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Bound states and the effective potential in field theories

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We calculate the leading contribution to the effective potential, $V(\phi)$, under the assumptions that there is a deep bound state and the corresponding pole dominates the one-particle irreducible vertices for spin-zero field theories. We find that V has a $\frac{3}{2}$-power branch point in the ϕ plane on the real axis. We show that the $\frac{3}{2}$ power is due to the fact that $V' = dV/d\phi$ satisfies an algebraic equation quadratic and cubic in V' for the cases considered. In the domain of ϕ for which V is real, the leading pole contribution is negative, allowing for the possibility of an instability in the normal vacuum.

I. INTRODUCTION

There is a recent revival of interest in the effective potential function in field theories.[1-3] The effective potential is a c-number function of c-number fields $V(\phi_i)$ defined to all orders in perturbation theory, the minimum of which determines the vacuum expectation value of the quantum fields. For theories with internal symmetries, spontaneous symmetry breaking may result when the minimum occurs for nonzero ϕ. Hence, $V(\phi)$ may be useful in searching for instabilities of the normal vacuum, especially when they occur beyond the tree-level approximation.

The function $V(\phi_i)$ is, in fact, the generating function for n-line one-particle-irreducible vertices (vertices for short) with all momenta zero.[1] For the case of a single scalar field σ, $V(\sigma)$ is given by the formula

$$V(\sigma) = -\sum_n \frac{\sigma^n}{n!}\Gamma^{(n)}(0), \tag{1.1}$$

where

$$\Gamma^{(n)}(0) = \Gamma^{(n)}(p_1, \ldots, p_n)|_{\text{all } p=0}. \tag{1.2}$$

An n-line Feynman graph is included in $\Gamma^{(n)}$ if it is connected and, further, does not become disconnected as a result of cutting a single line. This means, in particular, that there are no propagators on external lines.

There have been a number of papers investigating $V(\phi)$ in the loop expansion.[1-3] The importance of the loop expansion lies in the fact that it is an expansion in a parameter that multiplies the whole Lagrangian and is not dependent on a particular separation of free field part and interacting part. In other words, it does not commit one to an *a priori* choice of the vacuum. One can then survey $V(\phi)$, find the minimum, make a separation into free field and interactions, and do perturbation theory as usual.

In this paper we calculate $V(\phi)$ under the assumption that the theory in question has a dynamically generated state and that a corresponding pole dominates $\Gamma^{(n)}(0)$. The idea is to investigate the properties of $V(\phi)$ that can be traced to the existence of a deeply bound state. We give some examples for which Eq. (1.1) can be summed and find two general features for all cases considered. First is that the bound-state contribution to $V(\phi)$ is negative, making an instability in the normal vacuum possible. A reliable determination of the true minimum is hampered by limitations in our approach. Second is that $V(\phi)$ has a branch point in ϕ at a position ϕ_{br} determined by pole parameters. The behavior of $V(\phi)$ in the neighborhood of the branch point is

$$V(\phi) \propto (\phi - \phi_{\text{br}})^{3/2}. \tag{1.3}$$

The $\frac{3}{2}$ power can be understood mathematically by the following circumstance. We are able to show that $V' \equiv dV/d\phi$ satisfies an algebraic equation, quadratic and cubic in V' for the cases examined, of the form

$$F(V', c\phi) = 0, \tag{1.4}$$

where c is a quantity depending on pole parameters. V' has square-root branch points at the turning points of F. Integrating to get V gives the $\frac{3}{2}$ power.

There is an interesting similarity between these results and those in the paper by Coleman, Jackiw, and Politzer (CJP).[3] They studied spin-zero theories with O(N) invariance for large N and showed that it is possible to obtain the $1/N$ expansion as a loop expansion involving an auxiliary field. For the case of 4-dimensional space-time, they found (i) a branch point in $V(\phi)$ for real ϕ and that it was of the $(\phi - \phi_{\text{br}})^{3/2}$ type [the $\frac{3}{2}$ power is not stated but follows from their Eqs. (2.8) and (2.11)], and (ii) that the theory has a dynamically generated state. Unfortunately, the state is a tachyon which signals something is wrong with the theory or approximation. If we give our analysis the same difficulty by blindly letting the mass of the bound state be pure imaginary the branch point in V is still present. That is, a pole in the vertices can control the singularity structure of V regardless of the viability of the theory or approximation. Hence, we suggest that the CJP results (i) and (ii) are not unrelated but that the branch point in V is a manifestation of the dynamical pole. It would, of course, be desirable from our point of view to have an example for which a dynamical state is found for real positive mass.[3a]

Feynman perturbation theory cannot generate bound states when carried to a finite order. Infinite subsets of graphs can. However, it would be a formidable problem to find an approximation that would generate poles in all appropriate channels of $\Gamma^{(n)}(p_1, \ldots, p_n)$. The standard Bethe-Salpeter equation in the ladder approximation would be of no use. It generates an amplitude that violates crossing. This shortcoming can be overcome,[4] but at the cost of a formidable set of coupled equations even for the 4-line amplitude. Further, these crossing-symmetric equations are not based on a loop expansion.

Our approach here is to ignore this problem and try to learn about $V(\phi_i)$ from the general properties that follow from the existence of a bound state. In this paper we only look at spin-zero fields and a spin-zero bound state. We leave out internal symmetries but allow for a multiplicative quantum number that can be thought of as parity and is thus denoted in this paper. The simplest case is that of pseudoscalar fields and a scalar bound state which is treated in Sec. II. In Sec. III we include a scalar field to this case. In Sec. IV we treat the case of scalar and pseudoscalar fields with a pseudoscalar bound state.

The basis for our approximation is the assumption that poles dominate the $\Gamma^{(n)}(0)$ for a sufficiently deep bound state. To calculate $V(\phi_i)$ we need factorization at the pole. For one of the cases in this paper—Sec. II—the pole in question is a pole of the vertices and Green's functions and hence is a particle state. Factorization for this case is in the domain of common lore. In the remaining sections, the poles in vertices are not poles of Green's functions and hence do not correspond to particles. For these cases there is a particle corresponding to a field that couples to the vertex pole and mixes with it, giving a "dynamical particle" in Green's functions at a shifted mass. Factorization still holds for the vertex pole. We feel that this is sufficiently unfamiliar to the interested reader to warrant a pedagogical appendix (Appendix A) on how this all works.

In the vast majority of stability equations in physics, it is far easier to determine whether a system is stable or not than to determine the detailed consequences of an instability. Loop expansion studies of simple field theories are an exception; if a normal vacuum is unstable, a stable vacuum can often be found and the theory solved at the true minimum. Our approach in this paper is not an exception. We do not know how to determine the detailed consequences of an instability we may find.

This analysis can be generalized to more relevant field theories. If the criteria for stability can be firmed up this may have useful applications in the quark model where the physical particles are postulated to be bound states. Another area is the further study of spontaneous breaking of gauge theories in which an instability in the normal vacuum follows from dynamical considerations rather than being put in via a Higgs field.[5]

Questions of determining stability aside, we feel the singularities of $V(\phi)$ may have some interest. Since their positions depend on bound-state parameters we speculate that it may be possible to formulate the bound-state problem in terms of criteria based on the singularities of $V(\phi)$. It is far easier to calculate vertices at momenta zero than for finite momenta, and hence we can envisage real advantages if this approach can be developed.

II. PSEUDOSCALAR FIELD, SCALAR BOUND STATE

We first consider the case

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu_\phi^{\,2}}{2}\phi^2 - f\phi^4 + \text{counterterms}. \qquad (2.1)$$

The only properties of $\mathcal{L}$ we require for this discussion are that $\mathcal{L}$ have a mass term and an interaction even in the field (and hence we call ϕ pseudoscalar). This Lagrangian is the only such renormalizable theory involving one spin-zero field. In order to calculate $V(\phi)$ we must assume that for some value of f there is an S-wave bound state of 2ϕ's (hence, the bound state is scalar). We wish to sidestep the question of the possible existence of the state for this particular theory. This is the simplest model that illustrates the pole assumption and the combinatorics needed to calculate $V(\phi)$. The argument generalizes to the interaction $\mathcal{L}_{\text{int}} = f_{ijkl}\phi_i\phi_j\phi_k\phi_l$. In this case, the required assumption is that there is a bound state in one internal-symmetry channel of 2ϕ's.

It is well known that n-line amplitudes $A^{(n)}$ have poles at the mass of physical particles in every channel that communicates and that the residue of the pole factors into two amplitudes $A^{(n_1)}$ and $A^{(n_2)}$, where $n_1 + n_2 = n + 2$. In perturbation expansions of Lagrangian theories this is automatic if the physical state corresponds to a field. These properties also hold for bound states in the theory for which there is no corresponding field and it is this case that we are interested in. The vertices $\Gamma^{(n)}$ by definition have no poles corresponding to fields but they do have poles if there is a bound state. For the theory at hand the scalar pole position in $\Gamma^{(n)}$ is at the physical mass of the bound state, denoted m_B. (This is not true for the cases that follow in the next sections.) We will show in this section that there is a well-defined contribution to $\Gamma^{(n)}(0)$ with m_B dependence, $\Gamma^{(n)}(0)\propto(1/m_B^2)^{n-3}$ $(n\geq 4)$, with corrections down by a factor of m_B^2, and hence, this is the leading term in the expansion in m_B^2 (a dimensionless variable will be defined below). These leading contributions to the $\Gamma^{(n)}$'s are easily summable to give the effective potential in this approximation.

