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Classical limit and generalizations of the homogeneous quasipotential equation for scalar interactions*

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The classical limit of Todorov's relativistic Schrödinger equation for scalar interactions is presented in a Hamiltonian context. The consequences of limitations on the coupling-constant size that exist in this quantum equation for bound states appear as orbital limitations in the classical case. A systematic method of generalizing the classical equations which removes these orbital restrictions is developed. The corresponding quantum equations do not display any limitations on the coupling-constant size. An exactly soluble example is given that displays very deep binding. In this example the total center-of-mass energy of two equal-mass bound particles goes to zero as the coupling constant goes to infinity.

I. INTRODUCTION

The purpose of this paper is to obtain a generalization of Todorov's homogeneous quasipotential equation for scalar interactions.[1] As shall be demonstrated, this generalization, although not unique, permits an extension of Todorov's equation to the region of large coupling strengths. In particular, in an exactly soluble model closely related to scalar quantum electrodynamics with scalar photons, we demonstrate the existence of very deep binding for which the total center-of-mass (c.m.) energy of the two equal-mass bound particles goes to zero as the coupling constant goes to infinity. The generalization (5.56) of Todorov's relativistic Schrödinger equation is not limited to scalar QED, but can be adapted to any scalar type of interaction including the nucleon-nucleon interaction.

Although this result refers to the relativistic two-body bound-state problem in quantum mechanics, the bulk of this paper is concerned with the classical Hamiltonian formalism. In constructing a generalization of the quasipotential equation, care must be taken so that not only a systematic but also a consistent approach be applied. Employing the classical limit as a vehicle to arrive at a generalization is a useful and traditional aid. It is also of interest in itself to examine the classical limit of this two-body relativistic quantum equation and its generalizations.

The remainder of this section is devoted to a review of Todorov's derivation of the quasipotential equation for scalar interactions as well as discussing some of the motivations for considering generalizations. Limitations on the size of the coupling constant found in Todorov's quasipotential equation also appear in its static limit, the Klein-Gordon equation. In Sec. II the consequences of such limitations are examined in the classical limit of the Klein-Gordon equation in the on-energy-shell, off-mass-shell approach characteristic of Todorov's quasipotential equation. Both the covariant and the three-dimensional Hamiltonian approaches are discussed. Before giving the classical Hamiltonian description of the two-body quasipotential equation, we give the coordinate and proper-time definitions that appear to be inherent in this approach. Also given in Sec. III is an alternative way of arriving at the effective quasipotential Hamiltonian by a c.m. reduction of a sum of two one-body Hamiltonians. This is an essential step in constructing a generalization of the quasipotential approach. The subsidiary condition (1.3) characteristic of this quasipotential approach is included in the two-body Hamiltonian by way of a Lagrange multiplier. The equations that arise from this Hamiltonian fit in quite naturally with the coordinate and proper-time definitions given earlier. Orbital restrictions are also discussed in the two-body case.

In Sec. IV a generalization of the Klein-Gordon equation is presented. Again, using the classical Hamiltonian procedure appears most convenient for the discussion. The orbital limitations in the classical case mentioned above as well as the coupling-constant limitations that appear in the Klein-Gordon equation do not appear in the generalization given here. In Sec. V the generalization of the two-body Hamiltonian is given. The generalization is made on the two-body Hamiltonian formalism discussed in Sec. III that provided an alternative derivation of the quasipotential Hamiltonian. The generalized version of the quasipotential equation is developed on the basis of this classical Hamiltonian formulation.

A. Todorov's quasipotential equation for scalar interactions

Todorov's quasipotential equation for the two-body elastic scattering amplitude T has the following form in the center-of-mass (c.m.) frame[1]:

$$T_w(\vec{q}, \vec{p}) + V_w(\vec{q}, \vec{p}) + \int \frac{d^3k}{(2\pi)^3} V_w(\vec{q}, \vec{k}) \frac{1}{2w(\vec{k}^2 - b^2 - i\epsilon)} T_w(\vec{k}, \vec{p}) = 0. \tag{1.1}$$

The quasipotential V_w is defined by this equation.

The variables w, $\vec{p}$, and $\vec{q}$ are respectively the center-of-mass energy and relative momenta in the initial and final states. The total c.m. momentum is $P = (w, \vec{0})$. The relative four-momenta p and q in the initial and final states are spacelike and are orthogonal to P; $p \cdot P = q \cdot P = 0$ where $p = (0, \vec{p})$ and $q = (0, \vec{q})$ in the c.m. frame. These two vectors have the same length b^2 in the c.m. frame where

$$b^2 = \frac{1}{4w^2}[w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2]. \tag{1.2}$$

Even though Eq. (1.1) is three-dimensional in appearance, it is Lorentz-invariant. We shall repeat some of Todorov's arguments to this effect here and in IB, as we shall be referring to them frequently in this paper when discussing generalizations of the quasipotential equation. The equation relating the two-body scattering amplitude T and the quasipotential V is defined as on the energy shell but off the mass shell. The additional assumption is made that the initial momenta (p_1 and p_2) and the final momenta (q_1 and q_2) of the particles satisfy

$$p_1^2 - p_2^2 = q_1^2 - q_2^2 = m_2^2 - m_1^2. \tag{1.3}$$

The total momentum vector $P = p_1 + p_2 = q_1 + q_2$ is, of course, timelike, $P^2 = w^2 < 0$. The relation (1.3) allows us to define the c.m. energies E_1 and E_2 of particles 1 and 2 in terms of this invariant:

$$\begin{aligned} E_1 &= -\frac{1}{w} P\cdot p_1 = -\frac{1}{w} P\cdot q_1 = \frac{w^2 + m_1^2 - m_2^2}{2w}, \\ E_2 &= -\frac{1}{w} P\cdot p_2 = -\frac{1}{w} P\cdot q_2 = \frac{w^2 + m_2^2 - m_1^2}{2w}. \end{aligned} \tag{1.4}$$

Another invariant is the energy variable E defined by

$$E = \frac{E_1 E_2 + b^2}{w} = \frac{w^2 - m_1^2 - m_2^2}{2w}. \tag{1.5}$$

The spacelike four-momenta p and q mentioned above are defined in a general frame as

$$p = \frac{E_2}{w} p_1 - \frac{E_1}{w} p_2, \quad q = \frac{E_2}{w} q_1 - \frac{E_1}{w} q_2, \tag{1.6}$$

and their orthogonality to P follows from (1.3). This orthogonality of the relative momenta to the total momentum P is a central feature of the quasipotential equation (1.1), as its following covariant form indicates:

$$T_P(q, p) + V_P(q, p) + \int \frac{d^4k}{(2\pi)^3} V_P(q, k) G_P(k) \delta(P\cdot k) T_P(k, p) = 0. \tag{1.7}$$

In the c.m. frame $G_P(k) = wG_w(\vec{k}) = \frac{1}{2}[1/(\vec{k}^2 - b^2 - i\epsilon)]$. The intermediate four-momentum k as well as the relative momenta p and q are orthogonal to P. The net effect of this is to yield a three-dimensional one time formulation of the two-body problem with the set of hyperplanes perpendicular to P acting as the time variable.

B. The relativistic two-body Schrödinger equation

A Schrödinger type of equation is obtained from (1.1) by defining the wave function

$$\phi_{\vec{q}}(\vec{p}) = \delta(\vec{p} - \vec{q}) + G_{w_q}(\vec{p}) T(\vec{p}, \vec{q}), \tag{1.8}$$

where

$$w_q = (m_1^2 + \vec{q}^2)^{1/2} + (m_2^2 + \vec{q}^2)^{1/2}.$$

Substituting (1.7) into (1.1) leads to

$$2w(\vec{q}^2 - b^2(w))\phi_w(\vec{q}) + \int \frac{d^3k}{(2\pi)^3} V_w(\vec{q}, \vec{k}) \phi_w(\vec{k}) = 0. \tag{1.9}$$

For potentials of the form $V_w = V_w(\vec{p} - \vec{q})$ this takes the local form

$$\left(\frac{-\nabla^2}{2m_w} + \mathcal{V}_w(\vec{r})\right) \psi_w(\vec{r}) = \frac{b^2}{2m_w} \psi_w(\vec{r}), \tag{1.10}$$

where

$$\mathcal{V}_w(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{4m_1 m_2} V_w(\vec{k}) \tag{1.11}$$

and $m_w = m_1 m_2 / w$ is the relativistic reduced mass.

