P_2-\mathcal{S}_{12}^{2}P_t) \\
&\qquad -4(s-m_1^2-m_2^2)(s-m_3^2-m_4^2)\phi^2] \\
&\quad +16\phi^2P_s[m_1m_2\mathcal{S}_{34}^{-}+\mathcal{S}_{12}^{-}m_3m_4] \\
&\quad +8\phi^2[m_1m_3(2(t-m_2^2-m_4^2)P_s-4_s\mathcal{T}_{24}^{2})+m_2m_4(2(t-m_1^2-m_3^2)P_s-4_s\mathcal{T}_{13}^{2})-m_2m_3(2(s+m_1^2-m_2^2)P_4 \\
&\qquad +2(s-m_3^2-m_4^2)\mathcal{S}_{12}^{2})+m_1m_4(2(s+m_2^2-m_1^2)P_s-2(s-m_3^2-m_4^2)\mathcal{S}_{12}^{2})]\}(\mathcal{T}_{13}\mathcal{T}_{24})^2/(\mathcal{S}_{12}\mathcal{S}_{34})^2 32t^2\phi^2.
\end{aligned}
$$

$$(3.48)$$

The $\mathcal{S}_{12}^{2}\mathcal{S}_{34}^{2}$ term in the first square bracket vanishes; the terms linear in ϕ inside this bracket turn out to be equal, and add so as to just cancel the ϕ^2 term in the bracket. The first square bracket vanishes entirely, therefore. The remaining two brackets are both multiplied by a ϕ^2 which cancels the ϕ^2 pole in the denominator of Eq. (3.48). When kinematic t singularities are removed, the t^2 in the denominator goes away, but the over-all factors of $(\mathcal{T}_{13}\mathcal{T}_{24})^2$ remain so that the surviving brackets do not contribute at pseudothreshold. To summarize, the $(J=1)$ contribution is of the form

$$(J=1)=-\bar{h}(12\to V)\bar{h}(34\to V)\{[2m_1m_3/\mathcal{T}_{13}{}^2+2m_2m_4/\mathcal{T}_{24}{}^2](-4t)\}(\mathcal{T}_{13}\mathcal{T}_{24})^2/16t^2\,, \tag{3.49}$$

plus the $\sin\theta_s$ terms of Eq. (3.45), plus the f_1 terms of Eq. (3.47), plus the terms in the last two square brackets, Eq. (3.48). Only the $(J=1)$ terms shown explicitly in Eq. (3.49) survive at pseudothreshold, and they just cancel the $(J=0)$ contribution at pseudothreshold, Eq. (3.42), Q. E. D.

F. $S_2=S_4=1,\ \lambda_2=\pm 1,\ \lambda_4=0$

The quantities

$$[\langle +1,-1|(J)|1+110\rangle\tfrac{1}{2}(1-\cos\theta_t)\pm\langle +1,-1|(J)|1-110\rangle\tfrac{1}{2}(1+\cos\theta_t)](\tfrac{1}{2}\sin\theta_t)^{-3} \tag{3.50}$$

are free of kinematical s singularities, and the linear combination with the lower sign is the one which does not superconverge, by the rule given in the Introduction. The $J=0$ contribution to the other linear combination is

$$\begin{aligned}(J=0)&=h(12\to\Pi)h(34\to\Pi)[(\cos\chi_1+\cos\chi_2)(1-\cos\theta_t)+(\cos\chi_1-\cos\chi_2)(1+\cos\theta_t)](-\sin\chi_4/2^{3/2}\sin^3\theta_t)\\&=\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)[P_1-P_2P_t/\mathcal{T}_{24}{}^2](\mathcal{T}_{13}\mathcal{T}_{24})^2m_4t^{3/2}/4\sqrt{2}\,\phi\\&=\bar{h}(12\to\Pi)\bar{h}(34\to\Pi)2(t+m_2{}^2-m_4{}^2)\mathcal{T}_3{}^2m_4t^{3/2}/4\sqrt{2}\,.\end{aligned} \tag{3.51}$$

The $(J=1)$ contribution to Eq. (3.50) may be obtained from

$$\begin{aligned}\langle +1,-1|(J=1)|1+110\rangle&=h(12\to V)h(34\to V)\\&\quad\times\{-\cos\theta_s[1+\cos(\chi_1-\chi_2)]\sin(\chi_3-\chi_4)+(\sin\chi_2+\sin\chi_1)\cos\chi_4\\&\qquad+\cos\theta_s\sin(\chi_1-\chi_2)\cos(\chi_3-\chi_4)\\&\qquad+\sin\theta_s[(1+\cos(\chi_1-\chi_2))(-\cos(\chi_3-\chi_4))-\sin(\chi_1-\chi_2)\sin(\chi_3-\chi_4)]\}/8\sqrt{2}\,.\end{aligned} \tag{3.52}$$

The equation for $\langle +1,-1|(J=1)|1-110\rangle$ is identical to Eq. (3.52) except that $\sin\chi_2$ and $\cos\chi_2$ are replaced by their negatives everywhere. Substituting $\langle +1,-1|(J=1)|1\pm 110\rangle$ into Eq. (3.50) and choosing the upper sign, we get

$$\begin{aligned}(J=1)&=h(12\to V)h(34\to V)\\&\quad\times\{\sin\theta_s[-\cos(\chi_3-\chi_4)+\cos\theta_t+\cos\theta_t(\cos(\chi_1-\chi_2)\cos(\chi_3-\chi_4)+\sin(\chi_1-\chi_2)\sin(\chi_3-\chi_4)-1)]\\&\qquad-\cos\theta_s\sin(\chi_3-\chi_4)+\sin\chi_1\cos\chi_4-\cos\theta_t\sin\chi_2\cos\chi_4\\&\qquad+\cos\theta_t\cos\theta_s[\cos(\chi_1-\chi_2)\sin(\chi_3-\chi_4)-\sin(\chi_1-\chi_2)\cos(\chi_3-\chi_4)]\}(\sqrt{2}\sin^3\theta_t)^{-1}\,.\end{aligned} \tag{3.53}$$