We start by reviewing the various n-line functions occurring in field theory. Let us denote the connected momentum-space Green's function by G:

$$\delta^4(\textstyle\sum p)G^{(n)}(p_1,\ldots,p_n) = \mathfrak{F}\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle|_{\text{connected}},$$

where $\mathfrak{F}$ denotes Fourier transform. It is possible to express $G^{(n)}$ in terms of $\Gamma^{(n')}$ and unrenormalized full propagators Δ as shown, for example, in Fig. 1. The general result is that $G^{(n)}$ is expressible as the sum of all possible tree structures in which the n'-line couplings are $\Gamma^{(n')}$ $(n'\leq n)$ and the internal and external lines are Δ.[6] The poles in the propagators have residue Z which is divergent in perturbation theory. Define a renormalized propagator Δ_R:

$$\Delta = Z^{1/2}\Delta_R Z^{1/2}, \tag{2.2}$$

and renormalized vertices

$$\Gamma_R^{(n')} = Z^{n'/2}\Gamma^{(n')}, \quad n'\neq 2. \tag{2.3}$$

If we further define

$$G^{(n)} = Z^{n/2}G_R^{(n)}, \tag{2.4}$$

then the renormalized quantities are finite. (It is understood that the $\delta\mu^2$ and δf counterterms cancel the appropriate divergences in $\Gamma_R^{(n')}$ and Δ_R.) The above statement expressing G in terms of Γ and Δ can be equally well made for the renormalized quantities.

Going one step further, we can divide out the propagators Δ_R on the external lines and thereby

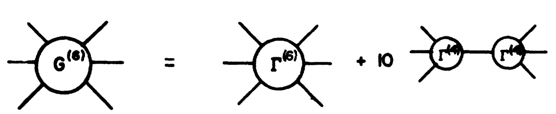

FIG. 1. Example of a one-particle analysis of a Green's function G. The lines on the right denote full propagators, Γ's are one-particle-irreducible vertices (vertices for short). The factor 10 indicates there are 10 different structures corresponding to different external labels.

define an off-mass-shell amplitude $A^{(n)}$ (see Ref. 7):

$$A^{(n)} = G_R^{(n)}(\Delta_R(p_1)\cdots\Delta_R(p_n))^{-1}. \tag{2.5}$$

Taking $A^{(n)}$ on-shell gives the S-matrix element. The one-particle analysis of $A^{(n)}$ now has propagators only on internal lines, and is of the form, all variables suppressed,

$$A^{(n)} = \sum \Gamma_R^{(n_1)}\cdots\Gamma_R^{(n_P)}(\Delta_R)^m, \tag{2.6}$$

where the sum goes over all independent tree structures. The indices satisfy the constraints

$$\sum_{i=1}^{P} n_i = n+2m, \quad P=m+1. \tag{2.7}$$

What we do now is apply the factorization property of physical states of amplitudes at the bound-state mass m_B to extract the leading term of $A^{(n)}(0)$ in our expansion in m_B^2. We will then show that this is, in fact, the leading term for $\Gamma_R^n(0)$.

Consider the amplitude $A^{(n)}(p_1,\ldots,p_n)$ as shown in Fig. 2(a). Choose momenta in pairs such that $(p_i+p_j)^2\to m_B^2$. Extract the residue of these poles as implied by Fig. 2(a). Then take pairs of pairs to their corresponding poles, $(p_i+p_j+p_k+p_l)^2\to m_B^2$. Continue until the freedom in choosing external momenta is exhausted. This procedure generates structures that look like tree graphs (not to be confused with tree graphs), with the rules shown in the first column of Fig. 3. These rules have the proviso that ϕ lines are only external and bound-state lines are only internal. All the momentum dependence is in the pole denominators and the external line couplings. $A^{(n)}$ will then have the factors

$n/2$ (ϕ, ϕ, bound state) vertices: $\beta(p_i^2, p_j^2, m_B^2)$,

$n-3$ pole factors: $-(p^2-m_B^2)^{-1}$,

$\frac{1}{2}n-2$ (3 bound state) vertices: $\gamma(m_B^2, m_B^2, m_B^2)$.

If we now take the external momenta to zero, this contribution gives $(n\geq 4)$

$$A^{(n)}(0) = N_n\beta^{n/2}\gamma^{(n/2-2)}(1/m_B^2)^{n-3}, \tag{2.8}$$

where $\beta=\beta(0,0,m_B^2)$, $\gamma=\gamma(m_B^2,m_B^2,m_B^2)$, and N_n is a combinatoric factor that counts the number of such graphs. N_n will be given later. Nonleading contributions to $A^{(n)}$ come from the higher terms in the Laurent expansion of the amplitude at each pole. These contributions will have at least one less power of $1/m_B^2$. To see this in more detail, suppose we picked up the finite part of the amplitude instead of the pole piece at one of the poles; then it is easy to see that the ratio of this to Eq. (2.8) would give either $m_B^2A_{4\phi}/\beta^2$, or $m_B^2A_{2\phi,2\text{bs}}/\beta\gamma$, or $m_B^2A_{4\text{bs}}/\gamma^2$, where the A's are 4-line nonpole amplitudes composed of ϕ and

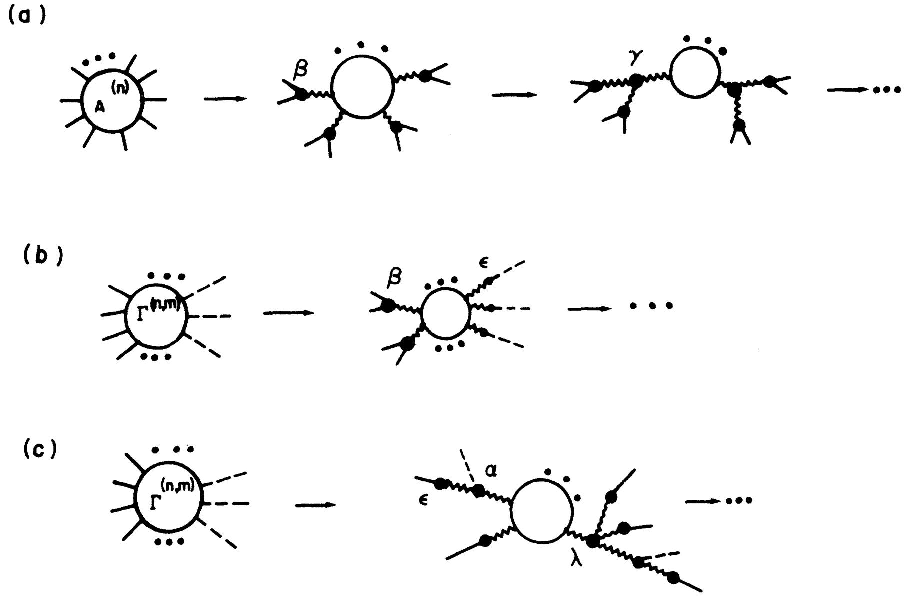

FIG. 2. The factorized residue structure at dynamical poles in vertices for the three cases (a), (b), and (c) corresponding to Secs. II, III, and IV in this paper. Wiggly lines represent dynamical poles; dots are couplings involving poles and external lines representing fields. Dashed lines are scalar fields; solid lines are pseudoscalar fields.

bound-state lines as indicated. These A's are dimensionless and we assume they are of order 1. Hence, for the first term in our expansion in m_B^2 to be good we must have

$$m_B^2/\beta^2 \ll 1, \quad m_B^2/\gamma^2 \ll 1. \tag{2.9}$$

Next we show that our $A^{(n)}(0)$, Eq. (2.8), is, in fact, the desired expression for $\Gamma_R^{(n)}(0)$. The point is this: The completely factored structure implied by Fig. 2 cannot be cut in two by cutting an internal ϕ line. If, for example, $\gamma(m_B^2, m_B^2, m_B^2)$ contained a one-particle-reducible graph, then there would be a direct coupling between a ϕ line (pseudoscalar) and bound-state line (scalar). Hence, Eq. (2.8) is obtained from the subset of graphs of $A^{(n)}$ that are one-particle-irreducible. A further check on this can be made by assuming the leading

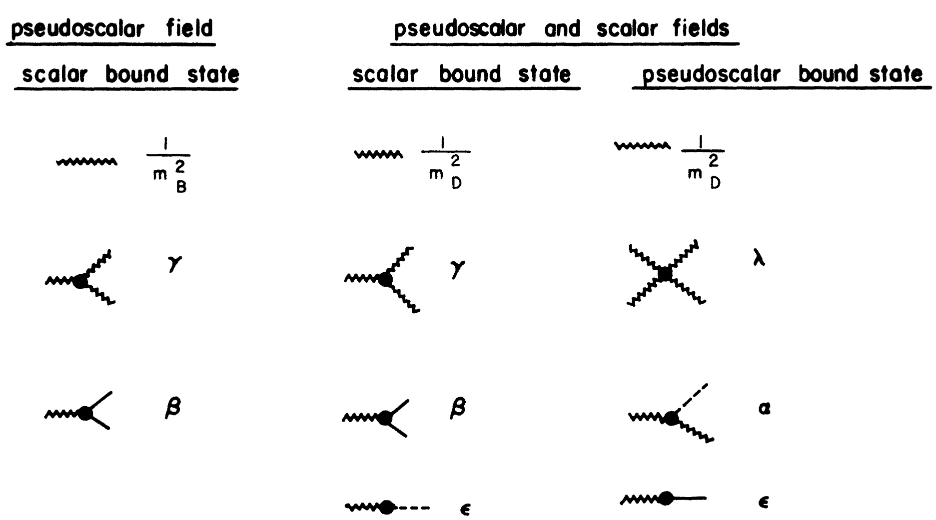

FIG. 3. Rules to find the residue of a dynamical pole. Factorized residues are tree structures composed of these elements. The wiggly lines—dynamical states—are internal only, the solid and dashed lines—pseudoscalar and scalar fields—are external only. The three columns correspond to the three sections of this paper, Secs. II, III, and IV.

term of $\Gamma^{(n)} \propto (1/m_B{}^2)^{n-3}$ and noting that the individual terms in the sum Eq. (2.6) have the behavior $(1/m_B{}^2)^{n-3-m}$. Hence, the terms with the number of propagators, m, greater than zero are nonleading.

