PHYSICAL REVIEW D VOLUME 12, NUMBER 4 15 AUGUST 1975

Physical degrees of freedom in two-dimensional quantum electrodynamics

A. Das
Department of Physics, Visva-Bharati University, Santiniketan-731235, West Bengal, India
(Received 28 January 1975)

Starting from a two-dimensional free spinor Lagrangian we arrive at a theory describing Goldstone bosons. Then, using a theorem by Coleman and the Higgs effect, it is shown that two-dimensional quantum electrodynamics is essentially a theory of massive scalar bosons.

Recently Willemsen[1] has advanced a simple argument to demonstrate that electrodynamics in two dimensions is essentially a theory of massive spin-zero bosons. In this note we shall place another simple model to elucidate this interesting feature of two-dimensional electrodynamics. Let us assume the existence of a free two-dimensional fermion theory given by the Lagrangian

$$\mathcal{L}=-\bar{\psi}\gamma_\mu\partial^\mu\psi\,, \tag{1}$$

with

$$\{\psi^\dagger_\alpha(x),\psi_\beta(y)\}_{x_0=y_0}=\delta_{\alpha\beta}\delta(x_1-y_1)\,. \tag{2}$$

The Lagrangian given by Eq. (1) is invariant under the constant-phase transformation

$$\psi\to e^{i\xi}\psi\,, \tag{3}$$

yielding the conserved current

$$j_\mu=i\bar{\psi}\gamma_\mu\psi\,. \tag{4}$$

If now the current is defined in the following limiting way:

$$j_\mu=\lim_{\epsilon\to 0}:i\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma_\mu\psi(x-\tfrac{1}{2}\epsilon): \tag{5}$$

and

$$j_\mu j_\nu=\lim_{\epsilon\to 0}:j_\mu(x+\tfrac{1}{2}\epsilon)j_\nu(x-\tfrac{1}{2}\epsilon):\,, \tag{6}$$

then we can define[2] a Sugawara theory given by

$$[j_0(x_1),j_1(y_1)]_{x_0=y_0}=-ic_1\partial_{x_1}\delta(x_1-y_1) \tag{7a}$$

and a stress tensor

$$\Theta_{\mu\nu}=\frac{1}{2c}:(j_\mu j_\nu+j_\nu j_\mu-g_{\mu\nu}j^\lambda j_\lambda):\,, \tag{7b}$$

where c is a positive number which fixes the normalization of j.

At this stage we can demand the existence of a massless boson in the theory.[3] From Eqs. (7) we have

$$\partial^\mu j_\mu=0 \tag{8}$$

and also

$$\partial_\mu j_\nu=\partial_\nu j_\mu\,, \tag{9}$$

which demands

$$j_\mu=\sqrt{c}\;\partial_\mu\varphi\,. \tag{10}$$

Here φ is a scalar field, and recalling Eqs. (7) and (8) we construct the theory of a massless boson given by

$$[\dot{\varphi}(x_1),\varphi(y_1)]_{x_0=y_0}=-i\delta(x_1-y_1)\,, \tag{11}$$

$$\Theta_{\mu\nu}=\tfrac{1}{2}(\partial_\mu\varphi\partial_\nu\varphi+\partial_\nu\varphi\partial_\mu\varphi)-g_{\mu\nu}\partial_\lambda\varphi\partial^\lambda\varphi\,, \tag{12}$$

and

$$\Box\varphi=0\,. \tag{13}$$

At this stage the theory given by Eqs. (11)–(13) leads to an uncomfortable situation owing to a theorem by Coleman[4] which we shall discuss later. However, to understand in what sense the massless scalar boson field given by Eq. (13) exists in two dimensions let us believe with Coleman[4] that we still define a particle as a normalizable eigenstate of $P_\mu P^\mu$ which are the states of one fermion and one antifermion (both in normalizable states moving to the left and still an eigenstate of $P_\mu P^\mu$).

