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Light-cone structure in the cavity approximation to the bag theory

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It is shown that the imposition of boundary conditions does not change the light-cone singularity of the free Dirac anticommutator function in three dimensions. A system of noninteracting quarks confined to a rigid cavity therefore exhibits Bjorken scaling.

In a recent paper[1] the problem of the deep-inelastic scattering of leptons off of a "bag"[2,3] was discussed in an approximation which ignores much of the dynamical nature of the bag's boundaries. The approximation, which was developed in Ref. 3, treats the bag as a (spherical) cavity of fixed radius R_0 nailed down at some point in space. The quantum modes in the cavity are populated with colored quarks so as to make possible construction of hadrons with the correct quantum numbers. The radius is not a free parameter but is fixed in terms of the field excitation as described in Ref. 3. For brevity we shall refer to this as the cavity approximation. The structure functions of deep-inelastic scattering were extracted by calculating highly virtual forward Compton scattering off of the occupied modes in the cavity.

The problem we wish to discuss here arose in that analysis but is probably also encountered in a semiclassical treatment of inelastic Compton scattering from any confined system. It arises within the cavity approximation and sheds some light on the validity of this approximation in the general context of the bag model.

The analysis of (I) was carried out in coordinate space, where a crucial role is played by the anticommutator function which carries the excitation from the point $(\vec{x}_1, t_1)$, where the virtual photon is absorbed, to the point $(\vec{x}_2, t_2)$, where it is emitted. (Henceforth we shall refer to the anticommutator function, somewhat loosely, as the propagator.) The approximation dictates the use of the "cavity propagator"—which, in particular, satisfies boundary conditions at the cavity walls. In (I) the cavity propagator, $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)$, was replaced by the free-space propagator $S(\vec{x}_1, t_1; \vec{x}_2, t_2)$. It was shown that in one spatial dimension the two are equivalent in the Bjorken limit; however, the replacement was not justified for the realistic three-dimensional case.

Here we show under very general circumstances and in three dimensions that the imposition of boundary conditions does not change the light-cone singularity of the free propagator, and that therefore the cavity and free propagators yield equivalent structure functions in the Bjorken limit. The discussion is based upon a multiple reflection expansion[4] for the cavity propagator, which expresses the propagation from $\vec{x}_1$ to $\vec{x}_2$ as a sum over successive reflections from the boundaries. Only the direct, lightlike propagation from $\vec{x}_1$ to $\vec{x}_2$ contributes in the Bjorken limit.

The situation is somewhat analogous to the canonical light-cone expansions for conventional interacting field theory developed some years ago.[5] In the cavity approximation the boundary acts like an external c-number field. The interactions of the quarks with the boundary are "soft" in the sense that they do not modify the leading light-cone singularity. The same conclusion was reached for certain conventional field theories in Refs. 5 provided the effects of fluctuations (renormalizations) are ignored. Fluctuations induced by the dynamics of the bag's boundary are the "renormalizations" which we are forced to ignore. The bag model deviates from the analogy because the wave function of the target (e.g., proton) is known in terms of the fields being studied—therefore the structure functions can be explicitly calculated in this model.[1]

Although we work in three spatial dimensions with Dirac fields and a spherical cavity, it will be obvious to the reader that our conclusions pertain to Bose fields as well, to other numbers of dimensions, and to any cavity in which it is permissible to take $(q^2)^{1/2}$ (the virtual photon mass) larger than the principal curvatures everywhere on the cavity boundary.

Another system which may be studied with these methods is the "cavity approximation" to the SLAC confinement model.[6] There, quark fields are confined to a (nearly) two-dimensional closed surface in three-dimensional space. In a "cavity approximation" the surface would be treated as a fixed

c-number function. Because the quarks are excluded from the interior of this shell there are no lightlike paths between points on the surface and the propagator of this model in fact vanishes on the light cone. Therefore Bjorken scaling will not emerge from a "cavity approximation" to this model.