Before considering generalizations of this relativistic Schrödinger equation for scalar interactions, we shall give some of the motivations. We consider the spectrum that arises from solving this equation for a potential $V(r)$ that is derived from a pure scalar analog of quantum electrodynamics.[1] The Lagrange function describing the interaction has the normal-ordered form

$$\mathcal{L}_I(x) = (g_1 : \psi_1^*(x)\psi_1(x) : + g_2 : \psi_2^*(x)\psi_2(x) :) A(x). \tag{1.12}$$

As $V_1 = -T_1$, the lowest-order approximation for the potential is

$$V = V_1 = -g_1 g_2/(\vec{q}-\vec{p})^2. \tag{1.13}$$

The coupling constants g_1 and g_2 have the dimension of mass. We introduce a dimensionless quantity α by setting

$$g_1 g_2 = 16\pi m_1 m_2 \alpha. \tag{1.14}$$

This leads to a relativistic Schrödinger equation of the form[1]

$$\left(\frac{-\nabla^2}{2m_w} - \frac{\alpha}{r}\right)\psi_w(\vec{r}) = \frac{b^2}{2m_w}\psi_w(\vec{r}). \tag{1.15}$$

This can be solved exactly giving the O(4)-symmetric result[1]

$$w^2 = m_1^2 + m_2^2 + 2m_1 m_2\left(1-\frac{\alpha^2}{n^2}\right)^{1/2}. \tag{1.16}$$

Consider the ground state $n=1$. If the coupling is strong enough so that $\alpha>1$, then the energy becomes complex. Such occurrences of complex energies are well known for point interactions in the case of one-body relativistic bound-state equations in which the Coulomb potential is a fourth component of the vector potential; the Dirac equation for hydrogenlike atoms is a prime example. For realistic atomic calculations such occurrences are academic because the finite extent of the nucleus smooths out the potential so that real energies occur for the ground state beyond $\alpha=1$. The appearance of complex energies for point interactions in the scalar case may, in fact, be an intrinsic limitation of the model. Atkinson and Crater[2] have recently examined the possibility that including radiative corrections in a nonperturbative way would eliminate or deflect this problem. A 1.2% shift upward of the value of α for which the energy becomes complex from $\alpha=1$ to $\alpha=1.012$ was found by this approach. This indicates, although it does not prove, that there are intrinsic limitations to the coupling-constant size in this model, and that rather basic changes are necessary if one is to obtain real bound-state energies for strongly coupled scalar interactions. This has significance beyond the simple model we are considering. For example, other scalar types of interactions such as the pseudoscalar nucleon-nucleon coupling differ from the interaction here mainly in their long range behavior. This long-range behavior does not affect the problem of complex energies.

The static limit of the homogeneous quasipotential equation (1.10) is the conventional Klein-Gordon equation. In the static limit $m_2 \to \infty$ and $m_w \to m_1 \equiv m$. Calling $\mathcal{E} = w - m_2$, one finds that the effective particle relativistic Schrödinger equation (1.10) becomes

$$\left(\frac{-\vec{\nabla}^2}{2m} + \mathcal{V}(\vec{r})\right)\psi(\vec{r}) = \frac{(\mathcal{E}^2 - m_2)}{2m}\psi(\vec{r}) \tag{1.17}$$

or

$$[p^2 + m^2 + 2m\mathcal{V}(\vec{r})]\psi(\vec{r}) = 0, \tag{1.18}$$

with the on-energy-shell condition $p^0 = \mathcal{E}$. For $\mathcal{V} = -\alpha/r$ the spectrum

$$\mathcal{E} = m\left(1-\frac{\alpha^2}{n^2}\right)^{1/2} \tag{1.19}$$

is easily obtained either directly from the static limit of (1.16) or by solving the eigenvalue equation (1.18). In a recent paper Dosch, Jensen, and Müller demonstrated that for a Dirac particle in an external scalar Coulomb field the energy remains real for all α.[3] An appropriate generalization of the relativistic Schrödinger equation (1.10) will not only give real energies for spin-zero particles in a strongly coupled external scalar field, but also will permit real energies for the strongly coupled two-body scalar interaction bound-state problem as well.

II. CLASSICAL LIMIT OF THE KLEIN-GORDON EQUATION

A. Four-dimensional proper-time formalism

It is of interest to examine what happens to the classical bound-state solution in the case of a particle in an external scalar field when the coupling becomes too large. The covariant "Hamiltonian" is[4]

$$\mathcal{H} = \frac{p_\mu p^\mu + m^2}{2m} + \mathcal{V}. \tag{2.1}$$

This "Hamiltonian" describes a particle off the mass shell ($p^2 + m^2 \neq 0$) and on the energy shell, meaning that p^0 is not regarded as an independent dynamical variable, but rather just a number (or in the quantum case, an eigenvalue). Alternatively, one can regard all four-momenta as independent with the modified mass-shell restriction $p^2 + m^2 + 2m\mathcal{V} = 0$ imposed after finding the equations of motion. The equations of motion are

$$\frac{\partial\mathcal{H}}{\partial p^\mu} = \frac{dr_\mu}{d\tau} = u_\mu = \frac{p_\mu}{m}, \quad -\frac{\partial\mathcal{H}}{\partial r^\mu} = \frac{dp_\mu}{d\tau} = -\frac{\partial\mathcal{V}}{\partial r^\mu}, \tag{2.2}$$

where τ is the proper-time variable. In the c.m. frame the potential is, like the quasipotential in (1.17), a function of r only so that

$$p^0 \equiv \mathcal{E} = \text{constant} = mu^0 = m\frac{dt}{d\tau} \tag{2.3}$$

and

$$\frac{md^2\vec{r}}{d\tau^2} = -\vec{\nabla}\mathcal{V}(r). \tag{2.4}$$

Combining these two equations leads to the equation of motion

$$\frac{d^2\vec{r}}{dt^2}=\frac{-m}{\mathcal{E}^2}\vec{\nabla}\mathcal{V}(r) \tag{2.5}$$

in terms of the lab time variable t. With $\mathcal{V}=-\alpha/r$ this equation is

$$\frac{d^2\vec{r}}{dt^2}=\frac{-m}{\mathcal{E}^2}\frac{\vec{r}}{r^3}. \tag{2.6}$$

B. Three-dimensional formalism

The modified mass-shell condition $\mathcal{K}=0$ and the on-energy-shell condition $\mathcal{E}=$ constant can be used directly to obtain the same equations of motion with the three-dimensional lab time Hamiltonian formalism. The three-dimensional Hamiltonian is

$$H=[\vec{p}^2+m^2+2m\mathcal{V}(r)]^{1/2}=\mathcal{E}=\text{ constant.} \tag{2.7}$$

This leads to the equations of motion

$$\frac{d\vec{r}}{dt}=\vec{\nabla}_p H=\frac{\vec{p}}{\mathcal{E}} \tag{2.8}$$

and

$$\frac{-d\vec{p}}{dt}=\vec{\nabla}_r H=\frac{m}{\mathcal{E}}\vec{\nabla}\mathcal{V}(r) \tag{2.9}$$

or

$$\frac{d^2\vec{r}}{dt^2}=\frac{-m}{\mathcal{E}^2}\vec{\nabla}\mathcal{V}(r), \tag{2.10}$$

which is the same as (2.5).