In Willemsen's demonstration[5] φ is a pseudoscalar particle, this choice being dictated by the use of the Sommerfeld ansatz

$$\partial_{x_1}\psi=i(\pi/2)\{j_1+\gamma^5\gamma^0,\psi\}\,.$$

The present theory is invariant under the constant-phase transformation given by Eq. (3), which guarantees the existence of a conserved current j_μ. Let us consider that in our two-dimensional world particles can be made to move to the left and to the right separately, i.e., we can consider the forward and backward light cone separately. We therefore introduce hyperbolic coordinates

$$u=x_0+x_1$$

and

$$v=x_0-x_1\,,$$

and then have

$$j_+(u)=\tfrac{1}{2}[j_0(u)+j_1(u)] \tag{14a}$$

$$=\lim_{\epsilon\to 0}[\psi_1^\dagger(u+\tfrac{1}{2}\epsilon)\psi_1(u-\tfrac{1}{2}\epsilon)-\langle\psi_1^\dagger(u+\tfrac{1}{2}\epsilon)\psi_1(u-\tfrac{1}{2}\epsilon)\rangle] \tag{14b}$$

and

$$j_-(v) = \tfrac{1}{2}[j_0(v) - j_1(v)] \tag{15a}$$

$$= \lim_{\epsilon\to 0}[\psi_2^\dagger(v+\tfrac{1}{2}\epsilon)\psi_2(v-\tfrac{1}{2}\epsilon) - \langle\psi_2^\dagger(v+\tfrac{1}{2}\epsilon)\psi_2(v-\tfrac{1}{2}\epsilon)\rangle] , \tag{15b}$$

where ψ_1 and ψ_2 are solutions of the model given by Eq. (1), i.e.,

$$\psi_1 = (2\pi)^{-1/2}\int dk\,\theta(k)(c_k e^{iku} + id_k^\dagger e^{-iku}) , \tag{16a}$$

$$\psi_2 = (2\pi)^{-1/2}\int dk\,\theta(-k)(c_k e^{ikv} - id_k^\dagger e^{-ikv}) . \tag{16b}$$

Hence

$$\partial_0 = \partial_u + \partial_v$$

and

$$\partial_x = \partial_u - \partial_v ,$$

and also

$$\partial^\mu j_\mu = \partial_u j_-(v) + \partial_v j_+(u) = 0 , \tag{17a}$$

$$\epsilon^{\mu\nu}\partial_\mu j_\nu = -\partial_u j_-(v) + \partial_v j_+(u) = 0 . \tag{17b}$$

At this stage we have the Sugawara theory with hyperbolic coordinates,

$$[j_+(u), j_+(u')] = -i(2\pi)^{-1}\partial_u\delta(u-u') , \tag{18a}$$

$$[j_-(v), j_-(v')] = -i(2\pi)^{-1}\partial_v\delta(v-v') , \tag{18b}$$

and

$$[j_+(u), j_-(v)] = 0 . \tag{18c}$$

Since $\Box = 4\partial_u\partial_v$, it follows easily that

$$\Box j_\mu = 0 . \tag{19}$$

Let us now see what happens if one attempts in the present model to obtain a formal equivalence between the two Hamiltonians of the fermion system and the boson system respectively, i.e., when at least

$$H_{\text{boson}} = \sum_{p>0} p a_p^\dagger a_p \tag{20}$$

$$= H_{\text{fermion}} = \sum_{k>0} k(c_k^\dagger c_k + d_k^\dagger d_k) \tag{21}$$

is obtained. In Eqs. (20) and (21) we have considered for simplicity operators for particles moving to the right. In writing down wave functions explicitly we shall introduce box renormalization (and remove zero-energy modes by hand) which would be a means to get rid of the bad infrared behavior in the theory.[1] Thus

$$\psi_1(u) = L^{-1/2}\sum_k \theta(k)(c_k e^{iku} + id_k^\dagger e^{-iku}) , \tag{22}$$

with the choice that $\theta(k) = 1$ when $k = 0$ (for the c's) and $\theta(k) = 0$ when $k = 0$ (for the d's). Then following Freundlich[6] we can write the annihilation operator

$$a_p = (-i)\sqrt{2\pi}\,(pL)^{-1/2}\int du\,e^{-ipu} j_1(u)$$

$$= (-i)\sqrt{2\pi}\,(pL)^{-1/2}\sum_k \theta(k)[c_k^\dagger c_{k+p} - d_k^\dagger d_{k+p} - i\theta(p-k) d_k c_{p-k}] ,$$

and then

$$H_{\text{boson}} = H_{\text{fermion}} - \pi L^{-1} Q(Q-1) , \tag{23}$$

where H_{boson} and H_{fermion} are given, respectively, by Eqs. (20) and (21), and $Q = \int du j_1(u)$ is the fermion charge operator. Freundlich[6] demonstrated that fermion states are Glauber coherent states of bosons, the boson vacuum being the ground state of the charge sector corresponding to the given fermion state. The scalar boson is the Goldstone particle and the degenerate vacua are the ground states of various charge sectors.[7]