We need to know N_n in Eq. (2.8). It is the number of independent tree structures in the sense of Fig. 2(a), e.g., $N_4=3$, $N_6=15$. It is convenient to write $N_n = N_n^{(\text{pair})} \times N_{n/2}^{(3)}$ where $N_{n/2}^{(3)}$ is the number of labeled bound-state tree structures that are stripped of the external pair of ϕ lines (see Fig. 2). The subscript $n/2$ denotes the number of "external" bound-state lines. $N_{n/2}^{(3)}$ is derived in Appendix B, Eq. (B2),

$$N_{n/2}^{(3)} = \frac{(n-4)!}{2^{(n-4)/2}[(n-4)/2]!}. \tag{2.10}$$

$N_n^{(\text{pair})}$ is just the number of ways of attaching n ϕ lines to $n/2$ identical bound-state lines:

$$N_n^{(\text{pair})} = \frac{n!}{2^{n/2}(n/2)!}. \tag{2.11}$$

The formula for $V(\phi)$ can be written

$$V(\phi) = -\sum_n \frac{(\phi Z^{-1/2})^n}{n!}[Z^{n/2}\Gamma^{(n)}(0)]. \tag{1.1$'$}$$

The quantity in square brackets is $\Gamma_R^{(n)}(0)$ for which we take $A^{(n)}$ in Eq. (2.8). We drop the $Z^{-1/2}$ multiplying ϕ and just remember that $V(\phi)$ is the generating function for the renormalized vertices $\Gamma_R^{(n)}(0)$. Equation (1.1′) becomes

$$V(\phi) = -\tfrac{1}{2}\Gamma_R^{(2)}(0)\phi^2 - \sum_{\substack{n=4\\ \text{even}}}^{\infty} \frac{(n-4)!\,n!}{2^{n-2}[(n-4)/2]!(n/2)!}\frac{\beta^2}{m_B{}^2}\left(\frac{\beta\gamma}{m_B{}^4}\right)^{n/2-2}\frac{\phi^n}{n!}. \tag{2.12}$$

Converting to a more useful index, $i = n/2 - 2$ gives

$$V(\phi) = -\tfrac{1}{2}\Gamma_R^{(2)}(0)\phi^2 - \frac{m_B{}^6}{\gamma^2}\xi^2 \sum_{i=0}^{\infty}\frac{(2i)!}{2^{2i+2}i!(i+2)!}\xi^i, \tag{2.12$'$}$$

where

$$\xi = \frac{\beta\gamma}{m_B{}^4}\phi^2. \tag{2.13}$$

Now $\Gamma_R^{(2)}(0) = \Delta_R(0)^{-1}$. Since there is no bound state in the channel, we have nothing to say about $\Delta_R(0)^{-1}$. In the absence of a mechanism to flip the sign of $\Gamma^{(2)}$, and thereby induce an instability, we take $\Gamma_R^{(2)}(0) \approx -m_\phi{}^2 < 0$.

In Appendix B we have defined the function entering in Eq. (2.12′):

$$W(\xi/2) = \xi^2 \sum_{i=0}^{\infty} \frac{(2i)!\,\xi^i}{2^{2i+2}i!(i+2)!} \tag{B3$'$}$$

$$= \tfrac{1}{6}[3\xi - 2 + 2(1-\xi)^{3/2}]. \tag{B6$'$}$$

Hence,

$$V(\phi) = -\tfrac{1}{2}\Delta_R{}^{-1}(0)\phi^2 - \frac{m_B{}^6}{6\gamma^2}[3\xi - 2 + 2(1-\xi)^{3/2}]. \tag{2.14}$$

What does this potential look like? The sign of ξ is fixed in Eq. (2.13) by the sign of $\beta\gamma$. Let us consider the two cases separately.

(i) $\beta\gamma > 0$. Define a dimensionless field $\hat{\phi}^2 = \xi = (\beta\gamma/m_B{}^4)\phi^2$. Further, define a dimensionless potential $\hat{V} = V/m_B{}^4$

$$\hat{V} = \frac{1}{2}\left(\frac{-\Delta_R{}^{-1}(0)}{\beta\gamma}\right)\phi^2 - \frac{m_B{}^2}{\gamma^2}W(\hat{\phi}^2/2), \tag{2.15}$$

$$W(\hat{\phi}^2/2) = \tfrac{1}{6}\{3\hat{\phi}^2 + 2[(1-\hat{\phi}^2)^{3/2} - 1]\}. \tag{2.16}$$

The physical domain of $\hat{\phi}$ is bounded by $0 \leq \hat{\phi}^2 \leq 1$ since $\hat{V}(\hat{\phi})$ has a branch point at $\hat{\phi} = \pm 1$. The first term in Eq. (2.15) is positive, the second negative. Hence there is a possibility of an instability. We list some elementary properties of this function:

(a) $\hat{\phi} = 0$ is a local minimum of $\hat{V}(\hat{\phi})$.

(b) The minimum of $\hat{V}(\hat{\phi})$ in the domain $[-1, +1]$ occurs either at $\hat{\phi} = 0$ or $\hat{\phi} = \pm 1$. The minimum occurs at $\hat{\phi} = 0$ if

$$\frac{m_B{}^2}{[-\Delta_R{}^{-1}(0)]}\frac{\beta}{\gamma} < 3. \tag{2.17}$$

(c) If inequality (2.17) is not satisfied, then at the minimum $(\hat{\phi} = \pm 1)$, $d\hat{V}/d\hat{\phi} \neq 0$.

(d) For $\hat{\phi}^2 > 1$, $\hat{V}(\hat{\phi})$ is complex. For $\hat{\phi} \to \infty$,

$$\operatorname{Re}\hat{V}(\hat{\phi}) \to \frac{1}{2}\left(\frac{-\Delta_R{}^{-1}(0)}{\beta\gamma} - \frac{m_B{}^2}{\gamma^2}\right)\hat{\phi}^2. \tag{2.18}$$

If our requirement for stability is that $\hat{V}(0)$ is the minimum in the physical domain, then the condition for stability is Eq. (2.17). If we further require that the minimum of $\operatorname{Re}\hat{V}(\hat{\phi})$ for all $\hat{\phi}$ occur at $\hat{\phi} = 0$, then Eq. (2.18) gives a stronger condition:

$$\frac{m_B{}^2\beta}{[-\Delta_R{}^{-1}(0)]\gamma} < 1. \tag{2.19}$$

Our conclusion then is that for $m_B{}^2 \to 0^+$, $\beta/\gamma \not\to 0$, the theory is stable. Returning to the original variable ϕ, the branch point is at

$$\phi_{\text{br}}{}^2 = m_B{}^4/\beta\gamma.$$

(ii) $\beta\gamma < 0$. Define $\hat{\phi}^2 = -\xi = -(\beta\gamma/m_B{}^4)\phi^2$. Now

$$\hat{V} = \frac{1}{2}\left(\frac{\Delta_R{}^{-1}(0)}{\beta\gamma}\right)\hat{\phi}^2 - \frac{m_B{}^2}{\gamma^2}W(-\hat{\phi}^2/2), \tag{2.20}$$

$$W(-\hat{\phi}^2/2) = \tfrac{1}{6}\{-3\hat{\phi}^2 + 2[(1+\hat{\phi}^2)^{3/2} - 1]\}. \tag{2.21}$$

Now the domain of $\hat{\phi}$ is unbounded, and $W < 0$ in the entire domain. For $\hat{\phi} \to \infty$,

$$\hat{V} \to -\frac{m_B^2}{\gamma^2}\frac{1}{3}\hat{\phi}^3. \qquad (2.22)$$

Hence, $\hat{V}$ has no lower bound and the theory is unstable.