C. Orbital restrictions

Using the modified mass-shell condition $\mathcal{K}=0$ together with $p=mu$ leads to

$$u^0=[\vec{u}^2+1+2\mathcal{V}(r)/m]^{1/2}=dt/d\tau=\mathcal{E}/m. \tag{2.11}$$

With $\vec{u}=\vec{v}\,dt/d\tau$ this in turn gives

$$\mathcal{E}=m\left[\frac{1+2\mathcal{V}(r)/m}{1-v^2}\right]^{1/2}. \tag{2.12}$$

If $\mathcal{V}=-\alpha/r$ then

$$\mathcal{E}=m\left(\frac{1-2\alpha/mr}{1-v^2}\right)^{1/2}. \tag{2.13}$$

Thus $v^2<1$ and $r>2\alpha/m$ give real energies. The classical orbit for bound states appear to have a lower bound on the radius. Furthermore, $mv^2/2<\alpha/r$ for $\mathcal{E}<m$. Notice that if $v^2\ll1$ and $2\alpha/mr\ll1$ then we have

$$\mathcal{E}=m+\frac{mv^2}{2}-\frac{\alpha}{r}, \tag{2.14}$$

the usual nonrelativistic expression.

Is this restriction on the lower limit of r in the classical limit related to the upper limit on α in the quantum case? We can see such a relation if we look at the classical limit ($\hbar\to0$) of the Klein-Gordon equation. Using $\psi=e^{iS/\hbar}$, the $\hbar=0$ limit is

$$\frac{(\vec{\nabla}S)^2}{2m}-\frac{\alpha}{r}=\frac{\mathcal{E}^2-m^2}{2m}=\frac{b^2}{2m}. \tag{2.15}$$

This leads to[4]

$$\frac{b^2}{2m}=\frac{-2\pi^2\alpha^2m}{(J_r+J_\theta+J_\phi)^2}, \tag{2.16}$$

where

$$J_i=\oint p_i\,dq_i \quad i=r,\theta,\phi \tag{2.17}$$

or

$$\mathcal{E}=m\left(1-\frac{\alpha^2}{n^2}\right)^{1/2}, \tag{2.18}$$

where $J_r+J_\theta+J_\phi=2\pi n$. This looks identical to the quantum result (1.19), however, n need not be an integer in (2.18). If we combine this equation with (2.13) then we obtain a relation between v and r, namely

$$v^2=1-\frac{1-2\alpha/mr}{1-\alpha^2/n^2}. \tag{2.19}$$

If $\mathcal{E}$ is to remain real then $\alpha^2/n^2<1$ and hence $v^2>0$ implies that $r<2n^2/m\alpha$. The other restriction on r ($r>2\alpha/m$) follows from $v^2<1$. Thus a bound orbit in this model is restricted to lie in the interval $r_1=2\alpha/m<r\leq2n^2/m\alpha=r_2$. Smaller r would lead to velocities greater than c or complex energies. For r in this range, $1>v^2\geq0$. Suppose that $\alpha^2=n^2=1-\epsilon$, where $\epsilon\ll1$. Then r must be very close to r_1, $r=r_1+a$. The constant a is determined by substitution into (2.19) to be r_1v^2, so that $r=r_1(1+\epsilon v^2)$. Hence as $\alpha/n\to1$ from below, the total energy of the bound particle approaches zero and its orbit approaches the circle $r=2\alpha/m$. Larger α or coupling strength would lead to complex energies or $v>c$.

III. CLASSICAL LIMIT OF THE QUASIPOTENTIAL EQUATION FOR SCALAR INTERACTIONS

A. Coordinate and proper-time definitions

In this section we shall give coordinate and proper-time definitions that appear to be inherent in the on-energy-shell, off-mass-shell quasipotential approach. The total momentum may be written as

$$P^\lambda=w\frac{dR^\lambda}{d\tau_w}=p_1^\lambda+p_2^\lambda=m_1\frac{dr_1^\lambda}{d\tau_1}+m_2\frac{dr_2^\lambda}{d\tau_2}, \tag{3.1}$$

where R^λ is the coordinate of the center of mass. r_1^λ, r_2^λ, τ_1, τ_2 are the respective coordinates and proper times of particles 1 and 2, and τ_w is the proper-time variable of the center of mass. In the c.m. frame, the proper-time variable τ_w is the c.m. time t. In this frame

$$d\tau_i = \frac{m_i}{E_i} dt = \frac{m_i}{E_i} d\tau_w, \; i = 1, 2. \tag{3.2}$$

This allows us to rewrite (3.1) as

$$w \frac{dR^\lambda}{d\tau_w} = E_1 \frac{dr_1^\lambda}{d\tau_w} + E_2 \frac{dr_2^\lambda}{d\tau_w}. \tag{3.3}$$

Maintaining the assumption (1.3) in the classical limit also gives E_1 and E_2 as functions of the invariant w, leading naturally to the definition of the c.m. coordinates given below:

$$R^\lambda = \frac{E_1 r_1^\lambda + E_2 r_2^\lambda}{w}. \tag{3.4}$$

The c.m. coordinates are found by weighting the individual particle coordinates with their c.m. energies. It is of interest to point out that this definition implicit in this quasipotential approach is similar to one of the more acceptable definitions of relativistic center of mass as shown by Pryce.[5] It should be noted, however, that E_1 and E_2 are not operators or dynamical variables but are invariants equal to the energies of particles 1 and 2 only in the c.m. frame.

In the c.m. frame $P = (w, \vec{0})$ so that

$$P^0 = w = w \frac{dR^0}{dt} = E_1 \frac{dr_1^0}{dt} + E_2 \frac{dr_2^0}{dt} = E_1 + E_2. \tag{3.5}$$

The relative time variable $r^0 = r_1^0 - r_2^0$ vanishes in this frame if we require R^μ and r^μ to be orthogonal. The coordinates r^λ and proper time τ_e of the fictitious relative particle can be deduced from the definition given here of the relative momentum[1]:

$$\begin{aligned} p^\lambda &= \frac{E_2 p_1^\lambda - E_1 p_2^\lambda}{w} \\ &= \frac{m_1 E_2}{w} \frac{dr_1^\lambda}{d\tau_1} - \frac{m_2 E_1}{w} \frac{dr_2^\lambda}{d\tau_2} \equiv \frac{m_w dr^\lambda}{d\tau_e}. \end{aligned} \tag{3.6}$$

The variables τ_e and τ_w can be related by using the definition of m_w:

$$\frac{dr^\lambda}{dt_e} = \frac{E_2}{m_2} \frac{dr_1^\lambda}{d\tau_1} - \frac{E_1}{m_1} \frac{dr_2^\lambda}{d\tau_2} = \frac{E_1 E_2}{m_1 m_2} \frac{d}{d\tau_w} (r_1^\lambda - r_2^\lambda). \tag{3.7}$$

This allows us to identify the relative coordinates and proper-time variable by

$$r^\lambda = r_1^\lambda - r_2^\lambda, \quad d\tau_e = \frac{m_1 m_2}{E_1 E_2} d\tau_w. \tag{3.8}$$

In the static limit $m_2 \to \infty$ the proper-time variable $\tau_e \to \tau_1$.

In the c.m. frame the independent dynamical variables are the spatial components. Hence, the classical and quantum Poisson bracket relations involve just these variables.