We now return to the bag. The spin-averaged structure functions for electroproduction off of a cavity containing quark radiation are given by[1]

$$W_{\mu\nu}=\frac{M}{2\pi}\int_{-\infty}^{\infty}dt\int_{\text{bag}}d^3x_1\int_{\text{bag}}d^3x_2\psi(\vec{x}_1,t;\vec{x}_2,0;q)$$
$$\times\langle T|\bar{q}(\vec{x}_1,t)\gamma_\mu QS_c(\vec{x}_1,t;\vec{x}_2,0)\gamma_\nu q(\vec{x}_2,0)-\bar{q}(\vec{x}_2,0)\gamma_\nu QS_c(\vec{x}_2,0;\vec{x}_1,t)\gamma_\mu Qq(\vec{x}_1,t)|T\rangle\,. \tag{1}$$

Q is the quark charge matrix. $\psi(\vec{x}_1,t;\vec{x}_2,0;q)$ is a familiar phase:

$$\psi(\vec{x}_1,t_1;\vec{x}_2,t_2;q)\equiv e^{iq^0(t_1-t_2)-i\vec{q}\cdot(\vec{x}_1-\vec{x}_2)}\,. \tag{2}$$

$q(\vec{x},t)$ is the cavity quark field and $S_c(\vec{x}_1,t_1;\vec{x}_2,t_2)$ is the anticommutator function constructed from $q(\vec{x},t)$:

$$S_c(\vec{x}_1,t_1;\vec{x}_2,t_2)\equiv\{q(\vec{x}_1,t),\bar{q}(\vec{x}_2,t_2)\}\,. \tag{3}$$

Other structure functions, e.g. for neutrino production, have similar expressions.

Important regions of coordinate space are determined by the phase ψ which modulates the current correlation function in Eq. (1), and by the condition that the anticommutator is causal,

$$|t|\geq|\vec{x}_1-\vec{x}_2|\,. \tag{4}$$

In the Bjorken limit, the virtual photon's energy becomes infinite ($q^0\to\infty$) with

$$|\vec{q}|=q^0+M\xi\,,\qquad \xi\equiv-q^2/2P\cdot q\,. \tag{5}$$

If $(\vec{x}_1-\vec{x}_2)$ is nearly parallel $\vec{q}$ and if t is nearly equal to $|\vec{x}_1-\vec{x}_2|$ then the phase $\psi(\vec{x}_1,t_1;\vec{x}_2,t_2;q)$ remains stationary in the limit. Singularities in S_c along the light cone in directions tangent to the vector $\vec{q}$ therefore contribute to the structure function in the limit.

In the case specifically of interest to us the propagator will also receive singular contributions on surfaces within the light cone but isolated a fixed distance, T, away from the light cone:

$$t\geq|\vec{x}_1-\vec{x}_2|+T\,. \tag{6}$$

It is necessary to review arguments that the light-cone behavior of S_c is dominant in the Bjorken limit even in the presence of singularities elsewhere in coordinate space.[7]

Contributions to S_c which satisfy Eq. (6) are multiplied by a phase which is not less oscillatory than

$$\psi(\vec{x}_1,t;\vec{x}_2,0;q)=e^{iq^0T}\psi(\vec{x}_1,|\vec{x}_1-\vec{x}_2|;\vec{x}_2,0;q)\,, \tag{7}$$

where $\psi(\vec{x}_1,|\vec{x}_1-\vec{x}_2|;\vec{x}_2,0;q)$ is the stationary phase which multiplies light-cone contributions,

$$\psi(\vec{x}_1,|\vec{x}_1-\vec{x}_2|;\vec{x}_2,0;q)=e^{-iM\xi|\vec{x}_1-\vec{x}_2|}\,. \tag{8}$$

Consider, now, an experimental measurement of $W_{\mu\nu}$ with a percentage resolution of Δ in q^0 which we take to be Gaussian. The experimental structure function ($\overline{W}_{\mu\nu}$) is a weighted average of the one we calculate:

$$\overline{W}_{\mu\nu}(q^0,\xi)\equiv\int\frac{dq'^0}{\Delta\sqrt{\pi}}\,e^{-(q^0-q'^0)^2/\Delta^2q^{02}}W_{\mu\nu}(q'^0,\xi)\,. \tag{9}$$

Contributions like Eq. (7) are damped by a factor

$$e^{-(\Delta q^0T)^2/4}$$

in the Bjorken limit. Only values of $T\lesssim 1/q^0$ are important. In this sense the light cone dominates the Bjorken limit.