We compare the two square brackets in Eq. (3.53) to the first and third square brackets of Eq. (3.32). We then use the result (3.33), as well as the middle step in Eq. (3.53), to simplify Eq. (3.53):

$$\begin{aligned}(J=1)&=h(12\to V)h(34\to V)\\&\quad\times\{\sin\theta_s[(-P_3P_4-4m_3m_4\phi+P_t\mathcal{S}_{34}{}^2)/\mathcal{S}_{34}{}^2+P_t(-2\mathcal{T}_{13}^-\phi f_1)/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2]/(\sqrt{2}\,\mathcal{T}_{13}\mathcal{T}_{24}\sin^3\theta_t)\\&\qquad+\bar{h}(12\to V)\bar{h}(34\to V)[2m_3P_4P_s/\mathcal{S}_{34}{}^2-2m_4P_3P_s/\mathcal{S}_{34}{}^2-2m_1P_4+2m_2P_4P_t/\mathcal{T}_{24}{}^2](\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_2\mathcal{S}_{34})^2\\&\qquad+[(2m_1P_3P_t-2m_3P_1P_t)\mathcal{T}_{24}{}^2P_s{}^2+P_tP_s(2m_1P_3-2m_3P_1)2\mathcal{T}_{24}^-\phi\\&\qquad\quad+(2m_4P_2P_t-2m_2P_4P_t)\mathcal{T}_{13}{}^2P_s{}^2+P_tP_s(2m_4P_2-2m_2P_4)2\mathcal{T}_{13}^-\phi]\}(\phi/2)^{1/2}/(\mathcal{T}_{13}\mathcal{T}_{24}\sin\theta_t)^3(\mathcal{S}_{12}\mathcal{S}_{34})^2\\&=\bar{h}(12\to V)\bar{h}(34\to V)2(s\phi)^{1/2}[(-2\phi\mathcal{S}_{34}^-)/\mathcal{S}_{34}{}^2-2\mathcal{T}_{13}^-P_t\phi f_1/(\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2](\mathcal{T}_{13}\mathcal{T}_{24})^2/8\sqrt{2}(t\phi)^{3/2}\\&\quad+\bar{h}(12\to V)\bar{h}(34\to V)\\&\quad\times\{[2m_3(\mathcal{S}_{34}{}^2P_2+2\phi(s+m_4{}^2-m_3{}^2))/\mathcal{S}_{34}{}^2-2m_4(\mathcal{S}_{34}{}^2P_1-2\phi(s+m_3{}^2-m_4{}^2))/\mathcal{S}_{34}{}^2-2m_1P_4\\&\qquad+2m_2(\mathcal{T}_{24}{}^2P_3+2\phi(t+m_4{}^2-m_2{}^2))/\mathcal{T}_{24}{}^2](\mathcal{T}_{13}\mathcal{T}_{24}\mathcal{S}_{12}\mathcal{S}_{34})^2\\&\qquad+[2m_1(\mathcal{T}_{13}{}^2P_4-2\phi(t+m_3{}^2-m_1{}^2))-2m_3(\mathcal{T}_{13}{}^2P_2+2\phi(t+m_1{}^2-m_2{}^3))]\mathcal{T}_{24}{}^2(\mathcal{S}_{12}{}^2\mathcal{S}_{34}{}^2-4s\phi)\\&\qquad+[2m_4(\mathcal{T}_{24}{}^2P_1-2\phi(t+m_2{}^2-m_4{}^2))-2m_2(\mathcal{T}_{24}{}^2P_3+2\phi(t+m_4{}^2-m_2{}^2))]\mathcal{T}_{13}{}^2(\mathcal{S}_{12}{}^2\mathcal{S}_{34}{}^2-4s\phi)\\&\qquad+P_tP_s(2m_1P_3-2m_3P_1)2\mathcal{T}_{24}^-\phi+P_tP_s(2m_4P_2-2m_2P_4)2\mathcal{T}_{13}^-\phi\}/8\sqrt{2}\,\phi t^{3/2}(\mathcal{S}_{12}\mathcal{S}_{34})^2\,.\end{aligned} \tag{3.54}$$

The first square bracket in Eq. (3.54) contains the $\sin\theta_s$ terms. These superconverge by themselves, because the factor of $t^{3/2}$ cancels when kinematic t singularities are removed, but the factors of $\mathcal{T}_{13}$ are left untouched. Inside the curly bracket, Eq. (3.54), the terms independent of ϕ mutually cancel, leaving the curly bracket linear in ϕ and Eq. (3.54) free of kinematical ϕ poles. We get

$$\begin{aligned}(J=1)=\bar{h}(12\to V)\bar{h}(34\to V)\{&\sin\theta_s \text{ terms} - 4[\mathcal{T}_{24}{}^2(m_1+m_3)\mathcal{T}_{13}^- + \mathcal{T}_{13}{}^2(m_2+m_4)\mathcal{T}_{24}^-](\mathcal{S}_{12}{}^2\mathcal{S}_{34}{}^2-4s\phi)\\ &-4s\mathcal{T}_{13}{}^2\mathcal{T}_{24}{}^2(2m_1P_4-2m_3P_2+2m_4P_1-2m_2P_3)\\ &+P_tP_s[2\mathcal{T}_{13}^-(m_1(s+m_3{}^2-m_4{}^2)+m_3(s+m_1{}^2-m_2{}^2))2\mathcal{T}_{24}^-\\ &\quad+2\mathcal{T}_{24}^-(m_4(s+m_2{}^2-m_1{}^2)+m_2(s+m_4{}^2-m_3{}^2))2\mathcal{T}_{13}^-]\}/8\sqrt{2}\,t^{3/2}(\mathcal{S}_{12}\mathcal{S}_{34})^2 .\end{aligned} \quad (3.55)$$

The factors of $t^{3/2}$ cancel when kinematic singularities are removed form the $J=0$ and 1 contributions, Eqs. (3.50) and (3.55), leaving expressions which vanish at pseudothreshold, Q. E. D.