We remind the reader that these conclusions are based on our keeping the leading term in the expansion in m_B^2. What have we left out? In terms of $\hat{V}$, corrections will be of the form[8]

$$\frac{m_B^2}{\gamma^2}\left(\frac{m_B^2}{\beta^2}f_1(\hat{\phi}) + \frac{m_B^2}{\beta\gamma}f_2(\hat{\phi}) + \frac{m_B^2}{\gamma^2}f_3(\hat{\phi})\right).$$

III. SCALAR AND PSEUDOSCALAR FIELDS, SCALAR BOUND STATE

We now investigate what happens if we add a massive scalar field σ to the case just described. That is, we assume again that there is a scalar bound state but we now have two fields, ϕ and σ, that interact via terms:

$$\mathcal{L}_{\text{int}} = g_1\sigma^3 + g_2\phi^2\sigma + f_1\phi^2\sigma^2 + f_2\phi^4 + f_3\sigma^4. \qquad (3.1)$$

The effective potential for this case is

$$V(\phi,\sigma) = -\sum_{m,n}\Gamma_R^{(m,n)}(0)\frac{\phi^m\sigma^n}{m!\,n!}, \qquad (3.2)$$

where $\Gamma_R^{(m,n)}(p_1,\ldots,p_m|q_1,\ldots,q_n)$ is the vertex for m pseudoscalars and n scalars. Again, we find the contribution to $\Gamma_R^{(m,n)}(0)$ that dominates when there is a deep bound state.

There is an added complication that occurs when there is a field σ with the same quantum numbers as the dynamical state. The position of a dynamical pole in vertices does not coincide with the pole position of physical states in Green's functions. To get a physical picture of this, it is convenient to think of turning on the coupling to the scalar field weakly so that we may think of it as a small perturbation on the previous problem. Also choose a situation in which all particles are stable for the sake of this argument. We can then expect a shift in the position of a dynamical pole in vertices because the amount of binding will be affected by the inclusion. Now there are two points we wish to note: (i) The shift will be the same for all dynamical vertex poles in all channels to a new mass we denote m_D and the residues will still factorize as before. (ii) Green's functions will have two poles in every scalar channel; one having its genesis in the free σ propagator at mass m_σ and one having its genesis in the dynamical state at mass $m_{\sigma'}$. The mixing of the two results in a shift of $m_{\sigma'}$ away from m_D. The two poles at m_σ and $m_{\sigma'}$ must appear in the scalar propagator with positive residues. Positivity of residues requires that the propagator have a sign change between $q^2 = m_\sigma^2$ and $m_{\sigma'}^2$. In fact, it has a simple zero at $q^2 = m_D^2$.

Property (i) is needed in order to calculate the effective potential in our approach. It is not obvious since these vertex poles do not correspond to particles. It is outside the scope of this paper to give a detailed proof of property (i). We will, though, show that property (ii) is a consequence of property (i). It is instructive to see how the cancellation of vertex poles occurs in constructing Green's functions.

A. Cancellation in two-body amplitudes

We show how the vertex pole cancels out of the amplitude for the simplest case of pseudoscalar-pseudoscalar scattering.[9] The general case is discussed in Appendix A. Figure 4(a) shows the one-particle analysis of $A^{(4,0)}$, where

$$A^{(4,0)} = \Gamma_R^{(4,0)} + [\Gamma_R^{(2,1)}\Delta_{\sigma R}(s)\Gamma_R^{(2,1)} + (t+u \text{ parts})]. \qquad (3.3)$$

Also shown are relations between the vertices and the self-energy in Figs. 4(b) and 4(c). The σ propagator is given by

$$\Delta_{\sigma R}^{-1}(s) = -\Gamma_R^{(0,2)}(s)$$
$$= s - m_\sigma^2 - [\Sigma(s) - \Sigma(m_\sigma^2) - (s-m_\sigma^2)\Sigma'(m_\sigma^2)]. \qquad (3.4)$$

Now assume the vertices have a pole at $s = m_D^2$. Then factorization gives

$$\Gamma_R^{(4,0)}(p_1,p_2,p_3,p_4) = \frac{\beta(p_1^2,p_2^2,m_D^2)\beta(p_3^2,p_4^2,m_D^2)}{s-m_D^2}, \qquad (3.5)$$

$$\Gamma^{(2,1)}(p_1,p_2|p) = \frac{\beta(p_1^2,p_2^2,m_D^2)\epsilon(m_D^2)}{s-m_D^2}, \qquad (3.6)$$

$$\Sigma(s) = \frac{\epsilon(m_D^2)\epsilon(m_D^2)}{s-m_D^2}, \qquad (3.7)$$

where $s = (p_1+p_2)^2$. Inserting Eqs. (3.4), (3.5), (3.6), and (3.7) in Eq. (3.3) gives the cancellation

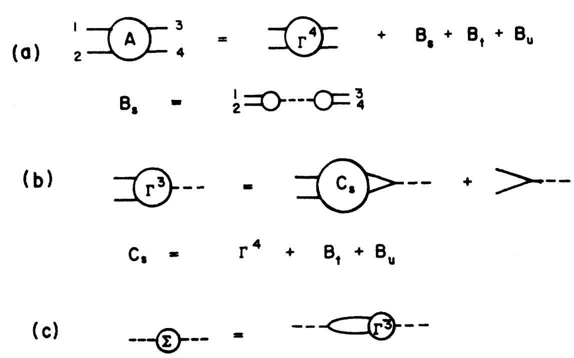

FIG. 4. (a) One-particle analysis of A^4; (b), (c) relations between vertices and the self-energy. Dashed lines are scalars; solid lines are pseudoscalars.

of the vertex poles, making the amplitude $A^{(4,0)}$ finite at $s=m_D^2$.

The inverse σ propagator is shown schematically in Fig. 5(a). We have identified the higher-mass state as the σ, and the lower one as the σ'. These two states are on equal footing. One cannot say that one is the bound state and one is the state corresponding to the field. If we were to turn off the coupling between σ's and ϕ's, holding m_σ constant, then m_σ would be the mass of the decoupled σ field, $m_{\sigma'}$ would approach m_D, and we would recover the case of the previous section.

Suppose the lower zero of $\Delta_{\sigma R}{}^{-1}$ occurred for $s<0$ as shown in Fig. 5(b). At first sight it appears that there is a tachyon. However, this is not the case. There is an instability and we need to look no further than the σ^2 term in $V(\phi,\sigma)$ to see it:

$$V=-\tfrac{1}{2}\Gamma_R^{(0,2)}(0)\sigma^2+\cdots, \tag{3.8}$$

where $\Gamma_R^{(0,2)}(0)=\Delta_R{}^{-1}(0)>0$. The coefficient of σ^2 is negative and therefore the vacuum is unstable. This is not a new result since the instability is a consequence of a physical particle mass going imaginary. However, it leads to the observation that as the position of the dynamical vertex pole approaches zero energy, if it couples to a field, then the theory becomes unstable.

B. Pole approximation to $\Gamma_R^{(m,n)}(0)$

We now identify for each m and n the factorized structure that gives the maximum power of $1/m_D^2$. For $n=0$, the problem was, of course, solved in the previous section, Eq. (2.8). Locating the dominant term for general n is straightforward and is shown in Fig. 2(b). As in the last case, take momenta of pairs of ϕ lines to the vertex poles, and for the σ lines couple them each directly to a pole term. The further residue analysis of this vertex follows the same procedure as the last section. We can write this answer by inspection, the factors being

$\frac{1}{2}m$ (ϕ,ϕ, bound state) vertices: $\beta(p_i^2,p_i^2,m_D^2)$,
n (σ, bound state) coupling: $\epsilon(m_D^2)$,
$m+2n-3$ pole factors: $-(p^2-m_D^2)^{-1}$,
$\frac{1}{2}m+n-2$ (3 bound state) vertices: $\gamma(m_D^2,m_D^2,m_D^2)$.

This gives the expression of $\Gamma_R^{(m,n)}(0)$:

$$\Gamma_R^{(m,n)}(0)=N_m^{(\text{pair})}N_{m/2+n}^{(3)}\beta^{m/2}\epsilon^n \times\gamma^{m/2+n-2}(1/m_D^2)^{m+2n-3}\,. \tag{3.9}$$

The N's are defined by Eq. (2.10) and (2.11), β,γ have the same meaning as before, and $\epsilon=\epsilon(m_D^2)$. Insert Eq. (3.9) in the definition of $V(\phi,\sigma)$, Eq. (3.1), and employ the summation indices n, and $i=m/2+n-2$:

$$V(\phi,\sigma)=-\tfrac{1}{2}\Gamma_R^{(2,0)}(0)\phi^2-\tfrac{1}{2}\bar{\Gamma}_R^{(0,2)}(0)\sigma^2 -\sum_{i=0}^{\infty}\frac{(2i)!\,\gamma^i}{2^{2i+2}i!\,(i+2)!\,m_D^{4i+2}} \times\sum_{n=0}^{i+2}\frac{(\phi^2\beta)^{i+2-n}(2\epsilon\sigma)^n(i+2)!}{n!\,(i+2-n)!}\,. \tag{3.10}$$

Although σ^2 appears in the sum ($i=0,n=2$), an additional term $\bar{\Gamma}$ is added and will be discussed shortly. The sum over n is just the binomial expansion of $(\phi^2\beta+2\epsilon\sigma)^{i+2}$. The sum over i is now the same as in the last case, Eq. (2.12′), if we