We proceed now to construct an expansion of the cavity propagator about the light cone. The construction imitates Huygens's principle for the propagation of light. If a disturbance occurs at a point $(\vec{x}_2,t_2)$ in the cavity it can propagate to the point $\vec{x}_1$ in a time $\Delta t\equiv|t_1-t_2|=|\vec{x}_1-\vec{x}_2|$. This propagation is indistinguishable from propagation in free space. Modifications due to confinement occur with successive reflections of the disturbance from the cavity walls. This is shown schematically in Fig. 1, where Path I is the lightlike path and Path II is the shortest path via reflection. For fixed $\vec{x}_1$ and $\vec{x}_2$ no reflection arrives sooner than some fixed time $T(\vec{x}_1,\vec{x}_2)$ after the direct ray, so they all satisfy Eq. (6) with $T=T(\vec{x}_1,\vec{x}_2)$. The remainder of this section is concerned with making this simple argument quantitative and dispensing with the dependence of $T(\vec{x}_1,\vec{x}_2)$ or $\vec{x}_1$ and $\vec{x}_2$.

The framework for this discussion is a multiple reflection expansion for the cavity propagator. We consider (massive) Dirac fields in three spatial dimensions, obeying bag boundary conditions.[3] Other fields or other boundary conditions are to be treated similarly. The propagation function of interest is the anticommutator function defined by Eq. (3). It satisfies the following boundary conditions:

$$S_c(\vec{x}_1, t; \vec{x}_2, t) = \gamma^0 \delta^3(\vec{x}_1 - \vec{x}_2), \tag{10}$$

$$(i\vec{n}_1 \cdot \vec{\gamma} - 1) S_c(\vec{x}_1, t_1; \vec{x}_2, t_2) = 0$$
$$\text{if } \vec{x}_1 \text{ is on the cavity boundary.} \tag{11}$$

Here $\vec{n}_1$ is the interior unit normal at the point $\vec{x}_1$ on the boundary. Of course $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)$ obeys the Dirac equation inside the cavity

$$(i\vec{\partial\!\!\!/}_1 - m) S_c(\vec{x}_1, t_1; \vec{x}_2, t_2) = S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)(-i\overleftarrow{\partial\!\!\!/}_2 - m) = 0 \tag{12}$$

and possesses the symmetry property

$$S_c(\vec{x}_1, t_1; \vec{x}_2, t_2) = S_c(\vec{x}_2, t_2; \vec{x}_1, t_1). \tag{13}$$

The cavity propagator obeys an integral equation relating its values interior to the cavity to its boundary values. The kernel is the free-space Dirac anticommutator function $S(\vec{x}_1, t_1; \vec{x}_2, t_2)$ which obeys Eqs. (10), (12), and (13) but not Eq. (11):

$$iS_c(\vec{x}_1, t_1; \vec{x}_2, t_2) = iS(\vec{x}_1, t_1; \vec{x}_2, t_2) + \int_{t_2}^{t_1} d\tau \int d^2\xi\, S(\vec{x}_1, t_1; \vec{\xi}, \tau) S_c(\vec{\xi}, \tau; \vec{x}_2, t_2). \tag{14}$$

Here $\vec{\xi}$ is a point on the cavity boundary. The ξ integral covers the bag boundary at each time τ. This equation is continuous as $\vec{x}_1$ approaches a point on the boundary. Equation (14) is valid only for $t_1 > t_2$. For $t_1 < t_2$, $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)$ may be defined from Eq. (13). Equation (14) may be iterated to obtain the multiple-reflection expansion for $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)$:

$$iS_c(\vec{x}_1, t_1; \vec{x}_2, t_2) = iS(\vec{x}_1, t_1; \vec{x}_2, t_2) + \int_{t_2}^{t_1} d\tau \int d^2\xi\, S(\vec{x}_1, t_1; \vec{\xi}, \tau) S(\vec{\xi}, \tau; \vec{x}_2, t_2)$$
$$- i \int_{t_2}^{t_1} d\tau \int_{t_2}^{\tau} d\tau' \int d^2\xi \int d^2\xi'\, S(\vec{x}_1, t_1; \vec{\xi}, \tau) S(\vec{\xi}, \tau; \vec{\xi}', \tau') S(\vec{\xi}', \tau'; \vec{x}_2, t_2) + \cdots. \tag{15}$$

The physical interpretation of Eq. (15) is simple and important: The influence of an event at $\vec{x}_2$ and t_2 upon a point $\vec{x}_1$ at time t_1 proceeds by successive reflections from the boundary. The terms in Eq. (15) represent successively zero, one, two, etc. reflections. For a massless field the free-space anticommutators are δ functions on the appropriate light cone and the intuitive picture of propagation by specular reflection is exact. For a massive field dispersion complicates the intuitive picture but not the proof of light-cone dominance.