We have now completed our detailed proof that six of the 11 meson-meson SCR's are satisfied by Eq. (1.16) at $t=(m_1-m_3)^2$.

IV. UNIQUENESS

The coupling scheme which we have just investigated in detail in preceding sections is not unique, but is rather the simplest scheme which will satisfy SCR's at $t=(m_1-m_3)^2$. We shall demonstrate in this section that the SCR's are also satisfied by three other schemes [Figs. 5(a), 5(b) and a modification of Fig. 5(a) to be described later]; and there may exist further schemes which we have not yet found.

Evidently the uniqueness of our couplings needs some further investigation, but we can list some reasons for optimism that uniqueness problems will be minimal, and we will not be inundated with alternative coupling choices. First of all, the superconvergence property of couplings is not preserved under linear superposition. Either our original coupling, or the coupling of Fig. 5(a), or the coupling of Fig. 5(b) will superconverge; but a linear combination of these couplings will not superconverge. The foregoing statement is readily proved in the equal-mass limit. This limit is especially simple becuase the helicity crossing matrices reduce to $\pi/2$ rotations, and one can work with the opened butterfly diagram without projecting out the contributions from the individual J's.

There is another reason why uniqueness problems may not be serious. We consider finite-energy sum rules and assume semilocal duality.[14] Let us imagine that the $\mathcal{L}=0$ couplings are crossed to a t channel with $|\lambda|<2$ and are averaged over some small region of the S variable. Then by semilocal duality, what results is the t-channel Regge residue, essentially. Now imagine the $\mathcal{L}=1, 2, \ldots$ contributions similarly crossed to the t channel and energy-averaged. These contributions must give the same Regge residue with the same helicity dependence; presumably then, the $\mathcal{L}=0$ and $\mathcal{L}>0$ coupling schemes cannot be too different, or in other words uniqueness problems will not be severe.

Now let us verify the superconvergence of the "spin-one exchange" scheme of Fig. 5(a). The first diagram of Fig. 5(a) represents our original coupling; the second diagram differs from our original coupling by the insertion of the following two Clebsch-Gordan coefficients:

$$3\sum_l \langle \tfrac{1}{2}m_1 | l | \tfrac{1}{2}m_1' \rangle (-1)^l \langle \tfrac{1}{2}m_2 | -l | \tfrac{1}{2}m_2' \rangle . \quad (4.1)$$

m_1' and m_2' are contracted with the quark lines coupling to J in Fig. 5(a); m_1 and m_2 are contracted with quark lines coming from S_1 and S_2, respectively. The factor of 3 in Eq. (4.1) is just right to give superconvergence. This can be proved simply by studying the s-channel couplings: There is no need to form the box or go through the trouble of crossing to the t channel. Insertion (4.1) amounts to a spin-one exchange between the quark and antiquark forming the resonance J. This exchange will exert different "forces" in channels $J=1$ and 0. A relatively simple Clebsch-Gordan calculation now shows that this new "force," when added to our original diagram, preserves the relative magnitude of the $J=0$ and $J=1$ couplings, but reverses their relative sign. Since the $J=0$ and

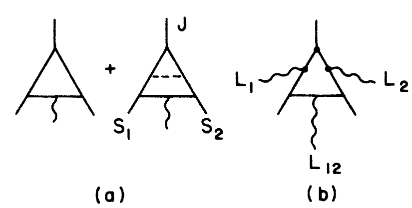

FIG. 5. Two superconvergent coupling schemes. (a) The spin-1 exchange scheme. The dotted line represents a spin-1 particle exchanged between the two quarks. (b) The 3-L scheme. Each wiggly line represents an angular momentum of 1.

$J=1$ couplings always occur squared in SCR's, the relative sign will not affect superconvergence, either in the equal mass or in the general mass case, Q. E. D.

We have an immediate corollary to the above for couplings to $\mathcal{L}>0$ mesons. A spin-one exchange between the quark lines in an $\mathcal{L}>0$ coupling will change the relative signs of all $\mathcal{S}=0$ and $\mathcal{S}=1$ couplings, but again will not affect superconvergence.

Now let us consider the coupling of Fig. 5(b), which we dub the "3-L" coupling because of its three orbital angular momentum lines. This coupling differs from our original coupling by the addition of a Clebsch-Gordan coefficient

$$\langle \tfrac{1}{2} m_1 | 0 | \tfrac{1}{2} m_1' \rangle = (-1)^{1/2 - m_1} \delta(m_1, m_1')/\sqrt{3} \qquad (4.2)$$

to the vertical quark line from S_1 to J, and a similar Clebsch-Gordan coefficient to the vertical quark line from S_2 to J. The initial state now contains three angular momenta $L_1 = L_2 = L_{12} = 1$. The total angular momentum L of the initial state can now reach the value $L=3$ as well as $L=1$. L is obtained by recoupling L_1, L_2, and L_{12} in all possible ways to obtain a total L: $L = 1 \otimes 1 \otimes 1 = 1$ or 3. The Clebsch-Gordan coefficients do not allow $L = 2$ or 0, so that parity is conserved.

Proof. The proof is a lot easier using tensors rather than Clebsch-Gordan coefficients. Let us try to form various total L's from three unit vectors $\hat{z}$. (Since each L_i has $L_{zi}=0$, $L_i=1$, each L_i transforms like a unit vector $\hat{z}$ pointing along the z axis.) In order to obtain a resultant L of 0 or 2, one needs the cross product $\hat{z}\times\hat{z}$, but this vanishes, Q. E. D. For $\mathcal{L}=0$, $L=3$ contributes only when the external spins can add up to form a resultant ≥ 3, i.e., when $S_1 = S_2 = J = 1$.

The superconvergence properties of the 3-L model may be deduced from the following theorem: The 3-L helicity coupling for $(S_1\lambda_1)+(S_2\lambda_2)\to(J\lambda)$ is identical to the corresponding 1-L coupling except for (a) a factor of $\frac{1}{3}$, and (b) a sign change if $\lambda=0$.