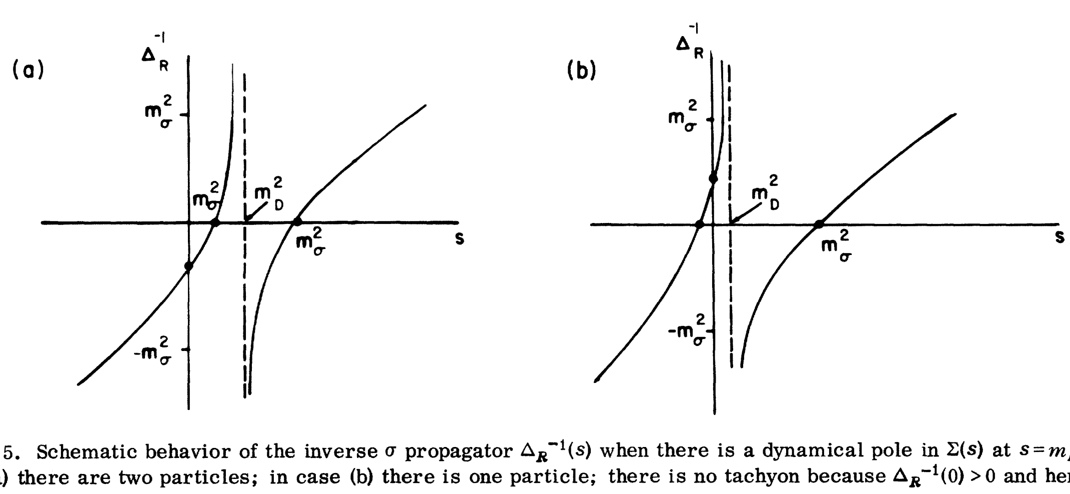

FIG. 5. Schematic behavior of the inverse σ propagator $\Delta_R{}^{-1}(s)$ when there is a dynamical pole in $\Sigma(s)$ at $s=m_D^2$. In case (a) there are two particles; in case (b) there is one particle; there is no tachyon because $\Delta_R{}^{-1}(0)>0$ and hence the normal vacuum is unstable, contrary to assumption.

make the replacements $\phi^2 \to \phi^2 + 2\epsilon\sigma/\beta$, $m_B{}^2 \to m_D{}^2$:

$$V(\phi,\sigma) = -\tfrac{1}{2}\Gamma_R^{(2,0)}(0)\,\phi^2 - \tfrac{1}{2}\tilde{\Gamma}_R^{(0,2)}(0)\,\sigma^2 - \frac{1}{6}\frac{m_D{}^6}{\gamma^2}\left[3\xi - 2 + 2(1-\xi)^{3/2}\right]\,, \qquad (3.11)$$

where

$$\xi = \gamma(\beta\phi^2 + 2\epsilon\sigma)/m_D{}^4\,. \qquad (3.12)$$

The coefficients of the quadratic terms are

$$-\tfrac{1}{2}\Gamma_R^{(2,0)}(0) = -\tfrac{1}{2}\Delta_{\phi R}{}^{-1}(0) \approx \tfrac{1}{2}m_\phi{}^2\,, \qquad (3.13)$$

$$-\frac{1}{2}\left(\tilde{\Gamma}_R^{(0,2)}(0) + \frac{\epsilon^2}{m_D{}^2}\right) = -\tfrac{1}{2}\Delta_{\sigma R}{}^{-1}(0)\,. \qquad (3.14)$$

We note that if $\tilde{\Gamma}_R$ were zero, the coefficient of σ^2 would be negative and the theory would be unstable. We have already discussed this and we want the theory to be stable under that criteria so we may proceed. Note that $\tilde{\Gamma}_R$ is down by a factor $m_D{}^2/\epsilon^2$ compared to the leading term. Therefore, we are really including a term from the next order in the expansion in $m_D{}^2$ in order to make the vacuum stable in the σ^2 term.

As in the previous section, there are two cases to examine, depending on the sign of $\beta\gamma$.

(i) $\beta\gamma > 0$. Define $\hat{\phi}^2 = (\beta\gamma/m_D{}^4)\phi^2$, $\hat{\sigma} = (2\epsilon\gamma/m_D{}^4)\sigma$ giving

$$\hat{V} = \frac{1}{2}\left(\frac{-\Delta_{\phi R}{}^{-1}(0)}{\beta\gamma}\right)\hat{\phi}^2 + \frac{1}{2}\left(\frac{-\tilde{\Gamma}_R^{(0,2)}(0)\,m_D{}^4}{4\epsilon^2\gamma^2}\right)\hat{\sigma}^2 - \frac{1}{6}\frac{m_D{}^2}{\gamma^2}\,W\big((\hat{\phi}^2 + \hat{\sigma})/2\big)\,, \qquad (3.15)$$

where W is defined by Eq. (B6′), Sec. II and $\hat{V} = V/m_D{}^4$ as before.

(ii) $\beta\gamma < 0$. Define $\hat{\phi}^2 = -(\beta\gamma/m_D{}^4)\phi^2$ and $\hat{\sigma} = -(2\epsilon\gamma/m_D{}^4)\sigma$, giving

$$\hat{V} = \frac{1}{2}\left(\frac{\Delta_{\phi R}{}^{-1}(0)}{\beta\gamma}\right)\hat{\phi}^2 + \frac{1}{2}\left(\frac{-\tilde{\Gamma}_R^{(0,2)}(0)\,m_D{}^4}{4\epsilon^2\gamma^2}\right)\hat{\sigma}^2 - \frac{1}{6}\frac{m_D{}^2}{\gamma^2}\,W\big(-(\hat{\phi}^2 + \hat{\sigma})/2\big)\,. \qquad (3.16)$$

Figure 6 shows the branch point curve in the $(\hat{\phi},\hat{\sigma})$ plane for these two cases. Case (ii) is unstable for the same reason as before. For case (i) let us suppose that the previous criteria for stability of the vacuum are satisfied. There is still the possibility of instabilities due to excursions in the $\hat{\sigma}$ direction. For large negative σ, $W \propto -(-\hat{\sigma})^{3/2}$. The explicit σ^2 term in Eq. (3.15) is positive and will overpower $\boldsymbol{W}$, and so it appears to be stable. However, this is a very risky argument, since the term $\tilde{\Gamma}_R\hat{\sigma}^2 m_D{}^4/4\epsilon^2\gamma^2$ is higher order in $m_D{}^2/\epsilon^2$ and is only the $\hat{\sigma}^2$ term of a function of $\hat{\phi}$ and $\hat{\sigma}$ as yet uncalculated. Although this term is needed to give stability for small $\hat{\sigma}$, it cannot be trusted in a condition for stability for large $\hat{\sigma}$.

IV. PSEUDOSCALAR BOUND STATE

We now look at the field theories of the previous sections again but this time assume the dynamical state is pseudoscalar instead of scalar. The same type of arguments go through but V for this case looks quite different.

A. Pseudoscalar field

First we treat the case of one pseudoscalar field, no scalar field, Eq. (2.1). Assume the vertices $\Gamma_R^{(m)}$ have poles at $p^2 = m_D{}^2$ in the pseudoscalar channels. The Green's functions will have poles at $m_\phi{}^2$ and $m_{\phi'}{}^2$ which straddle $m_D{}^2$ as in Fig. 5(a).

Figure 2(c) shows how to locate the contribution

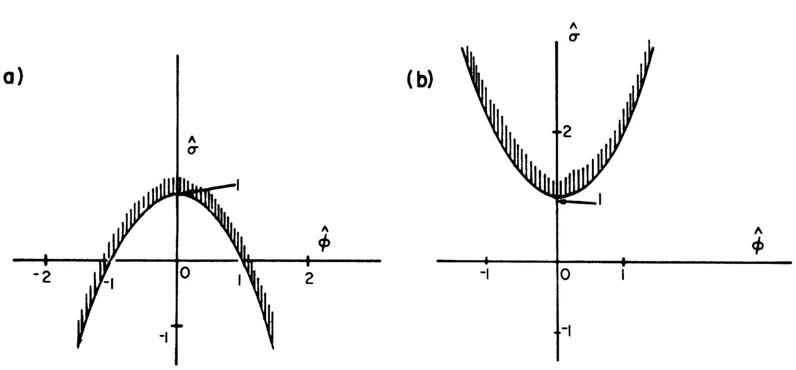

FIG. 6. Branch point in the $(\hat{\phi},\hat{\sigma})$ plane for a scalar bound state. The shaded area indicates V is complex. For (a) $\beta\gamma > 0$, (b) $\beta\gamma < 0$.

to $\Gamma_R^{(m)}$ that has the maximum power of $1/m_D^2$. (Delete the dashed-line scalar couplings until later.) The only difference between this case and the scalar-field–scalar-bound-state case is that the 3-bound-state couplings are replaced by 4-bound-state couplings. The completely factorized vertex is a tree graph structure constructed from the rules in the third column of Fig. 3 (delete α coupling), with the proviso that ϕ lines (solid) are only external, bound-state lines (wiggly) are only internal. This structure has the following factors:

m (ϕ–bound-state) couplings: $\epsilon(m_D^2)$,
$\frac{3}{2}m-2$ pole factors: $-(p_i^2-m_D^2)^{-1}$,
$\frac{1}{2}m-1$ (4-bound-state) couplings:
$$\lambda(m_D^2, m_D^2, m_D^2, m_D^2, s_j, t_j, u_j),$$
where
$$s_j+t_j+u_j=4m_D^2.$$