Starting with the (quantum) Poisson bracket relation

$$[r_{i\lambda}, p_{j\mu}] = i\,\delta_{ij}\, g_{\lambda\mu}, \quad \lambda, \mu = 1, 2, 3 \tag{3.9}$$

we find

$$[r_\lambda, p_\mu] = i g_{\lambda\mu}, \quad \lambda, \mu = 1, 2, 3 \tag{3.10}$$

and

$$[R_\lambda, P_\mu] = i g_{\lambda\mu} \quad (\lambda, \mu = 1, 2, 3). \tag{3.11}$$

Since the r's commute and the energy variables are not operators, the c.m. coordinates commute with one another:

$$[R_\lambda, R_\mu] = 0 \tag{3.12}$$

B. The effective free-particle Hamiltonian and the two-body Hamiltonian

It is instructive and, for our purpose of constructing a generalization of (1.10), necessary to obtain (1.10) by an alternative approach to that given in IB. First of all, one should recognize that for a free effective particle the effective "Hamiltonian" in (1.10) is

$$\mathcal{H}^0 = \frac{\vec{p}^2 - b^2}{2 m_w}. \tag{3.13}$$

This may be written in the alternative form

$$\mathcal{H}^0 = \frac{\tilde{p}_\mu \tilde{p}^\mu + m_w^2}{2 m_w}, \tag{3.14}$$

where $\tilde{p}^\mu = p^\mu + (E/w) p^\mu$, as the relative momentum p^μ is orthogonal to P_μ. In the c.m. frame $\tilde{p}^\mu = (E, \vec{p})$. Here E plays the role of the energy of the fictitious effective particle; this is reflected in the equation $E^2 = m_w^2 + b^2$. The "Hamiltonian" (3.14) is identically zero on the mass shell when $\vec{p}^2 = b^2$.

We can obtain the effective one-particle "Hamiltonian"(3.13) as a c.m. reduction of a sum of two one-body "Hamiltonians" for free single-particle systems. If we take two free-particle "Hamiltonians"

$$\mathcal{H}_i^0 = \frac{p_i^2 + m_i^2}{2 m_i}, \quad i = 1, 2 \tag{3.15}$$

and define

$$\frac{w}{m_1 + m_2} (\mathcal{H}_1^0 + \mathcal{H}_2^0) = \mathcal{H}^0, \tag{3.16}$$

then we obtain (3.14) in the c.m. frame. This follows from the fact that

$$p_1 = \frac{E_1}{w} P + p, \quad p_2 = \frac{E_2}{w} P - p, \quad P \cdot p = 0, \tag{3.17}$$

as well as $E_i^2 = m_i^2 + b^2$. This latter condition in turn follows from the condition (1.3) in the c.m. frame.[6] Notice that in the static limit $m_2 \to \infty$ (3.16) reduces to $\mathcal{H}_1^0$.

If the particles are not free then we define

$$\mathcal{H} = \frac{\vec{p}^2 - b^2}{2m_w} + \mathcal{V}_w. \tag{3.18}$$

C. The subsidiary condition and the two-body "Hamiltonian"

The result (3.18) was obtained with the aid of the subsidiary condition (1.3). In this section we shall derive an equivalent form of the above "Hamiltonian" using the subsidiary condition by way of a Lagrange multiplier. This will have the additional advantage of confirming in a Hamiltonian context the definition of the relative proper-time variable τ_e given earlier and its relation to the subsidiary condition. It will also lead naturally to the definition of p_1, p_2, and p in terms of coordinates given in Eqs. (3.1) and (3.6). The "Hamiltonian" is taken as

$$\mathcal{H}' = \frac{w}{M}\left(\frac{p_1^2 + m_1^2}{2m_1} + \frac{p_2^2 + m_2^2}{2m_2}\right) + \mathcal{V}_w + \frac{w\lambda}{2M}(p_1^2 - p_2^2 - m_2^2 + m_1^2), \tag{3.19}$$

where $M = m_1 + m_2$. The Lagrange multiplier is $w\lambda/2M$. With the subsidiary term included we can treat all variables as independent (provided, of course, that the modified mass-shell condition $\mathcal{H}' = 0$ and the subsidiary conditions are imposed after finding the equation of motion). In finding the equation of motion only one time variable is needed. We take this time variable to be τ_e, the proper-time variable of the effective particle. The equations of motion are

$$\frac{\partial \mathcal{H}'}{\partial p_{1\mu}} = u_1^\mu = \frac{dr_1^\mu}{d\tau_e} = \frac{w}{M} p_1^\mu \left(\frac{1}{m_1} + \lambda\right),$$
$$\frac{\partial \mathcal{H}'}{\partial p_{2\mu}} = u_2^\mu = \frac{dr_2^\mu}{d\tau_e} = \frac{w}{M} p_2^\mu \left(\frac{1}{m_2} - \lambda\right), \tag{3.20}$$

and

$$\frac{\partial \mathcal{H}'}{\partial r_{1\mu}} = \frac{-dp_1^\mu}{d\tau_e} = \frac{\partial \mathcal{V}_w}{\partial r_{1\mu}},$$
$$\frac{\partial \mathcal{H}'}{\partial r_{2\mu}} = \frac{-dp_2^\mu}{d\tau_e} = \frac{\partial \mathcal{V}_w}{\partial r_{2\mu}}. \tag{3.21}$$

From the second set of equations we obtain several relations. First of all, in the c.m. frame $\mathcal{V}_w = \mathcal{V}_w(\vec{r}_1 - \vec{r}_2)$. Hence, as expected, $p_1^0 = E_1$ and $p_2^0 = E_2$ are constants. Also it follows that the c.m. momentum remains zero:

$$\frac{d}{d\tau_e}(\vec{p}_1 + \vec{p}_2) = \frac{d\vec{P}}{d\tau_e} = -\vec{\nabla}_1 \mathcal{V}_w - \vec{\nabla}_2 \mathcal{V}_w = 0. \tag{3.22}$$

The equation of motion of the relative momenta is given by

$$\frac{d}{d\tau_e}\vec{p} = \frac{d}{d\tau_e}\left(\frac{E_2\vec{p}_1 - E_1\vec{p}_2}{w}\right) = -\frac{(E_1 + E_2)}{w}\vec{\nabla}\mathcal{V}_w = -\vec{\nabla}\mathcal{V}_w. \tag{3.23}$$

This same equation of motion could be obtained from the effective one-body Hamiltonian (3.18) using p and r as independent variables with τ_e as the relative proper-time variable, provided that $p^\mu = m_w\, dr^\mu/d\tau_e$ is consistent with the above Hamiltonian (3.19). This can be verified directly from the equations of motion (3.20) and the definition of relative momentum (1.6) by an appropriate and consistent choice of λ. The variable must be chosen so that

$$\frac{d}{d\tau_e}(r_1^\mu - r_2^\mu) = \frac{p^\mu}{m_w} = \frac{E_2 p_1^\mu}{m_2 m_1} - \frac{E_1 p_2^\mu}{m_1 m_2}. \tag{3.24}$$

This gives two equations for λ:

$$\frac{E_2}{m_2} = \frac{w}{M}(m_1\lambda + 1), \quad \frac{E_1}{m_1} = \frac{w}{M}(1 - \lambda m_2). \tag{3.25}$$

Both of these equations are satisfied by

$$\lambda = \frac{m_1 E_2 - m_2 E_1}{m_1 m_2 w}. \tag{3.26}$$

This same value of λ also retrieves for us the relation $p_i^\mu = m_i\, dr_i^\mu/d\tau_i$ by substitutions into Eq. (3.20). In terms of the c.m. time variable t (which is the proper-time variable τ_w in the c.m. frame), the equation of motion (3.23) takes on the form

$$\frac{d\vec{p}}{d\tau_e} = \frac{E_1E_2}{m_1m_2}\frac{d}{dt}\left(\frac{m_1m_2}{w}\frac{E_1E_2}{m_1m_2}\frac{d\vec{r}}{dt}\right) = -\vec{\nabla}\mathcal{V}_w, \tag{3.27}$$

or

$$\frac{d^2\vec{r}}{dt^2} = \frac{-w m_1 m_2 \vec{\nabla}\mathcal{V}_w}{E_1^2 E_2^2}. \tag{3.28}$$

In the static limit $m_2 \to \infty$ this reduces to (2.6).