Proof. Part (a) of the theorem is an obvious consequence of the factors of $(1/\sqrt{3})$ in Eq. (4.2). As for part (b), notice that the Clebsch-Gordan coefficient (4.2) would be just a Kronecker delta acting on the quark line were it not for the phase factor ± 1 for $m_i' \gtrless 0$. Now suppose $\lambda=+1$. Then $m_1' = m_2' = +\frac{1}{2}$ necessarily, and the two Clebsch-Gordan coefficients will contribute a net phase $(+1)(+1)=+1$. Similarly for $\lambda=-1$, the two Clebsch-Gordan coefficients will contribute a net phase $(-1)(-1)=+1$. Only for $\lambda=0$ will there be a net phase change of (-1), Q. E. D. Now let us form an s-channel helicity amplitude by multiplying together an $S_1+S_2\to(J,\lambda)$ coupling, an $S_3+S_4 \to (J,\mu)$ coupling, and a $(2J+1)\, d^J_{\lambda\mu}(\theta_s)$, and summing over J. If we do this for both 3-L and 1-L couplings and compare the resulting expansions, we find from the foregoing theorem that the d^1_{01} and d^1_{10} terms have opposite sign in the two expansions, whereas the remaining terms have the same sign. The d^1_{01} and d^1_{10} rotation matrices are the only ones involving $\sin\theta_s$. Hence switching from 1-L to 3-L couplings merely changes the sign in front of all $\sin\theta_s$ terms. (We ignore the over-all trivial renormalization by a factor of $\frac{1}{3}^2$.) From the discussion in the preceding section, especially the remark at the end of part B, the terms proportional to $\sin\theta_s$ always vanish at the superconvergence point $t=(m_1-m_3)^2$. Hence the 3-L couplings superconverge, even in the general mass case.

The 3-L coupling may be generalized readily from J's with $\mathcal{L}=0$ to J's with $\mathcal{L}>0$. We take $L_1 = L_2 = 1$, and let permissible values of L_{12} be determined by L and parity (which imply $L_{12} = \mathcal{L}\pm 1$). In the limit of degenerate internal masses we can work with the butterfly form without projecting out the contributions from each J. We also assume elastic scattering ($m_1=m_3$, $m_2=m_4$) so that the superconvergence point is $t=\theta_s=0$ and all rotation matrices $d(\theta_s)$ on internal lines reduce to Kronecker deltas. Then the phase (± 1) associated with the $L_1(L_2)$ insertion will be canceled by the phase (± 1) associated with the $L_3(L_4)$ insertion on the same quark line. Hence the 3-L scheme superconvergences for $\mathcal{L}>0$, at least in the special case of degenerate internal masses and elastic scattering.

It is possible to combine features of the spin-one exchange and 3-L schemes so as to obtain a hybrid coupling scheme which also superconverges. Suppose we attach an $L=1$ line to each side of each quark triangle in Fig. 5(a), just above the points where the S_1 and S_2 lines join on to the triangle. The first diagram will then look like Fig. 5(b); the second diagram will look like Fig. 5(b) with a spin-one exchange line inserted above L_1 and L_2. From the preceding discussion, the hybrid scheme will be identical to the 3-L scheme except for a reversal of the relative phase between couplings to $J=0$ and $J=1$. Here as for the spin-one exchange scheme discussed earlier, this phase reversal will not affect superconvergence, Q. E. D.

One coupling scheme which does *not* superconverge is worth discussing, because it has been used in a fit to $\mathcal{L}=1$ meson decays carried out by Colglazier and Rosner.[15] Motivated by a specific model for the breaking of $SU(6)_W$, they proposed an ($\mathcal{L}=1$ meson)$\to$($\mathcal{L}=0$ meson) + ($\mathcal{L}=0$ meson) coupling which is a sum of three terms. In the degenerate internal masses limit, their first two terms yield scattering amplitudes having the butterfly form, Fig. 2, with specific choices for the interiors of the central blobs. Their third coupling, however, yields boxlike diagrams with the blobs attached to the vertical rather than horizontal sides of the box. By studying the $\Pi V \to \Pi V$ amplitude generated by this box, we have been able to verify that the third coupling destroys superconvergence (at least in the limit of elastic scattering and degenerate internal masses.) Colglazier and Rosner were able to discard this coupling and still obtain a good fit. The data on spin-dependence of $\mathcal{L}=1$ decays used in their fit are still fragmentary, but this result is encouraging.

V. *W* SPIN AND SU(3)

In this section we will show how to exhibit explicitly the W-spin content of the several superconvergent coupling schemes considered in Secs. III and IV. We then discuss possible symmetries of each scheme in the order $SU(6)_W$ and W spin, chiral $SU(3)\times SU(3)$, and finally SU(3).

The 3-L couplings may be made $SU(6)_W$ invariant.[16,17] In order to exhibit the W-spin content of the 3-L scheme, Fig. 5(b), we form the vector sum of each external spin with the orbital angular momentum immediately adjacent to it as one goes (say) clockwise around the quark loop. I.e., we form $J\otimes L_2$, $S_2\otimes L_{12}$, $S_1\otimes L_1$. [See Fig. 6(a).] Call the three resultant vectors W_J, W_2, W_1, because as a matter of fact the resultants are just the usual W spins of particles J, S_2, S_1 One easily verifies that each W has the right magnitude and z component to qualify as the W spin of its associated S (or J). The z component is correct because each L has zero azimuthal quantum number; hence W and S have the same z component. As for the magnitude, one can easily verify the usual "W-S flip." The magnitude of W can be only 0 or 1 because it must couple to a quark loop. If $S=0$, then $W=1$, since $L=1$. If $S=1$ and $\lambda=0$, then $W=0$ because (we use a tensor notation, with $\vec{\epsilon}$ representing S and $\hat{z}$ representing L) $\vec{\epsilon}\times\hat{z}=0$ when $\lambda=0$. If $S=1$ and $\lambda=\pm 1$, then $W=1$ because $\vec{\epsilon}\cdot\hat{z}=0$. Therefore, W has both the correct magnitude and the correct z component to be the W spin of S.