The completely factorized residue still has momentum dependence as indicated in the arguments of λ. Since the bound state is pseudoscalar there is a requirement on the momenta in pseudoscalar channels. But there is no requirement on the momenta of an even number of ϕ's. We are going to take all momenta to zero and use this as an approximation to $\Gamma_R^{(n)}(0)$. However, we cannot take the momenta to zero in λ as it stands since it is only defined for $s_j+t_j+u_j=4m_D^2$. What we must do is this: Choose each momenta $p_{\text{even}\,\phi}^2=\frac{4}{3}m_D^2$; then each λ will be evaluated at the symmetry point; then, treating λ as a constant
$$\lambda=\lambda(m_D^2, m_D^2, m_D^2, m_D^2;\ \tfrac{4}{3}m_D^2, \tfrac{4}{3}m_D^2, \tfrac{4}{3}m_D^2),$$
take all momenta to zero. This choice may seem arbitrary but it is the only one in which $\Gamma(0)$ is extrapolated from a crossing-symmetric form. The vertex is then
$$\Gamma_R^{(m)}(0)=N_m^{(4)}\epsilon^m\lambda^{m/2-1}\left(\frac{1}{m_D^2}\right)^{3m/2-2}, \tag{4.1}$$
where $\epsilon=\epsilon(m_D^2)$. $N_m^{(4)}$ is a combinatoric factor derived in Appendix B that counts the number of such structures:
$$N_m^{(4)}=\frac{(3m/2-3)!}{(m/2-1)!\,6^{m/2-1}}. \tag{4.2}$$
Changing the index to $i=m/2-1$, $V(\phi)$ becomes
$$V(\phi)=-\tfrac{1}{2}\tilde{\Gamma}_R^{(2)}(0)\phi^2-\frac{m_D^2}{\lambda}\xi\sum_{i=0}^{\infty}\frac{(3i)!\,\xi^i}{i!\,6^i(2i+2)!}, \tag{4.3}$$
where
$$\xi=\frac{\epsilon^2\phi^2\lambda}{m_D^6}. \tag{4.4}$$

We have added a term $\tilde{\Gamma}$ in order to make the ϕ^2 term positive for the same reasons discussed in Sec. III. Without it, the ϕ propagator would be like that shown in Fig. 5(b). From Appendix B we have
$$U(\xi)\equiv\sum_{j=1}^{\infty}\frac{(3j-3)!\,\xi^j}{6^{j-1}(j-1)!\,(2j)!} \tag{B17$'$}$$
$$=\tfrac{1}{2}-\tfrac{1}{2}[(X^{1/3}-X^{-1/3})^2+1]^2, \tag{B19$'$}$$
where
$$X=\left(1-\frac{9\xi}{8}\right)^{1/2}+\left(\frac{-9\xi}{8}\right)^{1/2}.$$
Hence
$$V(\phi)=-\tfrac{1}{2}\tilde{\Gamma}_R^{(2)}(0)\phi^2-\frac{m_D^4}{\lambda}U(\xi). \tag{4.5}$$

The inclusion of a scalar field results in a minor modification of this formula so we give that result before discussing the behavior of $V(\phi)$.

B. Inclusion of a scalar field

Figure 2(c) shows how to locate the maximum power of $1/m_D^2$ when a scalar field is included. The complete rules are given in column 3 of Fig. 3. The coupling to the scalar field is $\alpha(p_i^2, m_D^2, m_D^2)$. We denote the vertices for m pseudoscalar and n scalars by $\Gamma_R^{(m,n)}$. For each structure $\Gamma_R^{(m,0)}$ we need to find the number of ways of attaching n σ lines to the $3m/2-2\equiv m_L$ bound-state lines. We denote this number by $N_{m_L,n}^{\sigma}$:
$$N_{m_L,n}^{\sigma}=\frac{(m_L+n-1)!}{(m_L-1)!}. \tag{4.6}$$
To derive it draw n distinguishable dots in a row and partition with m_L-1 lines making m_L bins. Permute all lines and dots giving the numerator. Divide by the number of permutations of the lines.

For each σ line there will be the factor α/m_D^2. Therefore, the potential will get a contribution
$$\frac{\Gamma_R^{(m,0)}\phi^m N_{m_L,n}^{\sigma}}{m!\,n!}\left(\frac{\alpha\sigma}{m_D^2}\right)^n. \tag{4.7}$$
Summing over n gives
$$-\Gamma_R^{(m,0)}(0)\frac{\phi^m}{m!}\sum_{n=0}^{\infty}\frac{(3m/2-3+n)!}{(3m/2-3)!\,n!}\left(\frac{\alpha\sigma}{m_D^2}\right)^n$$
$$=-\Gamma_R^{(m,0)}(0)\frac{\phi^m}{m!}\left(1-\frac{\alpha\sigma}{m_D^2}\right)^{-(3m/2-2)} \tag{4.8}$$
$$=-N_m^{(4)}\epsilon^m\lambda^{m/2-1}\left(\frac{1}{m_D^2-\alpha\sigma}\right)^{3m/2-2}\frac{\phi^m}{m!}. \tag{4.9}$$

Comparison with Eq. (4.1) shows that the only modification of V is the replacement $m_D^2\to m_D^2-\alpha\sigma$.

Therefore,

$$V(\phi,\sigma) = -\tfrac{1}{2}\bar{\Gamma}_R^{(2,0)}(0)\phi^2 - \tfrac{1}{2}\Gamma_R^{(0,2)}(0)\sigma^2 - \frac{(m_D{}^2-\alpha\sigma)^2}{\lambda}U(\xi), \tag{4.10}$$

where

$$\xi = \frac{\epsilon^2\phi^2\lambda}{(m_D{}^2-\alpha\sigma)^3}. \tag{4.11}$$

The only σ^2 term is shown explicitly, $\Gamma_R^{(0,2)}(0) \approx -m_\sigma{}^2$.

Let us look at $V(\phi,\sigma)$ for the two cases $\lambda>0$, $\lambda<0$.

(i) $\lambda > 0$. Define $\hat{\phi}^2 = (9\lambda\epsilon^2/8m_D{}^6)\phi^2$, $\hat{\sigma} = (\alpha/m_D{}^2)\sigma$, $\hat{V} = V/m_D{}^4$; then

$$\hat{V} = -\frac{4\bar{\Gamma}_R^{(2,0)}(0)m_D{}^2}{9\lambda\epsilon^2}\hat{\phi}^2 - \frac{\Gamma_R^{(0,2)}(0)}{2\alpha^2}\hat{\sigma}^2 - \frac{(1-\hat{\sigma})^2}{\lambda}U\left(\frac{8\hat{\phi}^2}{9(1-\hat{\sigma})^3}\right). \tag{4.12}$$

The singularity of U at $\hat{\phi}^2 = (1-\hat{\sigma})^3$ is shown in Fig. 7(a). The singularity is a $\frac{3}{2}$-power branch point. U is positive in the region below the singular curve, and hence this term gives a negative contribution to $\hat{V}$. This term is zero for $\hat{\phi}=0$ for all values of $\hat{\sigma}$. For large negative $\hat{\sigma}$, $\hat{\phi}\neq 0$, $-(1-\hat{\sigma})^2U \to$ negative constant. A detailed search for an instability depends on a balance between this term and the first two terms which cannot be trusted for large $\hat{\phi}$ and $\hat{\sigma}$.

(ii) $\lambda < 0$. Define $\hat{\phi}^2 = -(9\lambda\epsilon^2/8m_D{}^6)\phi^2$, $\hat{\sigma}$ and $\hat{V}$ same as above; then

$$V = \frac{4\bar{\Gamma}_R^{(2,0)}(0)m_D{}^2}{9\lambda\epsilon^2}\hat{\phi}^2 - \frac{\Gamma_R^{(0,2)}(0)}{2\alpha^2}\hat{\sigma}^2 - \frac{(1-\hat{\sigma})^2}{\lambda}U\left(\frac{-8\hat{\phi}^2}{9(1-\hat{\sigma})^3}\right). \tag{4.13}$$

The singularity is now at $\hat{\phi}^2 = -(1-\hat{\sigma})^3$ as shown in Fig. 7(b). Now U is negative in the region below the singular curve, and the third term again gives a negative contribution to $\hat{V}$.