To proceed we note from Eq. (15) that $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2) - S(\vec{x}_1, t_1; \vec{x}_2, t_2)$ vanishes unless there are points $(\vec{\xi}, \tau)$ on the bag's surface which are both in the backward light cone of $(\vec{x}_1, t_1)$ and in the forward light cone of $(\vec{x}_2, t_2)$. Figure 2 shows the region (for one spatial dimension, higher dimensions being entirely analogous but harder to draw) for fixed $\vec{x}_2$, inside of which $S_c(\vec{x}_1 t_1; \vec{x}_2, t_2)$ and $S(\vec{x}_1, t_1; \vec{x}_2, t_2)$ are identical. For all points $(\vec{x}_1, t_1)$ inside the rectangle $ABCD$ the propagators are identical. In the Bjorken limit only points tangent to the light cone about $(\vec{x}_2, t_2)$, parallel to $\vec{q}$, and at distances of order $1/q^0$ about the light cone will contribute. In Fig. 2 this is the region bounded by the line HH' above, the section of the lines AB and AC below, and the cavity boundary LL' elsewhere. Throughout this region, except for the small region near the cavity boundary, the cavity and free-space

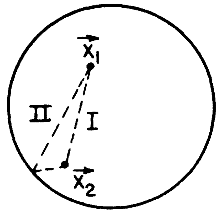

FIG. 1. Lightlike path (I) and shortest path involving reflection (II) between two points $\vec{x}_1$ and $\vec{x}_2$ in a spherical cavity.

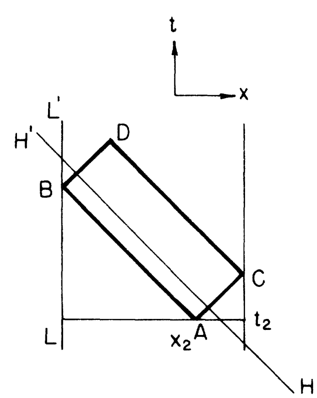

FIG. 2. Space-time domain in two dimensions within which cavity and free-space propagators are identical. If $(\vec{x}, t)$ is within the rectangle $ABCD$ the anticommutator $S_c(\vec{x}, t; \vec{x}_2 t_2)$ is identical to the free-space anticommutator function.

propagators are identical. To complete the proof that $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)$ may be replaced by $S(\vec{x}_1, t_1; \vec{x}_2, t_2)$ in the Bjorken limit it is necessary to show that the contribution of the small region where a reflection is important (bounded by the line BD below, by HH' above, and by the cavity boundary LL' at the side) vanishes in the Bjorken limit.

Consider some large (fixed) value of $q^0(\sim q^2)$ and define a sphere within the cavity by the relation

$$r < R(\epsilon) \equiv (1-\epsilon)R_0 . \tag{16}$$

The integrals of Eq. (1) may be divided into three regions: (1) $|\vec{x}_1| < R(\epsilon)$, $|\vec{x}_2| < R(\epsilon)$; (2) $|\vec{x}_1| < R(\epsilon)$, $\vec{x}_2| > R(\epsilon)$ or vice versa; and (3) $x_1 > R(\epsilon)$, $x_2 > R(\epsilon)$. Consider region (1): All paths from $\vec{x}_1$ to $\vec{x}_2$ via reflection from the boundaries satisfy

$$|t_1 - t_2| > |\vec{x}_1 - \vec{x}_2| + 2\epsilon R_0 .$$

The arguments regarding light-cone dominance presented above indicate that only $(x_1 - x_2)^2 \gtrsim 1/q^2$ will contribute to $\overline{W}_{\mu\nu}$. We therefore choose $4\epsilon^2 R_0^{\,2} > 1/q^2$. No reflection is close enough to the light cone to contribute; by Eq. (15) $S_c(\vec{x}_1, t_1; \vec{x}_2, t_2)$ and $S(\vec{x}_1, t_1; \vec{x}_2, t_2)$ are identical for region (1).