With a bit more labor, one can verify that the three $S_i\otimes L_i=W_i$ Clebsch-Gordan coefficients in Fig. 6(b) can be deleted. They are canceled by the numbers which come in when one recouples Fig. 5(b) to obtain Fig. 6(b). Hence the 3-L vertex equals a simple quark loop with three W-spin legs attached. We conclude that the 3-L scheme is consistent with $SU(6)_W$ invariance for couplings.

We can only conclude that the 3-L scheme is *consistent* with $SU(6)_W$. The 3-L scheme *does not require* $SU(6)_W$, because SCR's allow $SU(6)_W$ to be broken by scale factors. SCR's are homogeneous in the couplings $V\Pi\to J$ and $VV\to J$. Therefore, we could multiply every $V\Pi\to J$ coupling by a common scale factor. The SCR's would still be satisfied, but $SU(6)_W$ would be broken. It might be possible to rule out such scale-broken $SU(6)_W$ by considering each external line as an internal line in some diagram with ≥ 5 legs, then writing an SCR in the $(\text{mass})^2$ of what used to be the external line.

Our original "1-L" coupling, Fig. 3, can also be recoupled so as to display manifest W-spin invariance [though not consistency with $SU(6)_W$]. First we insert two adjacent angular momentum lines into the 1-L vertex as in Fig. 6(b), where $L_a=L_b=1$. This insertion changes nothing, because the two lines add zero total orbital momentum to the diagram. [*Proof*: Recouple as in Fig. 6(c). The total added orbital angular momentum can only be zero, as shown in Fig. 6(c), because forming an orbital angular momentum of one is analogous to forming $\hat{z}\times\hat{z}$.] Now diagram 6 may be recoupled so as to reveal the W-spin content, in the same manner as diagram 5(b) was recoupled. One will obtain Fig. 6(a) with (L_1, L_{12}, L_2) relabeled (L_b, L_a, L), and a change of coupling order at the $(W_J W_b J)$ vertex. Instead of $\langle W_J\lambda | J\lambda L_b 0\rangle$ at this vertex we have $\langle W_J\lambda | L_b 0 J\lambda\rangle$. These two Clebsch-Gordan coefficients differ by a phase factor $(-1)^{1+J-W_J}$ (the 1 coming from $L_b=1$). This phase is enough to destroy consistency with $SU(6)_W$, though W-spin invariance still holds.

The 1-L vertex is consistent with the weaker symmetry chiral $SU(3)\times SU(3)$.[18]

Proof. The factor $(-1)^{1+J-W_J}$ is $(+1)$ for couplings $S_1S_2\to\Pi$ or V_z and (-1) for couplings $S_1S_2\to V_\pm$. $V_\pm$, $V_z+\Pi$, and $V_z-\Pi$ belong to different irreducible representations of chiral $SU(3)\times SU(3)$. Therefore, this symmetry does not relate the $S_1S_2\to V_\pm$ couplings to the $S_1S_2\to\Pi$ or V_z couplings, Q. E. D.

The "spin-one exchange" vertex, Fig. 5(a), is not consistent with chiral $SU(3)\times SU(3)$.

Proof. Recall that the linear combinations $(V_z+\Pi)$ and $(V_z-\Pi)$ belong to different irreducible representations, and that the spin-one and 1-L vertices have opposite relative phase between the $S_1S_2\to V$ and $S_1S_2\to\Pi$ couplings. Therefore, a chiral-invariant 1-L coupling $V_+V_-\to(V_z+\Pi)$ becomes a chiral-noninvariant spin-one exchange coupling $V_+V_-\to(V_z-\Pi)$, Q. E. D.

Evidently, from the foregoing discussion, it is not too hard to construct SCR schemes which violate the higher symmetries $SU(6)_W$ and chiral

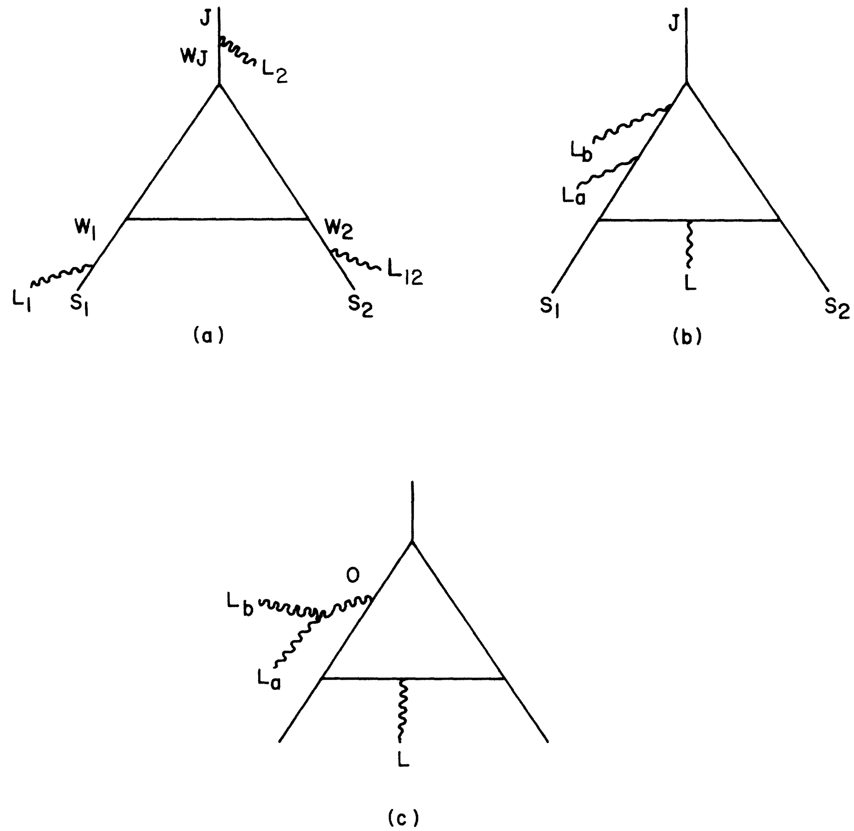

FIG. 6. (a) The 3-L vertex recoupled to reveal its W-spin content. (b) The 1-L vertex adjusted for conversion to W-spin. (c) Figure (b) recoupled.