ACKNOWLEDGMENTS

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APPENDIX A: DYNAMICAL POLE ASSUMPTION

In this appendix we show that if vertices have a dynamical pole at energy m_D with factorizing residues, and if there is a field with the same quantum numbers as the pole, the pole is not present in Green's functions and hence does not correspond to a particle state. Our demonstration here is for the case of two spin-zero fields ϕ_1 and ϕ_2 that carry different values of a conserved multiplicative quantum number, e.g., parity. Choose ϕ_1 to be the field coupling to the dynamical pole. Consider a momentum-space connected Green's function G with n_1 external ϕ_1 lines and n_2 external ϕ_2 lines. The single-particle analysis of G consists in expressing G_R (R for renormalized) in terms of structures topologically equivalent to tree graphs with full renormalized vertices in place of the bare couplings and full renormalized propagators in place of the free propagators. This takes the form (all variables suppressed)[6]

$$G_R^{(n_1,n_2)} = (\Delta_R^{(1)})^{n_1}(\Delta_R^{(2)})^{n_2} \times \sum \Gamma_R^{(\mu_1,\nu_1)}\cdots\Gamma_R^{(\mu_p,\nu_p)}(\Delta_R^{(1)})^{m_1}(\Delta_R^{(2)})^{m_2}. \tag{A1}$$

There is a propagator for each external line, and

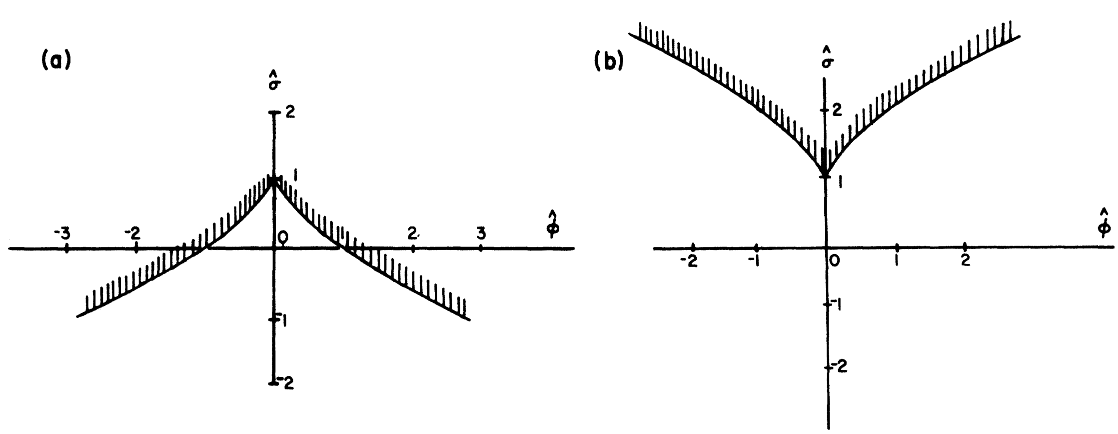

FIG. 7. Branch point in the $(\hat{\phi},\hat{\sigma})$ plane for a pseudoscalar bound state. The shaded area indicates V is complex. For (a) $\lambda>0$, (b) $\lambda>0$.

a structure of P vertices connected by m_1 and m_2 internal lines. The sum goes over all such structures. For each term in the sum, the indices are related as follows:

$$\sum_{i=1}^{P} \mu_i = n_1 + 2m_1,$$

$$\sum_{j=1}^{P} \nu_j = n_2 + 2m_2,$$

$$P = m_1 + m_2 + 1.$$

The dynamical pole assumption means the following: Select any $\Gamma_R^{(\mu_i,\nu_j)}$ and partition the $\mu_i + \nu_j$ lines into two sets of size a and a', $a + a' = \mu_i + \nu_j$. This selects a particular channel $a \leftrightarrow a'$, call it the s_a channel. If this channel has the quantum numbers of the field ϕ_1 then there is a pole with residue that factorizes:

$$\Gamma_R^{(\mu_i,\nu_j)} = \frac{\gamma^{(a)}\gamma^{(a')}}{s_a - m_D^2} + \text{finite}, \tag{A2}$$

where $\gamma^{(a)}$ and $\gamma^{(a')}$ are functions of momenta in the a and a' partitions. For the special case in which a (or a') is one ϕ_1 particle, the $\gamma^{(a)}$ is just a number, $\gamma^{(1)}$. In particular,

$$\Gamma_R^{(2,0)} = \frac{\gamma^{(1)}\gamma^{(1)}}{s_1 - m_D^2} + \text{finite}. \tag{A3}$$

The renormalized propagator $\Delta_R^{(1)}$ is

$$\Delta_R^{(1)-1}(s) = s - m_1^2 - [\Sigma(s) - \Sigma(m_1^2) - (s - m_1^2)\Sigma'(m_1^2)]$$
$$= -\Gamma_R^{(2,0)}(s), \tag{A4}$$

where $\Sigma(s)$ is the self-energy of particle 1 and $\Sigma' = (d/ds)\Sigma$.

Now to demonstrate the cancellation of vertex poles we select any one term out of the sum Eq. (A1) and consider any particular vertex $\Gamma_R^{(\mu_i,\nu_j)}$. Choose any partition of $\mu_i + \nu_j$, (a, a') that has the quantum numbers of the ϕ_1 field. Then $\Gamma_R^{(\mu_i,\nu_j)}$ will have a pole in that channel with factorizing residues, as in Eq. (A2). There are three cases to consider: (1) If one of the sets—a or a'—is one ϕ_1 line and this line is an external line of G, then the propagator on this external line has a simple zero which eliminates the pole. (ii) If one of the sets—a or a'—is one internal ϕ_1 line (call it the partition a'), then the vertex is connected to another vertex with partition (b', b), b' one ϕ_1 line, as shown in Fig. 8(a). There is another term in the sum Eq. (A1) shown in Fig. 8(b) that has a pole at m_D^2 with a residue of the opposite sign which we will show below. (iii) If the partition of the $\mu_i + \nu_j$ lines into two sets—which we now call a, b—does not involve a ϕ_1 line, then the situation is shown in Fig. 8(b). There is the contribution Fig. 8(a) that cancels it.

The cancellation of vertex poles between Figs. 8(a) and 8(b) follows immediately from Eqs. (A2), (A3), and (A4). In Fig. 8(a), the vertices have poles, the propagator a zero:

$$\frac{\gamma^{(a)}\gamma^{(1)}}{s - m_D^2}\ \frac{s - m_D^2}{-\gamma^{(1)}\gamma^{(1)}}\ \frac{\gamma^{(1)}\gamma^{(b)}}{s - m_D^2}.$$

The pole in Fig. 8(b) gives

$$\frac{\gamma^{(a)}\gamma^{(b)}}{s - m_D^2}.$$

APPENDIX B: TREES

The dynamical pole residue structures are trees in the mathematical sense; see for example Harary.[10] In this appendix we show how to count the number of independent trees for which the external lines are labeled. There are two separate cases we need: (A) 3-line couplings-scalars, and (B) 4-line couplings-pseudoscalars. This analysis deals with the residue structures only. The attaching of scalar and pseudoscalar lines to these structures is dealt with in the text of the paper. After counting trees we then give an alternate derivation of the functions that enter in V by using a generating function. We are able to show that $\partial V/\partial\phi$ satisfies an algebraic equation—quadratic for case A, cubic for case B—without ever determining explicitly the combinatoric factors. Hence, we can find V directly as promised in the Introduction.

A. Trees with 3-line couplings

Consider a tree with n external labeled lines denoted $T_n^{(3)}$. Each tree will have $n-3$ internal lines and $n-2$ 3-line couplings. The total number of lines is $2n-3$. The number of distinguishable trees is denoted $N_n^{(3)}$. For example, $N_2^{(3)} = 1$, $N_3^{(3)} = 1$, $N_4^{(3)} = 3$, $N_5^{(3)} = 15$. To find $N_n^{(3)}$ for general n, we proceed by induction. Consider the process of attaching a line labeled $n+1$ to every line—internal and external—of each n-line tree.

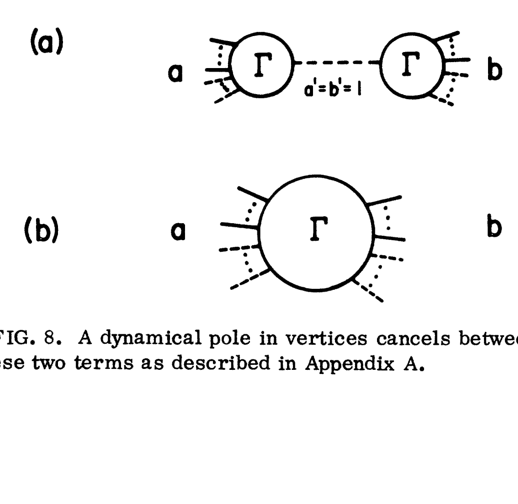

FIG. 8. A dynamical pole in vertices cancels between these two terms as described in Appendix A.

This generates each $(n+1)$-line tree once. Therefore,

$$N_{n+1}^{(3)} = (2n-3)\, N_n^{(3)} \quad . \tag{B1}$$

Since $N_2^{(3)} = 1$, we have

$$N_n^{(3)} = (2n-5)!! = \frac{(2n-4)!}{2^{n-2}(n-2)!} . \tag{B2}$$

Consider the function

$$W(x) \equiv \sum_{n=2}^{\infty} N_n^{(3)} \frac{x^n}{n!} = x^2 \sum_{i=0}^{\infty} \frac{(2i)!\, x^i}{2^i i!\,(i+2)!} . \tag{B3}$$

This is the form of the function that enters in V in Secs. II and III. Using the Gauss multiplication formula[11] for $n=2$

$$\Gamma(nz) = (2\pi)^{(1/2)(1-n)}\, n^{(nz-1/2)} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) \tag{B4}$$

we find

$$W(x) = \frac{x^2}{2} F(1, \tfrac{1}{2}; 3; 2x) , \tag{B5}$$

where F is the hypergeometric function[11]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(a+i)\Gamma(b+i)}{\Gamma(c+i)i!} z^i \quad .$$

This F is an elementary function, and we find

$$W(x) = \tfrac{1}{3}[3x - 1 + (1-2x)^{3/2}] \quad . \tag{B6}$$

We now show how to get Eq. (B6) without the step Eq. (B1). Consider an n-line tree: In Fig. 9 we have drawn the line labeled n and have noted that every tree can be classified into sets branching at line n with $n-i$ external lines on the left and $i-1$ lines on the right. The sum of these—each taking the value 1—gives the following equation for $N_n^{(3)}$:

$$N_n^{(3)} = \frac{1}{2} \sum_{i=2}^{n-1} \frac{(n-1)!}{(n-i)!\,(i-1)!} N_{n-i+1}^{(3)} N_i^{(3)} \quad . \tag{B7}$$

The combinatoric factor is just the number of ways partitioning $(n-1)$ lines into 2 sets of $(n-i)$ lines and $(i-1)$ lines. The factor of 2 comes from permuting the two sets. This equation looks far worse than the equivalent definition—Eq. (B1)—but as we will see it is not. It has important advantages: It is derived without probing the internal line structure of trees. The combinatoric factor is based on partitioning external lines. The formula Eq. (B7) is easily generalized whereas Eq. (B1) is not. Further, we can go from Eq. (B7) to Eq. (B6) without even explicitly determining $N_n^{(3)}$.

