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Self-stress and renormalization group

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An intimate connection exists between the vanishing of the self-stress of a particle (at rest), as required by Lorentz covariance, and a unique normalization of a universal function of relevant coupling constants in the theory in modern renormalization-group equations. This connection leads, unambiguously, to a correct version of the classic Pais-Epstein result for the self-stress in a self-consistent manner.

In this note we consider the old and fundamental problem associated with the self-stress of a particle in the language of the modern renormalization-group approach which carefully takes into consideration the breaking of scale invariance, in general, in relativistic quantum field theory. We have in mind the renormalization-group equations in the form of those of Callan and Symanzik.[1] The vanishing of the self-stress of a particle (at rest), as a requirement of Lorentz covariance, and its apparent and immediately connected difficulty in quantum field theory are well known and go back to an early classic work of Pais and Epstein.[2] The apparent and immediate difficulty resulted when the lowest-order expression for the self-mass of the electron, in quantum electrodynamics, was substituted in the Pais-Epstein result for the self-stress, which yielded a nonvanishing result for the latter. The reason for this apparent inconsistency was clarified long ago by Rohrlich,[3] who emphasized that regulators are to be introduced initially into the theory for self-consistency (thus breaking scale invariance); in this manner he obtained the unambiguous result that the self-stress of the electron does indeed vanish (even to lowest order in perturbation theory).[4] Here we discuss this problem from the renormalization-group approach.[1] An intimate connection exists between the vanishing of the self-stress of a particle and a unique normalization of a universal function of the relevant coupling constants in the theory in the renormalization-group equations. This leads, unambiguously, to a correct version of the Pais-Epstein expression in a self-consistent manner.

The self-stress of a particle (at rest) is formally defined[2-4] in a standard notation:

$$\begin{aligned} 3\langle s(0)\rangle &= \lim_{V\to\infty}\left(\frac{1}{V}\right)_0\!\left\langle p\left|\int d^3x\,\Theta_\mu{}^\mu(x)\right|p\right\rangle_0 - M \\ &= {}_0\langle p|\Theta_\mu{}^\mu(0)|p\rangle_0 - M \\ &= \lim_{VT\to\infty}\left(\frac{1}{VT}\right)_0\!\left\langle p\left|\int (dx)\Theta_\mu{}^\mu(x)\right|p\right\rangle_0 - M , \end{aligned} \tag{1}$$

where in the second and third lines we have made use of translational invariance in the limit of (space) $V\to\infty$ and (space-time) $VT\to\infty$, respectively. The state $|p\rangle_0$ denotes a state of a particle at rest with (renormalized) mass M. $\Theta_\mu{}^\mu$ denotes the trace of a conserved, symmetrized, and possibly improved energy-momentum tensor in the theory to formally define a dilation current.[5,6] The first expression on the right-hand side of (1) is understood to denote the connected part of the matrix element. For concreteness we consider M to denote the mass of a fermion in a theory governed by the interaction Lagrangian density: $G_0\overline{\Psi}\gamma_5\Psi\phi-(\lambda_0/4)\phi^4$, where the symbols have their usual meanings. A very convenient starting point for our purposes is the derivation of the Callan-Symanzik equations[1] in Ref. 5. Let M_0 and μ_0 denote the unrenormalized masses of the fermion and the boson, respectively. From the reduction formula, we obtain from the work of Ref. 5 that the first term on the right-hand side of (1) is given by

$$\left(ZM_0\frac{\partial}{\partial M_0}+Z'\mu_0{}^2\frac{\partial}{\partial \mu_0{}^2}\right)S^{-1}(p)\Big|_{\text{mass shell, at rest}} , \tag{2}$$

where $S(p)$ is the unrenormalized fermion propagator; Z and Z' are (over-all) properly chosen renormalization constants to make the resulting expression (2) cutoff-independent. The unique normalization condition we have mentioned above will now follow. By using the chain rule[5] and substituting the renormalized fermion propagator $\tilde{S}(p)$ for $S(p)$, etc., one immediately obtains for (2) at arbitrary p, with the definition

$$L\equiv ZZ_2{}^{-1}M_0(\partial/\partial M_0)+Z'Z_2{}^{-1}\mu_0{}^2(\partial/\partial\mu_0{}^2) ,$$

the following quantity:

$$\left[(LM)\frac{\partial}{\partial M}+(L\mu^2)\frac{\partial}{\partial\mu^2} + (LG)\frac{\partial}{\partial G}+(L\lambda)\frac{\partial}{\partial\lambda}-\left(\frac{LZ_2}{Z_2}\right)\right]\tilde{S}^{-1}(p) . \tag{3}$$

From the mass-shell condition $\tilde{S}^{-1}(p) \sim (\gamma \cdot p + M)$ for the renormalized propagator with unit amplitude, and the vanishing of $\langle s(0) \rangle$ in (1), we obtain the unique normalization condition from (3) (see also Ref. 7)

$$\left(Z Z_2^{-1} M_0 \frac{\partial M}{\partial M_0} + Z' Z_2^{-1} \mu_0^2 \frac{\partial M}{\partial \mu_0^2} \right) = M . \tag{4}$$