SU(3)×SU(3). W-spin symmetry, on the other hand, seems almost impossible to avoid, in any scheme where the external particles are attached directly to a quark loop. The same remark can be made about the generalization of W spin proposed by Melosh for couplings to $\mathcal{L}>0$ mesons, a generalization where the pion is allowed to have $W_z=\pm 1$ as well as $W_z=0$.[13,19] Consider how the coupling of Fig. 5(b) should be modified to accommodate $\mathcal{L}>0$. We must relabel the J-line $\mathcal{S}$ (the total quark spin of the intermediate resonance), relabel the L_{12}-line S_Q (say), and insert an $\mathcal{L}$ line. One end of the $\mathcal{L}$ line couples to $\mathcal{S}$ to form J; the other end couples to S_Q to form L_{12}. By parity conservation, $\mathcal{L}=L_{12}\pm 1$, therefore $S_Q=1$. Now suppose S_2 is a pion. One may exhibit explicitly the W-spin content of the diagram by recoupling as for the $\mathcal{L}=0$ case, except now $W_2=S_2\otimes S_Q$ rather than $S_2\otimes L_{12}$. Of course $W_2=1$ again, as for the $\mathcal{L}=0$ case, but W_{2z} may now take on the values ± 1 as well as 0, because S_{Qz} may equal ± 1 as well as 0. Hence we obtain the generalization of pion W_z suggested by Melosh, Q.E.D. Of course the present approach also generalizes W spin even when none of the external particles is a pion. Also, the butterfly form, Fig. 2, yields the selection rule $|\Delta\mathcal{L}_z|\leq 1$ proposed by Melosh for decays of $\mathcal{L}\geq 1$ mesons. The quark-antiquark pair which scatter to form $\mathcal{L}$ can carry at most one unit of helicity. Hence $|\mathcal{L}_z|\leq 1$, which implies that $|\Delta\mathcal{L}_z|\leq 1$, since S_1 and S_2 have $\mathcal{L}=0$.

Now let us consider the constraints which SCR's impose on the SU(3) structure of couplings. (The same remarks will apply to all three schemes.) Because of scale-breaking effects, SCR's relate only couplings of mesons having the same hypercharge [Cf. the discussion of scale-broken $SU(6)_W$ earlier in this section. SCR's are homogeneous in the couplings $S_1S_2\to(Y=1$ meson) and $S_1S_2\to(Y=0$ meson). Therefore, every $S_1S_2\to(Y=1$ meson) coupling may be multiplied by the same scale factor without violating any SCR; but SU(3) is broken thereby.] For example, SCR's relate the couplings $K^*\bar{K}\to\rho$, $K^*\bar{K}\to\omega$, and $K^*\bar{K}\to\phi$, but do not relate these couplings to the $K^*\pi\to K^*$ coupling. The first three couplings are related because they must emerge when one inserts a complete set of intermediate SU(3) states into the SU(3) quark loop for (say) $K^*\bar{K} \to K^*\bar{K}$. [Compare the way in which a complete set of spin intermediate states was inserted in Fig. 3(b).] Therefore, the first three couplings may be expressed in terms of two parameters, an over-all scale plus the ϕ-ω mixing angle.[20] SCR's do not determine the mixing angle, because for any mixing angle ϕ, ω, and ρ constitute a complete set.

This completes our investigation of meson-meson $\mathcal{L}=0$ unequal mass SCR's, their uniqueness, and the symmetries which they respect. Further work now in progress will extend the approach of this paper to scattering amplitudes involving external photons and fermions.

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APPENDIX: THE s-TO-u HELICITY CROSSING MATRIX

In this appendix we calculate the phase of the s-to-u channel helicity crossing matrix. This phase may be calculated from the corresponding phase for s-to-t channel crossing, but the calculation involves something more than a simple relabeling.

According to Cohen-Tannoudji, Morel, and Navelet (C-TMN), the s-to-t channel crossing matrix is[5]

$$\langle \lambda_3\lambda_1 | H_t | \lambda_4\lambda_2\rangle_C = \prod_{i=1}^{4} d_{\lambda_i'\lambda_i}(\chi_i)(-1)^P \langle \lambda_3'\lambda_4' | H_s | \lambda_1'\lambda_2'\rangle_C\,, \tag{A1}$$

where

$$P=(\sigma+2S_2+2S_4)+(\lambda_2'-\lambda_3')+(\lambda_3'-\lambda_3)+(\lambda_4'-\lambda_4)\,. \tag{A2}$$

$\sigma=1$ if S_1 and S_4 are fermions; $\sigma=0$ in all other cases. The first two parentheses on the right in Eq. (A2) come from C-TMN, Table X1 (except we have replaced unprimed helicities by primed helicities, because C-TMN use primes to denote t-channel helicities). The last two parentheses in Eq. (A2) come from a double use of the formula

$$(-1)^{\lambda'-\lambda} d_{\lambda'\lambda}(\chi)=d_{\lambda'\lambda}(-\chi)\,.$$

This double sign change is necessary because we are using Wang's crossing angles χ_i in Eq. (A1), rather than C-TMN's, and the two differ by a change of sign of χ_3 and χ_4.[5] The subscript C in Eq. (A1) (short for C-TMN) reminded us that C-TMN use helicity amplitudes which are defined without the "particle 2" factors introduced by Jacob and Wick.[12] If we indicate Jacob-Wick amplitudes by omitting any subscript, we have

$$\langle \lambda_3'\lambda_4' | H_s | \lambda_1'\lambda_2'\rangle = (-1)^{S_4-\lambda_4'+S_2-\lambda_2'} \langle \lambda_3'\lambda_4' | H_s | \lambda_1'\lambda_2'\rangle_C\,, \tag{A3}$$

and a similar formula for the t-channel amplitudes. (We have been using Jacob-Wick amplitudes throughout the body of this paper.) When we insert the additional phase coming from Eq. (A3) into Eqs. (A1) and (A2), we find that the s-to-t crossing phase is independent of the λ_i'. Hence this phase will cancel out of the SCR, Eq. (1.8), and we can ignore this phase, as we have done in Eq. (1.1).