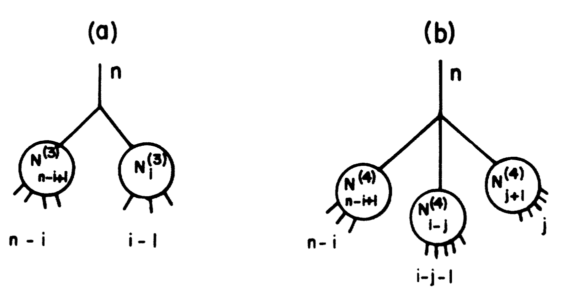

FIG. 9. Classification of tree graphs in terms of the branching at the external line labeled n for (a) 3-line couplings, (b) 4-line couplings as described in Appendix B.

Define $P_n = N_n^{(3)}/(n-1)!$, giving

$$P_n = \frac{1}{2} \sum_{i=2}^{n-1} P_{n-i+1} P_i , \quad n \geq 3 \quad . \tag{B8}$$

Multiply by x^n and sum $n=3$ to ∞. Interchanging the order of sums gives

$$\sum_{n=3}^{\infty} x^n P_n = \frac{1}{2x} \sum_{i=2}^{\infty} P_i x^i \sum_{n=i+1}^{\infty} P_{n-i+1} x^{n-i+1} \quad . \tag{B9}$$

Define

$$A = \sum_{n=2}^{\infty} P_n x^n = \sum_{n=2}^{\infty} N_n^{(3)} \frac{x^n}{(n-1)!} \quad ; \tag{B10}$$

then Eq. (B9) becomes

$$A - x^2 P_2 = \frac{1}{2x} A^2 , \tag{B11}$$

where $P_2 = 1$. We solve this equation and choose the root that gives $A \propto x^2$ for small x:

$$A = x[1 - (1-2x)^{1/2}] \quad . \tag{B12}$$

From the definition of A, Eq. (B10), it follows that

$$A(x) = x \frac{d}{dx} W(x) , \tag{B13}$$

and finally,

$$W(x) = \int_0^x \frac{A(x')\,dx'}{x'} \quad . \tag{B14}$$

The source of the $\frac{3}{2}$ power clearly comes from the integration over A.

B. Trees with 4-line coupling

For this case a tree with n external labeled lines, denoted $T_n^{(4)}$, has $n/2-2$ internal lines, $n/2-1$ couplings, and $3n/2-2$ total lines (n must be even). The number of trees $T_n^{(4)}$ we denote $N_n^{(4)}$, e.g., $N_2^{(4)} = 1$, $N_4^{(4)} = 1$, $N_6^{(4)} = 10$, $N_8^{(4)} = 280$. Finding this for general n is considerably more difficult than for the 3-line case. We claim the recurrence relation is (derived below)

$$N_{n+2}^{(4)} = \frac{(3n/2-2)(3n/2-1)}{2} N_n^{(4)} , \tag{B15}$$

giving

$$N_n^{(4)} = \frac{(3n/2-3)!}{6^{n/2-1}(n/2-1)!} . \tag{B16}$$

Define the function $U(x)$ that enters in the potential:

$$U(x) \equiv \sum_{\substack{n=2\\ \text{even}}}^{\infty} N_n^{(4)} \frac{x^{n/2}}{n!} = \sum_{j=1}^{\infty} \frac{(3j-3)!\, x^j}{6^{j-1}(j-1)!\,(2j)!} . \tag{B17}$$

Using Eq. (B4), $n=2$, and $n=3$, $U(x)$ can be expressed as a hypergeometric function:

$$U(x) = F(-\tfrac{1}{3}, -\tfrac{2}{3}; \tfrac{1}{2}; \tfrac{9}{8}x) - 1 , \tag{B18}$$

which in turn is

$$U(x) = \tfrac{1}{2} - \tfrac{1}{2}[(X^{1/3} - X^{-1/3})^2 - 1]^2 , \tag{B19}$$

where

$$X = (1 - \tfrac{9}{8}x)^{1/2} + (-\tfrac{9}{8}x)^{1/2} . \tag{B20}$$

We now give an argument to justify Eq. (B15). Note that for $T_n^{(4)}$, there are $3n/2 - 1 \equiv n_L$ total lines, and hence $N_{n+2}^{(4)}/N_n^{(4)} = n_L(n_L+1)/2$. This is the number of ways of placing two indistinguishable dots on the n_L lines of $T_n^{(4)}$. If we can make a correspondence between each $T_n^{(4)}$ with 2 dots and each $T_{n+2}^{(4)}$, then Eq. (B15) is verified. To do this, cut $T_n^{(4)}$ at each dot. Color the resulting cut trees red, white, and blue, white being the middle one. Draw a single 4-line vertex labeled $n+2$, red, white, and blue. Join the colored trees to the colored lines, assigning the label $n+1$ to the free white line. Repeat the joining but this time reverse the ends of the white tree. This procedure gives each tree twice.

This argument is a recipe, not a proof. A proof along these simplistic lines is tortuous and more powerful techniques are called for.[7] Rather than going that direction, for which we see little future, we show how to get Eq. (B19) without knowing $N_n^{(4)}$ explicitly analogously to the 3-line case.

By considering Fig. 9(b) we can write the following equation by inspection, analogously to Eq. (B7): For $n \geq 4$, we have

$$N_n^{(4)} = \frac{1}{3!} \sum_{\substack{i=3\\ \text{odd}}}^{n-1} \sum_{\substack{j=1\\ \text{odd}}}^{i-2} \frac{(n-1)!\, N_{j+1}^{(4)} N_{i-j}^{(4)} N_{n-i+1}^{(4)}}{j!\,(i-j-1)!\,(n-i)!} . \tag{B21}$$

Define new indices

$$\alpha = \frac{j+1}{2}, \quad \beta = \frac{i+1}{2}, \quad \rho = \frac{n}{2},$$

and

$$Q_\rho = \frac{N_{2\rho}}{(2\rho-1)!} , \tag{B22}$$

giving

$$Q_\rho = \frac{1}{6} \sum_{\beta=2}^{\rho} \sum_{\alpha=1}^{\beta-1} Q_\alpha Q_{\beta-\alpha} Q_{\rho-\beta+1}, \quad \rho \geq 2 . \tag{B23}$$

Multiply by x^ρ and sum $\rho = 2$ to ∞. Change the summation order

$$\sum_{\rho=2}^{\infty} \sum_{\beta=2}^{\rho} \sum_{\alpha=1}^{\beta-1} \to \sum_{\alpha=1}^{\infty} \sum_{\beta=\alpha+1}^{\infty} \sum_{\rho=\beta}^{\infty} . \tag{B24}$$

Define a generating function:

$$B(x) = \sum_{\rho=1}^{\infty} Q_\rho x^\rho = \sum_{\rho=1}^{\infty} \frac{N_{2\rho} x^\rho}{(2\rho-1)!} = \sum_{\substack{n=2\\ \text{even}}}^{\infty} \frac{N_n^{(4)} x^{n/2}}{(n-1)!} . \tag{B25}$$

Equations (B23), (B24), and (B25) give

$$B - xQ_1 = \frac{1}{6x} B^3 , \tag{B26}$$

where $Q_1 = 1$. There is one root that has $B \propto x$ for small x:

$$B = -(-2x)^{1/2}(X^{1/3} - X^{-1/3}) , \tag{B27}$$

where X is given in Eq. (B20). For x small and negative all factors are real and we mean the positive real root everywhere.

Comparing Eq. (B17) and Eq. (B25) shows that

$$2x \frac{dU}{dx} = B(x) . \tag{B28}$$

Hence,

$$U(x) = \frac{1}{2} \int_0^x \frac{B(x')}{x'} dx' . \tag{B29}$$

Since Eq. (B28) checks for the explicit functions, our recursion relation Eq. (B15) is correct.

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[3a] *Note added in proof.* The model in Ref. 3 has also been solved by H. Schnitzer, Phys. Rev. D 10, 1800 (1974). He further analyzes V for the case of a normal vacuum. A detailed comparison of the results of this paper and the O(N) model will be given in a forthcoming paper.

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[7] This particular off-shell extrapolation is sometimes called the truncated Green's function.

[8] We are implicitly assuming that the counterterms $\delta f \phi^4$ and $\delta\mu^2\phi^2$ are nonpole contributions to $V(\phi)$, and hence that they would be included in these higher orders in $m_B{}^2$. Whether this is true or not depends on the actual choice of conditions to determine δf and $\delta\mu^2$.

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