D. Orbital restrictions in the two-body bound-state problem

For our examination of the orbital restrictions in the two-body case, we consider the equal-mass case.[7] The modified mass-shell condition $\mathcal{H}=0$ together with $E^2=b^2+m_w{}^2$ leads to

$$E^2=\left(\frac{w^2-2m^2}{2w}\right)^2=m_w{}^2+\vec{p}^2+2m_w\mathcal{V}_w. \quad (3.29)$$

Furthermore,

$$\vec{p}^2=m_w{}^2\vec{u}^2=m_w{}^2\left(\frac{d\vec{r}}{d\tau_e}\right)^2=m_w{}^2\left(\frac{dt}{d\tau_e}\vec{v}\right)^2. \quad (3.30)$$

For equal masses,

$$\frac{dt}{d\tau_e}=\frac{w^2}{4m^2}, \quad m_w{}^2=\frac{m^4}{w^2}. \quad (3.31)$$

Hence

$$\vec{p}^2=\frac{w^2}{16}\vec{v}^2 \quad (3.32)$$

and (3.29) reduces to

$$w^2-4m^2=\frac{w^2v^2}{4}+8m_w\mathcal{V}_w. \quad (3.33)$$

This in turn leads to

$$w=2m\left(\frac{1+2\mathcal{V}_w/w}{1-v^2/4}\right)^{1/2}. \quad (3.34)$$

The nonrelativistic, weak coupling limit of this is

$$w=2m+\frac{1}{2}\frac{m}{2}v^2+\mathcal{V}_w. \quad (3.35)$$

With $\frac{1}{2}m$ as the nonrelativistic reduced mass, (3.35) is recognized as the rest mass plus nonrelativistic kinetic plus potential energy of the two-body system. The rest of our analysis will follow that of the static limit given in Sec. II C. With $\mathcal{V}_w=-\alpha/r$, w becomes complex if $r<2\alpha/w$ or if $v>2$. It must be remembered, of course, that v is the relative velocity of two particles as seen by the c.m. observer and need not be less than c, but of course must be less than $2c$. To relate this to the appearance in the quantum case of complex w for $\alpha/n>1$, we examine as before the $\hbar=0$ limit of the quasipotential equation (1.10). The two-body analog of (2.16) is

$$\frac{b^2}{2m_w}=\frac{-m_w\alpha^2 2\pi^2}{(J_r+J_\theta+J_\phi)^2}, \quad (3.36)$$

and with definitions similar to (2.17) we obtain

$$w^2=2m^2\left[1+\left(1-\frac{\alpha^2}{n^2}\right)^{1/2}\right]. \quad (3.37)$$

This leads as before to an inequality that must be satisfied for real w:

$$r_1=\frac{2\alpha}{w}<r\leq\frac{4\alpha}{w[1-(1-\alpha^2/n^2)^{1/2}]}=r_2. \quad (3.38)$$

As $\alpha/n\to 1$ from below, the total energy of the two-particle system approaches $\sqrt{2m}$ and its orbit can range from $r>(2n/m)^{1/2}$ where $v\to 2$ to $r=2(2n/m)^{1/2}$ where $v\to 0$.

IV. GENERALIZATION OF THE KLEIN-GORDON EQUATION AND ITS CLASSICAL LIMIT

A. The generalized classical covariant "Hamiltonian"

We now postulate the following generalization of (2.1):

$$\mathcal{H}=\frac{p_\mu p^\mu+\beta^2}{2\beta}. \quad (4.1)$$

This "Hamiltonian" follows from a Lagrangian of the form $\mathcal{L}=\beta u^2$.[8] The function β plays the role of mass but it is allowed to depend on r. The condition $\mathcal{H}=0$ means that the particle satisfies a modified mass-shell constraint. This constraint is used after imposing Hamilton's equations in their four-dimensional form.

$$\frac{\partial\mathcal{H}}{\partial p^\lambda}=\frac{dr_\lambda}{d\tau}=\frac{p^\lambda}{\beta}, \quad (4.2)$$

$$\frac{\partial\mathcal{H}}{\partial r^\lambda}=\frac{-dp_\lambda}{d\tau}=\frac{-\partial\beta}{\partial r^\lambda}\frac{1}{\beta^2}(p^2+\beta^2)+\frac{\partial\beta}{\partial r^\lambda}. \quad (4.3)$$

Using $p^2+\beta^2=0$, the last equation becomes

$$\frac{dp_\lambda}{d\tau}=\frac{d}{d\tau}\beta r_\lambda=\frac{-\partial\beta}{\partial r^\lambda}. \quad (4.4)$$

Since $\beta=\beta(r)$, this equation is complicated when using the proper-time variable τ. The equation is simpler in terms of the lab variable t. Since

$$\frac{dt}{d\tau}=\frac{p^0}{\beta}=\frac{\mathcal{E}}{\beta}, \quad \frac{d}{d\tau}\beta\frac{d\vec{r}}{d\tau}=\frac{\mathcal{E}}{\beta}\frac{d^2\vec{r}}{dt^2}, \quad (4.5)$$

and the equation of motion is

$$\frac{d^2\vec{r}}{dt^2}=\frac{-\beta\vec{\nabla}\beta}{\mathcal{E}^2}. \quad (4.6)$$

As a check, this same equation can be derived from the three-dimensional Hamiltonian formalism. With $\mathcal{H}=0$, we have

$$H=(\vec{p}^2+\beta^2)^{1/2}=\mathcal{E}=\text{constant}. \quad (4.7)$$

This gives

$$\frac{d\vec{r}}{dt}=\vec{\nabla}_p H=\frac{\vec{p}}{\mathcal{E}} \quad (4.8)$$

and

$$\frac{-d\vec{p}}{dt}=\vec{\nabla}_r H=\frac{\beta\vec{\nabla}\beta}{\mathcal{E}}=-\mathcal{E}\frac{d^2\vec{r}}{dt^2}, \quad (4.9)$$

which agrees with (4.6).

B. Orbital analysis

In this section we shall perform the same type of analysis that leads to orbit information for the Hamiltonian (2.1). From Hamilton's equations in their covariant form we have

$$u^0 = \frac{dt}{d\tau} = \frac{\mathcal{E}}{\beta}, \quad \vec{u} = \frac{\vec{p}}{\beta}. \tag{4.10}$$

Using the condition $p^2 + \beta^2 = 0$ this leads to

$$\frac{dt}{d\tau} = (\vec{u}^2 + 1)^{1/2}. \tag{4.11}$$

Since $\vec{u} = \vec{v}\,dt/d\tau$ we obtain

$$\frac{dt}{d\tau} = \frac{1}{(1 - v^2)^{1/2}} \tag{4.12}$$

or

$$\mathcal{E} = \frac{\beta}{(1 - v^2)^{1/2}}. \tag{4.13}$$

Equations (4.13) and (4.12) are to be compared with (2.11) and (2.12). The function $\beta(r)$ is chosen so that in the weak coupling, nonrelativistic limit $\mathcal{E}$ is given by (2.14) for $\mathcal{V} = -\alpha/r$. The choice of β is not unique. We will examine two choices:

$$\beta_{\text{I}} = m + \mathcal{V}, \quad \beta_{\text{II}} = m e^{\mathcal{V}/m}. \tag{4.14}$$

So for $\mathcal{V} = -\alpha/r$

$$\beta_{\text{I}} = m - \frac{\alpha}{r}, \quad \beta_{\text{II}} = m e^{-\alpha/mr}. \tag{4.15}$$

The first point of contrast between (4.13) and (2.13) is that the former gives real $\mathcal{E}$ for all values of r, unlike (2.13). Owing to this and earlier considerations in II C we do not expect any limit on α above which $\mathcal{E}$ becomes complex. This can be demonstrated for the first choice of β analytically. Writing $\mathcal{H} = 0$ as

$$p_\lambda p^\lambda + m^2 + 2m\mathcal{V} + \mathcal{V}^2 = 0, \tag{4.16}$$

with $\mathcal{V} = -\alpha/r$, the corresponding Hamilton-Jacobi equation is

$$\frac{(\nabla \vec{S})^2}{2m} - \frac{\alpha}{r} + \frac{\alpha^2}{2mr^2} = \frac{\mathcal{E}^2 - m^2}{2m}. \tag{4.17}$$

This differs from (2.15) by the presence of a repulsive term $\alpha^2/2mr^2$ and by an analytic continuation in l^2 to the energy equation

$$\mathcal{E} = m\left\{1 - \frac{\alpha^2}{[n + (l^2 + \alpha^2)^{1/2} - l]^2}\right\}^{1/2}, \tag{4.18}$$

where $l = 2\pi(J_\theta + J_\phi)$. $\mathcal{E}$ is real for all α and approaches zero only for $\alpha \to \infty$, unlike (2.18) which approaches zero for $\alpha/n \to 1$ and becomes complex for larger α.