Now let us consider the phase for s-to-u crossing. The C-TMN phase, Eq. (A2), is for a reaction with direct and crossed channels

$$\begin{aligned} &S_1S_2 \to S_3S_4\,,\\ &\bar{S}_4S_2 \to S_3\bar{S}_1\,, \end{aligned} \tag{A4}$$

whereas we want the phase for a reaction with direct and crossed channels

$$\begin{aligned} &S_1S_2 \to S_3S_4\,,\\ &S_1\bar{S}_4 \to S_3\bar{S}_2\,. \end{aligned} \tag{A5}$$

Since S_2S_4 are the particles which cross in reactions (A4), whereas S_1S_4 are the particles which cross in reactions (A5), we should interchange labels 1 and 2 in Eqs. (A1) and (A2). Then $t=(p_1-p_3)^2$ becomes $u=(p_1-p_4)^2$ and we get

$$\langle \lambda_3\lambda_2 | H_u | \lambda_4\lambda_1\rangle_C = \Pi d_{\lambda_i'\lambda_i}(\omega_i)(-1)^{P'} \langle \lambda_3'\lambda_4' | H_s | \lambda_2'\lambda_1'\rangle_C\,, \tag{A6}$$

$$P'=\sigma'+2S_1+2S_4+\lambda_1'-\lambda_2'+\lambda_3'-\lambda_3+\lambda_4'-\lambda_4\,, \tag{A7}$$

$\sigma'=1$ if S_2 and S_4 are fermions; $\sigma'=0$ in all other cases. The ω_i follow from the χ_i by interchanging 1 and 2 everywhere; e.g. if

$$\cos\chi_1=f(m_1,m_2,s,t)\,, \tag{A8}$$

then

$$\cos\omega_2=f(m_2,m_1,s,u)\,, \tag{A9}$$

etc.

Equations (A6) and (A7) are still not what we want, because these equations are for a reaction with direct and crossed channels [we relabel $1 \leftrightarrow 2$ in Eqs. (A4)]

$$\begin{aligned} &S_2S_1 \to S_3S_4\,,\\ &\bar{S}_4S_1 \to S_3\bar{S}_2\,. \end{aligned} \tag{A10}$$

Note the differences in coupling order between reactions (A5) and (A10). The amplitudes

$\langle\lambda_3'\lambda_4'|H_s|\lambda_1'\lambda_2'\rangle_C$ and $\langle\lambda_3'\lambda_4'|H_s|\lambda_2'\lambda_1'\rangle_C$ are *not* identical, even though there are no effects due to the "particle 2" factors of Jacob and Wick. In fact these two amplitudes are related to the same M functions by two slightly different sets of boosts and rotations. From C-TMN, Eq. (II-4), we have[5]

$$\langle\lambda_3'\lambda_4'|H_s|\lambda_1'\lambda_2'\rangle_C = \mathfrak{D}_{A_1\lambda_1'}(L_1)\mathfrak{D}_{A_2\lambda_2'}(L_2)\mathfrak{D}_{A_3\lambda_3'}(L_3\epsilon)\times\mathfrak{D}_{A_4\lambda_4'}(L_4\epsilon)M(A_1\cdots A_4)\,, \qquad (A11)$$

$$\langle\lambda_3'\lambda_4'|H_s|\lambda_2'\lambda_1'\rangle_C = \mathfrak{D}_{A_1\lambda_1'}(L_1')\mathfrak{D}_{A_2\lambda_2'}(L_2')\mathfrak{D}_{A_3\lambda_3}(L_3'\epsilon)\times\mathfrak{D}_{A_4\lambda_4'}(L_4'\epsilon)M(A_1\cdots A_4)\,. \qquad (A12)$$

The A_i are spinor indices on M function M. ϵ is a rotation through π about the y axis, needed for final-state particles. L_i is a Lorentz transformation which maps the center-of-mass unit four-vectors $\hat{x}, \hat{y}, \hat{z}, \hat{t}$ into (respectively) $\hat{x}$, a unit four-vector along $\hat{p}_1\times\hat{p}_3$, the unit four-vector $(|\vec{p}_i|/m_i;\ \gamma_i\hat{p}_i)$, and p_i/m_i. L_i' is identical to L_i, except that L_i' maps $\hat{y}$ into $\hat{p}_2\times\hat{p}_3=-\hat{p}_1\times\hat{p}_3$. Therefore, L_i and L_i' must be simply related, and to determine that relationship we decompose L_i following Moussa and Stora or C-TMN Eq. (II.5):[21]

$$L_i = B_i\Omega_i\epsilon'\,. \qquad (A13)$$

B_i is a boost which maps $\hat{t}$ into p_i/m_i but leaves $\hat{y}$ alone. ϵ' is a rotation by π about the y axis (for particles 2, 4 only) which cancels a corresponding rotation in Ω_i but again leaves $\hat{y}$ alone. The only factor which affects $\hat{y}$ is the pure rotation Ω_i, which maps $\hat{z}$ onto $\hat{p}_i$ and $\hat{y}$ onto a unit vector along $\hat{p}_1\times\hat{p}_3$. In order to change Ω_i into a rotation mapping $\hat{z}$ onto $\hat{p}_i$ and $\hat{y}$ onto a unit vector along $-\hat{p}_1\times\hat{p}_3$, we can simply preface Ω_i with a rotation by π around the z axis, and we get

$$L_i' = B_i\Omega_i\exp(-i\pi J_z)\epsilon'\,. \qquad (A14)$$

Now we commute the new $\exp(-i\pi J_z)$ factor through ϵ and ϵ', using

$$\exp(-i\pi J_z)(\epsilon \text{ or } \epsilon') = (\epsilon \text{ or } \epsilon')\exp(+i\pi J_z)\,. \qquad (A15)$$