Let us now consider the two-point function[5]

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{d^2k}{2\pi}\, e^{ik\cdot x}\,\delta(k^2)\theta(k_0)$$

$$= \int \frac{dk_1}{2\pi|k_1|} e^{-i|k_1|x_1}\cos(k_1 x_0) ,$$

which implies an infrared divergence forbidding the presence of a two-dimensional momentum-space δ-function singularity on the light cone. This is reminiscent of the Coleman demonstration[4] that there is no Goldstone boson in two dimensions.

Because of the Coleman theorem we are forced to adopt one of the following two procedures: either bring back zero-energy modes to our theory or introduce a coupling of some other massless fields. For the first procedure to be operative one needs extra boson operators which cannot be defined in terms of bilinear products of the fermion operators.[6] There is then complete operator identity between fermions and bosons, and the charge spectrum becomes continuous. However, one then cannot infer *the status of the physical degrees of freedom in the theory*. For the other procedure, we introduce a minimal coupling of photon fields in our theory. Let the Lagrangian for the massless scalar boson be given by

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi . \tag{24}$$

The theory is broken in the sense that under the invariant transformation $\varphi \to \varphi + \xi$ (where ξ is a constant) Eq. (24) yields a Noether current the generator of which vanishes owing to the presence of the space integral. The transformation $\varphi \to \varphi + \xi$ implements only the zero-energy modes of φ which cannot be regarded as canonical degrees of freedom, in momentum space the support of the constant ξ being the single Lorentz-invariant point $k_\mu = 0$.

We now introduce minimal coupling of photons into our theory, i.e., we demand the invariance of Eq. (24) under

$$\varphi(x) \to \varphi(x) + \frac{e}{c}\,\xi(x)$$

and

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \xi(x) .$$

Then for Eq. (24) we have

$$\mathcal{L} = -\tfrac{1}{2}\left(\partial_\mu \varphi - \frac{e}{c} A_\mu\right)\left(\partial^\mu \varphi - \frac{e}{c} A^\mu\right) - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} , \tag{25}$$

where we have added the last term, the free photon Lagrangian. The above expression can be simplified to the form

$$\mathcal{L} = -\tfrac{1}{4} F'_{\mu\nu} F^{\mu\nu\prime} - \frac{e^2}{2c}\, \Phi_\mu \Phi^\mu , \tag{26}$$

where

$$F'_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu$$

and

$$\Phi_\mu = A_\mu - \frac{e}{c}\,\partial_\mu \varphi .$$

Equation (26) is a Lagrangian for a massive scalar photon with mass $e/\sqrt{c}$ which is the result of a very simple Higgs mechanism.

I would like to thank P. Roy for some useful discussions. Financial support in the form of a senior fellowship from the Council of Scientific and Industrial Research (Government of India) is gratefully acknowledged.

---

[1] J. F. Willemsen, Phys. Rev. D 9, 3570 (1974).

[2] S. Coleman, D. Gross, and R. Jackiw, Phys. Rev. 180, 1359 (1969).

[3] Y. Freundlich and D. Lurié, Phys. Rev. D 1, 1660 (1970); Y. Freundlich, *ibid.* 1, 3290 (1970).

[4] S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[5] See Eqs. (4) amd (6) in Ref. 1.

[6] Y. Freundlich, Nucl. Phys. B36, 627 (1972).

[7] If we believe in the existence of the $N \to \infty$, $L \to \infty$ limit, where N is the number of fermions, then from (23) we observe that $H_{\text{boson}} = H_{\text{fermion}}$. This may be regarded as the outcome of the fact that when the number of particles is finite one cannot construct boson operators out of fermion operators. See R. J. Penny, J. Math. Phys. 6, 1031 (1965).