The equation for v^2 analogous to (2.18) is

$$v^2 = 1 - \frac{(1 - \alpha/mr)^2}{1 - \alpha^2/n'^2}, \tag{4.19}$$

where $n' = n + (l^2 + \alpha^2)^{1/2} - l$. Unlike the case for (2.19), this equation does not relate $v^2 < 1$ to a lower limit on r. There are, however, limits on r placed by $v^2 > 0$ given by

$$r_- = \frac{n'}{\alpha m}\left[1 - \left(1 - \frac{\alpha^2}{n'^2}\right)^{1/2}\right] \leq r \leq \frac{n'}{\alpha m}\left[1 + \left(\frac{1 - \alpha^2}{n'^2}\right)^{1/2}\right] = r_+. \tag{4.20}$$

C. Quantum spectrum: An example

Now we consider the generalization of the quantum Klein-Gordon equation. This static limit equation is

$$(p^2 + \beta^2)\psi = 0. \tag{4.21}$$

If we take $\beta = m - \alpha/r$ then the on-energy-shell equation is just the stationary-state Klein-Gordon equation (1.17) with

$$\mathcal{V} = \frac{-\alpha}{r} + \frac{\alpha^2}{2mr^2}. \tag{4.22}$$

If $\beta = m e^{-\alpha/mr}$, then

$$\mathcal{V}_{\text{eff}} = \frac{m}{2}(e^{-2\alpha/mr} - 1). \tag{4.23}$$

Solving the stationary-state Klein-Gordon equation with $\mathcal{V}_{\text{eff}}$ given in (4.22) leads to

$$\mathcal{E} = m\left(1 - \frac{\alpha^2}{(n + \frac{1}{2}\{[(2l+1)^2 + 4\alpha^2]^{1/2} - (2l+1)\})^2}\right)^{1/2}. \tag{4.24}$$

$\mathcal{E}$ is real for all α with $\mathcal{E} \to 0$ as $\alpha \to \infty$. A similar spectrum has been found by Dosch, Jensen, and Müller for the Dirac equation for a scalar potential.[3] Numerical studies indicate that if the effective potential (4.23) is used in place of (4.22) then real $\mathcal{E}$ is also obtained for all α. The equation we ultimately seek is a generalization of the two-body homogeneous quasipotential equation (1.10). Obviously this equation must yield the above type of spectrum in the static limit.

V. GENERALIZATION OF THE QUASIPOTENTIAL EQUATION FOR SCALAR INTERACTION AND ITS CLASSICAL LIMIT

A. Coordinate and proper-time definitions

In this section we shall give coordinates and proper-time definitions that are a generalization of those given in Sec. III A for the original quasipotential equation. First of all, p_1 and p_2 will be defined as

$$p_i^\lambda = \beta_i \frac{dr_i^\lambda}{d\tau_i}, \quad i=1,2. \tag{5.1}$$

If, as in (3.1), we define

$$P^\lambda = w\frac{dR^\lambda}{d\tau_w} = p_1^\lambda + p_2^\lambda = \beta_1 \frac{dr_1^\lambda}{d\tau_1} + \beta_2 \frac{dr_2^\lambda}{d\tau_2} \tag{5.2}$$

then

$$d\tau_i = \frac{\beta_i}{E_i}\, d\tau_w. \tag{5.3}$$

This allows us to write

$$w\frac{dR^\lambda}{d\tau_w} = E_1\frac{dr_1^\lambda}{d\tau_w} + E_2\frac{dr_2^\lambda}{d\tau_w}. \tag{5.4}$$

The energy variables E_1 and E_2 are defined as before to be

$$E_1 = \frac{-P\cdot p_1}{w} = \frac{w^2 - p_1^{\,2} + p_2^{\,2}}{2w},$$
$$E_2 = \frac{-P\cdot p_2}{w} = \frac{w^2 - p_2^{\,2} + p_1^{\,2}}{2w}. \tag{5.5}$$

Evidently, in the frame in which $\beta_i = \beta_i(r)$, the Hamiltonian formulation would lead to $\dot{p}_1^0 = \dot{E}_1 = 0 = \dot{p}_2^0 = \dot{E}_2$, and the original definition of R^λ given in equation (3.4) would be valid. This would also allow one to impose the same subsidiary condition (1.3). In the c.m. frame, the relative time variable $r_1^0 - r_2^0 = 0$ as before since (3.5) and the attendant arguments still hold. The relative momentum variable is defined as before and is equal to

$$p^\lambda = \frac{E_1E_2}{w}\frac{d}{d\tau_w}(r_1^\lambda - r_2^\lambda). \tag{5.6}$$

At this point, there are two logical choices for relations between $d\tau_e$ and $d\tau_w$. If $d\tau_e = (m_1m_2/E_1E_2)d\tau_w$ as before, then $p^\lambda = m_w dr^\lambda/d\tau_e$. That is, the relations between the relative momenta p, the relativistic reduced mass m_w, and the proper-time variable τ_e remain the same. The second choice would be

$$d\tau_e = \frac{\beta_1\beta_2}{E_1E_2}\, d\tau_w. \tag{5.7}$$

Then we have

$$p^\lambda = \beta_w \frac{d}{d\tau_e}(r_1^\lambda - r_2^\lambda), \tag{5.8}$$

where

$$\beta_w = \frac{\beta_1\beta_2}{w}. \tag{5.9}$$

We take the second choice, as it has the correct static limit (4.10), unlike the first choice.

B. Generalized two-body "Hamiltonian" with the subsidiary condition

The "Hamiltonian" we postulate that is a generalization of (3.19) is

$$\mathcal{H}' = \frac{w}{B}\left(\frac{p_1^{\,2}+\beta_1^{\,2}}{2\beta_1} + \frac{p_2^{\,2}+\beta_2}{2\beta_2}\right) + \frac{w\lambda}{2B}(p_1^{\,2} - p_2^{\,2} - m_2^{\,2} + m_1^{\,2}), \tag{5.10}$$

where $B = \beta_1 + \beta_2$. The subsidiary condition is the same as used in Eq. (3.19). This "Hamiltonian" leads to the following equations relating p_i and u_i:

$$\frac{\partial\mathcal{H}'}{\partial p_{1\mu}} = \frac{dr_1^\mu}{d\tau_e} = \frac{w}{B}p_1^\mu\left(\frac{1}{\beta_1} + \lambda\right),$$
$$\frac{\partial\mathcal{H}'}{\partial p_{2\mu}} = \frac{dr_2^\mu}{d\tau_e} = \frac{w}{B}p_2^\mu\left(\frac{1}{\beta_2} - \lambda\right). \tag{5.11}$$

Notice that a unique choice of the multiplier is found to be consistent with $d\tau_e = (\beta_1\beta_2/E_1E_2)d\tau_w$. Using (5.11) and (1.6) leads to

$$p^\mu = \frac{\beta_1\beta_2}{w}\frac{d}{d\tau_e}(r_1 - r_2)^\mu = \frac{1}{B}\left(\beta_2 p_1^\mu(\beta_1\lambda + 1) - \beta_1 p_2^\mu(1-\lambda\beta_2)\right) = \frac{E_2p_1^\mu - E_1p_2^\mu}{w}. \tag{5.12}$$