Equation (A15), following from the fact that ϵ and ϵ' are 180° rotations about the y axis, hence reverses the sign of every J_z eigenvalue. One then easily finds that $\mathfrak{D}_{A_i\lambda_i}(L_i')=\mathfrak{D}_{A_i\lambda_i}(L_i)(-1)^{\pm\lambda_i'}$. Hence from Eqs. (A11) and (A12),

$$\langle\lambda_3'\lambda_4'\theta|H_s|\lambda_1'\lambda_2'\rangle_C = \langle\lambda_3'\lambda_4'|H_s|\lambda_2'\lambda_1'\rangle_C(-1)^{\lambda_1'-\lambda_2'-\lambda_3'+\lambda_4'} = (-1)^{\lambda'-\mu'}\langle\lambda_3'\lambda_4'\theta'|H_s|\lambda_2'\lambda_1'\rangle_C\,. \qquad (A16)$$

I.e., the two amplitudes both equal the same boosted M function, except for a phase. This is the relation we need. If we have both amplitudes in the form of a partial-wave series, then of course the d^J's on the right- and left-hand sides of Eq. (A15) have the arguments $\theta'=\pi-\theta$ and θ respectively, as shown, because interchanging λ_1' and λ_2' changes the center-of-mass scattering cosine from $\cos\theta=\hat{p}_1\cdot\hat{p}_3$ to $\cos\theta'=\hat{p}_2\cdot\hat{p}_3=-\hat{p}_1\cdot\hat{p}_3=\cos(\pi-\theta)$. The distinction between θ and θ' is not relevant to our present application (since we do not want to partial-wave expand $\langle\lambda_3'\lambda_4'\theta'|H_s|\lambda_2'\lambda_1'\rangle_C$, but merely want to eliminate it from a crossing relation); but the distinction does become relevant if one wishes to check that Eq. (A16) gives the proper Bose or Fermi symmetry under interchange when $S_1\equiv S_2$.

Had we chosen to interchange final rather than initial helicities, we would have had to map $\hat{y}$ into $\hat{p}_i\times\hat{p}_4$ rather than $\hat{p}_2\times\hat{p}_3$. Since $\hat{p}_2\times\hat{p}_3=\hat{p}_1\times\hat{p}_4$, we have as an immediate corollary of Eq. (A16)

$$\langle\lambda_3'\lambda_4'|H_s|\lambda_1'\lambda_2'\rangle_C = (-1)^{\lambda'-\mu'}\langle\lambda_4'\lambda_3'|H_s|\lambda_1'\lambda_2'\rangle_C\,, \qquad (A17)$$

i.e., interchanging initial-state particles gives the same phase change as interchanging final-state particles.

Now we return to Eq. (A6), use Eq. (A16) to interchange the initial-state particles in the s and u channels, use Eq. (A3) to switch to a Jacob-Wick "particle 2" convention everywhere, and get

$$\langle\lambda_3\lambda_2|H_u|\lambda_1\lambda_4\rangle = \prod_{i=1}^{4}\bar{d}_{\lambda_i'\lambda_i}(\omega_i)(-1)^q\langle\lambda_3'\lambda_4'|H_s|\lambda_1'\lambda_2'\rangle\,, \qquad (A18)$$

where $q=\sigma'+2S_1+2S_4+\mu'-\lambda$. σ' and ω_i are defined in Eq. (A7), $\mu'=\lambda_3'-\lambda_4'$; $\lambda=\lambda_1-\lambda_4$.

In Sec. II we show that the su part of our direct channel amplitude is [except for a phase $(-1)^{-\lambda'}$] just the st part relabeled $3\leftrightarrow 4$. For the proof that the su part superconverges, we would like that part to remain just the st part relabeled $3\leftrightarrow 4$, even after the crossing matrix is applied. The crossing matrix in Eq. (A18), however, is not just the st crossing matrix relabeled $3\leftrightarrow 4$. [For one thing, we relabeled $1\leftrightarrow 2$ in Eqs. (A8) and (A9), not $3\leftrightarrow 4$.] The latter matrix would be (we drop phase factors which cancel out of SCR's anyway)

$$\begin{aligned}(3\leftrightarrow 4)\prod_{i=1}^{4}d_{\lambda_i'\lambda_i}(\chi_i) &= \prod{}_{\lambda_i'\lambda_i}d(\pi-\omega_i)\\ &= \prod d_{\lambda_i'\lambda_i}(\omega_i)(-1)^{S_i+\lambda_i'}\\ &\propto (-1)^{\lambda-\mu}\prod d_{\lambda_i'\lambda_i}(\omega_i)(-1)^{S_i+\lambda_i'}\\ &\propto \prod d_{\lambda_i'\lambda_i}(\omega_i)(-1)^{\lambda-\mu+\lambda'-\mu'}\\ &\propto \prod d_{\lambda_i'\lambda_i}(\omega_i)(-1)^{q+\lambda'-\mu}\,.\end{aligned}$$

(In the third line we used parity conservation to relate two u-channel amplitudes having helicities of opposite sign.) Thus the actual crossing matrix, Eq. (A18), differs from a $3 \leftrightarrow 4$ permutation of the st crossing matrix only by an extra term $\lambda' - \mu$ added to q. As mentioned following Eq. (2.9), the λ' just cancels the extra $(-1)^{-\lambda'}$ phase of the direct channel su part; the μ multiplies the crossed su part by an extra phase, but does not affect its superconvergence. (In comparing this appendix to Sec. II, note that we dropped the primes on direct channel helicities in Sec. II.)

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