If we let $q^2 \to \infty$, taking ϵ^2 to zero subsequently, $R(\epsilon) \to R_0$ and all points in the cavity satisfy the conditions for region (1). However, we must examine the possibility that the other regions [(2) and (3)] contribute so singularly to the structure function that they do not become unimportant even as $\epsilon \to 0$. This is excluded if the first reflection in the multiple-reflection expansion is no more singular than the free-space anticommutator function. Since the free-space anticommutator contributes a finite term to the structure function, the reflections will contribute finite terms multiplied by the volume over which reflection remains important and this vanishes as $\epsilon \to 0$.

This argument employs the method of images. For region (3) both $\vec{x}_1$ and $\vec{x}_2$ are very near the cavity boundary. For case (2) one coordinate, say $\vec{x}_1$, is near the boundary. Only points of reflection nearby $\vec{x}_1$ yield nearly lightlike paths from $\vec{x}_2$ to $\vec{x}_1$. This is illustrated in Fig. 3. In both cases for small ϵ the distance from the points to the boundary is small compared to the radius of curvature of the boundary (here R_0) and we may replace the boundary by the appropriate tangent plane (see Fig. 3). A rigorous justification of this approximation is beyond the scope of this paper. It is discussed for the scalar wave equation by Balian and Bloch,[4] and should be familiar to the reader as the consequence of standing very close to a nonplanar mirror. In this approximation the

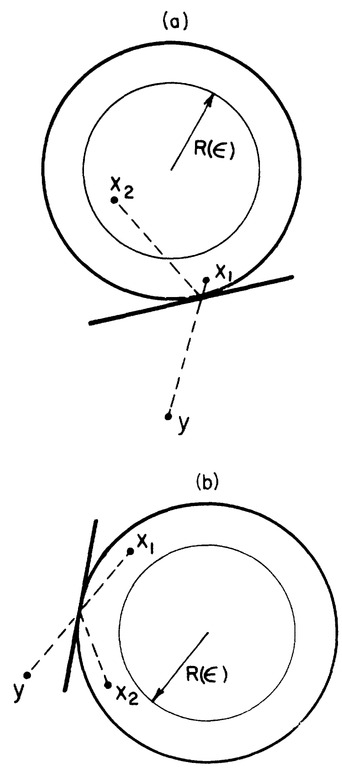

FIG. 3. (a) Method-of-images construction when one point is near the boundary. (b) Method-of-images construction when both points are near the boundary.

cavity propagator is given by

$$S_c(\vec{x}_1, t_1; \vec{x}_2, t_2) = S(\vec{x}_1, t_1; \vec{x}_2, t_2) + i\hat{n}\!\!\!/ S(\vec{x}_1, t_1; \vec{y}, t_2) , \tag{17}$$

where $\vec{y}$ is the image of $\vec{x}_2$ through the tangent plane and $\hat{n}$ is the unit normal to the tangent plane.[8] Equation (17) may be obtained from Eq. (15) by replacement of the boundary by the tangent plane.

Clearly the modification of the free-space propagator induced by the cavity boundary is as singular as but no more singular than the free-space propagator. The contribution of points in regions (2) and (3) therefore remains bounded and goes to zero in the Bjorken limit as $\epsilon \to 0$.

This completes the discussion of the equivalence of the free-space and cavity propagators in the Bjorken limit.

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[1]R. L. Jaffe, Phys. Rev. D 11, 1953 (1975). Hereafter we refer to this paper as (I).

[2]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3479 (1974).

[3]A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, Phys. Rev. D 10, 2599 (1974).

[4]The expansion is well known in potential theory; see R. Balian and C. Bloch, Ann. Phys. (N.Y.) 60, 401 (1970). The application to fermions is, so far as we know, new.

[5]J. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971); D. Gross and S. B. Treiman, *ibid.* 4, 1059 (1971); C. H. Llewellyn Smith, *ibid.* 4, 2392 (1971).

[6]W. A. Bardeen, M. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, Phys. Rev. D 11, 1094 (1975).

[7]This subject has been studied for various singular behaviors by R. L. Jaffe [Phys. Rev. D 6, 716 (1972)], D. Bhaumik, O. W. Greenberg, and R. N. Mohapatra [*ibid.* 6, 2989 (1972); 6, 1948(E) (1973)], and D. Bhaumik and O. W. Greenberg [*ibid.* 7, 3136 (1973)].

[8]Formula (17) is valid for the massless ψ field. For the massive case a similar expression can be obtained by solving the Dirac equation with the boundary condition (11) imposed on a plane. The anticommutator is the free-field one plus a correction term which is not singular on the light cone.