This in turn gives two equations for λ,

$$E_2 = \frac{w}{B}\beta_2(\beta_1\lambda + 1), \quad E_1 = \frac{w}{B}\beta_1(1-\lambda\beta_2), \tag{5.13}$$

and they are both satisfied by

$$\lambda = \frac{\beta_1E_2 - E_1\beta_2}{\beta_1\beta_2 w}. \tag{5.14}$$

This is to be compared with (3.26). Notice that if we had used the prefactor w/M instead of w/B in the choice of the "Hamiltonian" we could not have found a unique λ. This value of λ also gives us back the relations $p_i^\mu = \beta_i\, dr_i^\mu/d\tau_i$, $i=1,2$ by substituting into (5.11). The other equations of motion are

$$\frac{\partial\mathcal{H}'}{\partial r_{1\mu}} = \frac{-d}{d\tau_e}p_1^\mu, \quad \frac{\partial\mathcal{H}'}{\partial r_{2\mu}} = \frac{-d}{d\tau_e}p_2^\mu. \tag{5.15}$$

As $\beta = \beta(\vec{r}_1 - \vec{r}_2)$, we find as before that $d\vec{P}/d\tau_e = 0$ and

$$\frac{d\vec{p}}{d\tau_e} = \frac{E_2\, d\vec{p}_1/d\tau_e - E_1\, d\vec{p}_2/d\tau_e}{w} = \frac{-E_2\vec{\nabla}_1\mathcal{H}' + E_1\vec{\nabla}_2\mathcal{H}'}{w} = -\vec{\nabla}\mathcal{H}'. \tag{5.16}$$

In order to compute the right-hand side of this equation we must employ the modified mass-shell condition $\mathcal{H}' \equiv 0$ as well as the subsidiary condition (1.3). Rather than give the result at this point we shall rederive this equation of motion and the subsequent equation of motion for $\vec{r}$ by employing an effective one-body reduction of this two-body "Hamiltonian," imposing the subsidiary condition as an intermediate step as was done in Sec. III B for the original quasipotential equation.

C. The generalized effective one-body "Hamiltonian"

We wish to obtain a one-body reduction of the two-body "Hamiltonian" analogous to the reduction (3.16)–(3.18). Using the substitution $p_1 = E_1 P/w + p$, $p_2 = E_2 P/w - p$ the combination $\mathcal{H}_1 + \mathcal{H}_2$, where

$$\mathcal{H}_i = \frac{p_1^2 + \beta_i^2}{2\beta_i}, \quad i = 1, 2 \tag{5.17}$$

becomes

$$\mathcal{H}_1 + \mathcal{H}_2 = \frac{p^2 - E_1^2 + \beta_1^2}{2\beta_1} + \frac{p^2 - E_2^2 + \beta_2^2}{2\beta_2}. \tag{5.18}$$

If we impose again the subsidiary condition in the form $E_i^2 = m_i^2 + b^2$, then this becomes

$$\mathcal{H}_1 + \mathcal{H}_2 = \frac{p^2 - b^2}{2\mu_B} + \frac{\beta_1 + \beta_2}{2} \frac{-m_1^2}{2\beta_1} \frac{-m_2^2}{2\beta_2}, \tag{5.19}$$

where

$$\mu_B = \beta_1 \beta_2 / B. \tag{5.20}$$

Multiplying this by w/B leads to

$$\mathcal{H} \equiv \frac{w}{B}(\mathcal{H}_1 + \mathcal{H}_2) = \frac{\vec{p}^2 - b^2}{2\beta_w} + \mathcal{V}_{\text{eff}}, \tag{5.21}$$

where

$$\mathcal{V}_{\text{eff}} = \frac{\beta_2(\beta_1^2 - m_1^2) + \beta_1(\beta_2^2 - m_2^2)}{2\beta_w B}. \tag{5.22}$$

This effective "Hamiltonian" should reduce to (3.18) for a weak potential $\mathcal{V}_w$. The choices

$$\beta_i = m_i e^{\mathcal{V}_w m_j / w m_i}, \quad i \neq j \tag{5.23}$$

and

$$\beta_i = m_i + \frac{\mathcal{V}_w m_j}{w}, \quad i \neq j \tag{5.24}$$

ensure this weak potential limit. The first choice leads to

$$\mathcal{V}_{\text{eff}} = \frac{w}{2}\left(\frac{m_1 e^{\mathcal{V}_w m_1/w m_2}(e^{2\mathcal{V}_w m_2/w m_1} - 1) + m_2 e^{\mathcal{V}_w m_2/w m_1}(e^{2\mathcal{V}_w m_1/w m_2} - 1)}{m_1 e^{(\mathcal{V}_w/w)(2 m_2 m_1 + m_1/m_2)} + m_2 e^{(\mathcal{V}_w/w)(2 m_1 m_2 + m_2/m_1)}}\right), \tag{5.25}$$

and (5.21) becomes the original classical form (3.18) as

$$\begin{aligned} &\frac{w m_1}{2}(e^{2\mathcal{V}_w m_2/w m_1} - 1) \to \mathcal{V}_w m_2, \\ &\frac{w m_2}{2}(e^{2\mathcal{V}_w m_1/w m_2} - 1) \to \mathcal{V}_w m_1 \end{aligned} \tag{5.26}$$

and

$$\mathcal{V}_{\text{eff}} \to \mathcal{V}_w, \quad \beta_w \to m_w. \tag{5.27}$$

The mass dependence postulated in (5.23) and (5.24) insures the correct static limit. The static limit $(m_2 \to \infty)$ of β_w is $\beta_1 \equiv \beta$. The static limit of $\mathcal{V}_{\text{eff}}$ is $(m_1 \equiv m)$

$$\mathcal{V}_{\text{eff}} \to \frac{\beta^2 - m^2}{2\beta}, \tag{5.28}$$

which agrees with (4.1) on the energy shell.

D. The equations of motion

The equations of motion that follow from (5.21) are

$$\frac{d\vec{r}}{d\tau_e} = \frac{\vec{p}}{\beta_w}, \tag{5.29}$$

$$\frac{d\vec{p}}{d\tau_e} = -\vec{\nabla}\mathcal{H}, \tag{5.30}$$

and of course the modified mass-shell condition $\mathcal{H} = 0$. The subsidiary condition (1.3) is already incorporated into these equations. The relevant time variable is as before the variable τ_e which reduces to the proper time τ of the bound particle in the static limit. The static limit equations (4.2)–(4.4) would have been simpler if we had used the lab variable t. In the two-body case, the analog of t is τ_w which becomes t in the c.m. frame. We shall give the equations of motion in terms of this variable. As $d\tau_w = dt$ in the c.m. frame

$$d\tau_e = \frac{\beta_1 \beta_2}{E_1 E_2} dt \tag{5.31}$$

and

$$\vec{p} = \beta_w \frac{d\vec{r}}{d\tau_e} = \frac{E_1 E_2}{w} \frac{d\vec{r}}{dt}. \tag{5.32}$$

Hence

$$\frac{d^2\vec{r}}{dt^2}=\frac{w}{E_1E_2}\frac{d\vec{p}}{dt}=\frac{w}{E_1^2E_2^2}\beta_1\beta_2\frac{d\vec{p}}{d\tau_e}. \tag{5.33}$$

Using the modified mass-shell condition $\mathcal{H}=0$ when computing $\vec{\nabla}\mathcal{H}$ gives

$$\frac{d^2\vec{r}}{dt^2}=\frac{-w}{2E_1^2E_2^2}\left[\vec{\nabla}\beta_1\left(\frac{\beta_1\beta_2}{B}+\frac{\beta_2m_1^2+\beta_1m_2^2}{B^2}+\frac{\beta_2^2-m_2^2}{B}\right)+\vec{\nabla}\beta_2\left(\frac{\beta_1\beta_2}{B}+\frac{\beta_1m_2^2+\beta_2m_1^2}{B^2}+\frac{\beta_1^2-m_1^2}{B}\right)\right]. \tag{5.34}$$

The static limit ($m_2\to\infty$, $\beta_1\equiv\beta$, $m_1\equiv m$, $\vec{\nabla}\beta_2=0$; $E_1\equiv\mathcal{E}$) of this equation is

$$\frac{d^2\vec{r}}{dt^2}=\frac{-\beta\vec{\nabla}\beta}{\mathcal{E}^2}, \tag{5.35}$$

which agrees with (4.6).