The above normalization condition is usually chosen in the literature either for simplicity or for convenience. The important point, here, is that it is fixed by the vanishing of $\langle s(0) \rangle$ (i.e., by the requirement of Lorentz covariance) and the proper (re)normalization of $\tilde{S}(p)$ near the mass shell. To determine the constant Z, for example, we apply the total differential operator $[M(d/dM) + \mu^2(d/d\mu^2)]$ to $\tilde{S}^{-1}(p)$ while keeping G_0, λ_0, and an ultraviolet cutoff Λ (introduced in some convenient manner) fixed,[8] and we pass to the mass-shell condition $\tilde{S}^{-1}(p) \sim (\gamma \cdot p + M)$:

$$\begin{aligned} \left[M \frac{\partial}{\partial M} + \mu^2 \frac{\partial}{\partial \mu^2} + \beta_1 \frac{\partial}{\partial G} + \beta_2 \frac{\partial}{\partial \lambda} \right] \tilde{S}^{-1}(p) &= \left[\left(M \frac{d}{dM} + \mu^2 \frac{d}{d\mu^2} \right) \ln M_0 \right] M_0 \frac{\partial}{\partial M_0} \tilde{S}^{-1}(p) \\ &\quad + \left[\left(M \frac{d}{dM} + \mu^2 \frac{d}{d\mu^2} \right) \ln \mu_0^2 \right] \mu_0^2 \frac{\partial}{\partial \mu_0^2} \tilde{S}^{-1}(p) , \end{aligned} \tag{5}$$

where

$$\beta_1(G, \lambda) = \left(M \frac{d}{dM} + \mu^2 \frac{d}{d\mu^2} \right) G ,$$

$$\beta_2(G, \lambda) = \left(M \frac{d}{dM} + \mu^2 \frac{d}{d\mu^2} \right) \lambda .$$

We then obtain, for example,

$$\begin{aligned} Z &= Z_2 \left(M \frac{d}{dM} + \mu^2 \frac{d}{d\mu^2} \right) \ln M_0 \\ &\equiv Z_2 (1 + \delta_1(G, \lambda)) , \end{aligned} \tag{6}$$

etc. Let us solve M_0 in terms of M and renormalized quantities with an ultraviolet cutoff kept fixed: $M_0 = M Z_0$. Upon substituting this expression into the left-hand side of (1) [and equivalently into the left-hand side of (4)], using the expressions (2), (5), and (6), and in turn using the normalization condition (4) self-consistently, we obtain for the expression in question

$$-\frac{M^2}{M_0} \left[M \frac{\partial}{\partial M} + \mu^2 \frac{\partial}{\partial \mu^2} + \beta_1 \frac{\partial}{\partial G} + \beta_2 \frac{\partial}{\partial \lambda} \right] \left(\frac{M_0}{M} \right) + M(1 + \delta_1) . \tag{7}$$

We have also applied the chain rule on the right-hand side of (5) and equated the coefficients of the various derivatives in (5). In a self-consistent manner we finally obtain for (1)

$$\begin{aligned} 3 \langle s(0) \rangle = -M \Bigg[& \left(M \frac{\partial}{\partial M} + \mu^2 \frac{\partial}{\partial \mu^2} + \beta_1 \frac{\partial}{\partial G} + \beta_2 \frac{\partial}{\partial \lambda} \right) \ln \left(\frac{M_0}{M} \right) \\ & - (1 + \delta_1) \Bigg] - M , \end{aligned} \tag{8}$$

which vanishes identically, since its right-hand side is the Callan-Symanzik equation for $\ln(M_0/M)$ and it contains the anomalous term $\delta_1(G, \lambda)$. To lowest order, for example,

$$\frac{M_0 - M}{M} \sim -\frac{G^2}{32\pi^2} \ln \left(\frac{\Lambda^2}{M^2} \right) , \tag{9}$$

$$\delta_1 = \frac{G^2}{16\pi^2} , \tag{10}$$

and hence

$$3 \langle s(0) \rangle = -M \left(\frac{G^2}{16\pi^2} - \frac{G^2}{16\pi^2} \right) = 0 . \tag{11}$$

Clearly this result is true in any renormalizable field theory. It rests on the unique normalization condition (4) followed by a self-consistent analysis. Needless to say, the derivatives with respect to the couplings G and λ appear in (8) in spite of the fact that the latter are dimensionless.

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[1]C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[2]A. Pais and S. T. Epstein, Rev. Mod. Phys. 21, 445 (1949).

[3]F. Rohrlich, Phys. Rev. 77, 357 (1950).

[4]For some other early references see F. Villars, Phys. Rev. 79, 122 (1950); S. Borowitz and W. Kohn, *ibid.* 86, 985 (1952); Y. Takahashi and H. Umezawa, Prog. Theor. Phys. 11, 251 (1952); J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955); S. S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson, New York, 1961).

[5]S. Coleman and R. Jackiw, Ann. Phys. (N. Y.) 67, 552 (1971).

[6]It is not the purpose of this note to construct improved energy-momentum tensors. For a recent thorough study of this see, for example, D. Z. Freedman and E. J. Weinberg, Ann. Phys. (N. Y.) 87, 354 (1974) and further references therein.

[7]Compare also, for example, with P. Carruthers, Phys. Rev. D 2, 2265 (1970).

[8]See, for example, S. L. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972); S. L. Adler, Phys. Rev. D 5, 3021 (1972); 7, 1948(E) (1973) for all details. We are implicitly assuming the validity of such a cutoff procedure in this work.