For the propose of comparing (5.34) with the unmodified two-body classical equation (5.31), it is convenient to look at the equal-mass case. In that case (5.34) is ($\beta_1=\beta_2=\beta$)

$$\frac{d^2\vec{r}}{dt^2}=\frac{-16}{w^2}\beta\vec{\nabla}\beta. \tag{5.36}$$

For

$$\beta=me^{\mathcal{V}_w/w} \tag{5.37}$$

we have

$$\frac{d^2\vec{r}}{dt^2}=\frac{-16m}{w^2}\vec{\nabla}\mathcal{V}_we^{2\mathcal{V}_w/w}=\frac{-16m}{w^2}\frac{\alpha\vec{r}}{r^3}e^{-2\alpha/wr}. \tag{5.38}$$

The explicit result for $\mathcal{V}=-\alpha/r$ is displayed on the right-hand side. This is to be compared with the equal-mass version of (3.28) which is

$$\frac{d^2\vec{r}}{dt^2}=\frac{-16m}{w^2}\vec{\nabla}\mathcal{V}_w=\frac{-16m}{w^2}\frac{\alpha\vec{r}}{r^3}. \tag{5.39}$$

As expected, the two do not differ significantly for small $\mathcal{V}$.

E. Orbit considerations and the classical spectrum

For orbital analysis in the classical limit, it is convenient to use the equal-mass version of (5.24).[7] For $\beta_1=\beta_2=\beta$ we have

$$\mathcal{H}=\frac{\vec{p}^2-b^2}{2\beta_w}+\mathcal{V}_{\text{eff}}, \tag{5.40}$$

where

$$\mathcal{V}_{\text{eff}}=\frac{w}{2\beta^2}(\beta^2-m^2)=\frac{\beta^2-m^2}{2\beta_w}. \tag{5.41}$$

The off-mass-shell condition $\mathcal{H}=0$ together with $E^2=b^2+m_w^2$ gives

$$E^2=\left(\frac{w^2-2m^2}{2m}\right)^2=m_w^2+\vec{p}^2+\beta^2-m^2. \tag{5.42}$$

For $\beta=m(1+\mathcal{V}_w/w)$ this becomes

$$E^2=m_w^2+\vec{p}^2+2m_w\mathcal{V}_w+\frac{m^2\mathcal{V}_w^2}{w^2}, \tag{5.43}$$

and differs from the corresponding term (3.29) in the original approach by the additional $\mathcal{V}^2$ term.

By a series of steps very similar to (3.30)–(3.33) we find that

$$w^2=4m^2\frac{(1+\mathcal{V}_w/w)^2}{1-v^2/4}. \tag{5.44}$$

As before, the nonrelativistic, weak coupling limit of this is (3.35). With $\mathcal{V}_w=-\alpha/r$ this gives real w for all r, unlike (3.34). Unlike (3.37), we do not expect any limitations on the value of α. We demonstrate this for the above choice of β. Equation (5.42) leads to the following Hamilton-Jacobi equation:

$$\frac{(\vec{\nabla}S)^2}{2m_w}+\mathcal{V}_w+\frac{\mathcal{V}_w^2}{2w}=\frac{b^2}{2m_w}. \tag{5.45}$$

With $\mathcal{V}_w=-\alpha/r$ this becomes

$$\frac{(\vec{\nabla}S)^2}{2m_w}-\frac{\alpha}{r}+\frac{\alpha^2}{2wr^2}=\frac{b^2}{2m_w}. \tag{5.46}$$

This is solved in much the same way that (2.15) is solved in standard textbooks.[4] The replacement, as made in obtaining (4.18), of the angular momentum $l=2\pi(J_\theta+J_\phi)$ by $\lambda=(l^2+\alpha^2m^2/w^2)^{1/2}$ leads to [in analogy with (2.16)]

$$\frac{b^2}{2m_w}=\frac{-\alpha^2m_w}{2n'^2}, \tag{5.47}$$

where

$$n'=n+\left(l^2+\frac{\alpha^2m^2}{w^2}\right)^{1/2}-l. \tag{5.48}$$

This in turn gives

$$w^2(w^2-4m^2)=\frac{-4m^4}{n'^2}\alpha^2, \tag{5.49}$$

or

$$2m^2[1\pm(1-\alpha^2/n'^2)^{1/2}]. \tag{5.50}$$

Now in the classical limit of the original approach, n' was equal to n and was independent of α^2 and w. This meant that only the positive root was to be chosen. This can most clearly be demonstrated by differentiating (5.49) with respect to $\tilde{\alpha}^2$ with n' replaced by n and held fixed. We obtain

$$\frac{dw}{d\alpha^2} \equiv w' = \frac{-4m^4}{n^2} \frac{1}{w(w^2 - 2m^2)} . \tag{5.51}$$

Clearly w decreases with increasing α^2, as long as $w^2 > 2m^2$. This means that the positive root must be chosen, as the negative root would yield $w^2 < 2m^2$ with w^2 increasing as α increases. Furthermore, the positive root is continuously connected to the $\alpha = 0$ limit of $w^2 = 4m^2$, whereas the negative root is not.

In the generalized case, with $n' = n'(\alpha, w)$, choose the positive root until $\alpha^2/n'^2 = 1$ at which point $w^2 = 2m^2$. As α increases beyond this particular crucial value (call it α_0) α^2/n'^2 *decreases*, unlike the original case where α^2/n^2 continues to increase and w^2 becomes complex. Thus for α beyond this crucial value, choose the negative root in (5.50). Ultimately as $\alpha^2/n'^2 \to 0$ (corresponding to $\alpha \to \infty$) $w^2 \to 0$, and one has the limiting case of the binding energy equal to the total rest mass as the coupling strength becomes infinite. The main reason that this differs from the original approach is that w' continues to remain negative as w^2 decreases through $2m^2$. Rather than give a demonstration of these points using the classical approach we shall display these points explicitly in the quantum case in Sec. V G for a quantum equation for which this system is the classical limit.

For now, we use this assumption on the choice of roots in the expression (5.49) to obtain a relation between v^2 and r. In the following, the plus root is chosen for $0 \le \alpha < \alpha_0$ and the minus root is chosen for $\alpha_0 \le \alpha < \infty$. The value α_0 is defined by

$$\alpha_0 = n'(w = 2m, \alpha_0), \tag{5.52}$$

and is given by

$$\alpha_0 = 2(n - l) + (2n^2 - 2nl + 4l^2)^{1/2}. \tag{5.53}$$

Equating (5.43) and (5.49) gives us the limits on r,

$$r_- \equiv \frac{\alpha}{w\eta}[1 - (1 - \eta)^{1/2}] \le r \le \frac{\alpha}{w\eta}[1 + (1 - \eta)^{1/2}] \equiv r_+, \tag{5.54}$$

where

$$\eta = \tfrac{1}{2}[1 - (1 - \alpha^2/n'^2)^{1/2}] \text{ for } 0 \le \alpha < \alpha_0,$$
$$\eta = \tfrac{1}{2}[1 + (1 - \alpha^2/n'^2)^{1/2}] \text{ for } \alpha_0 \le \alpha < \infty. \tag{5.55}$$

Hence, η increases from zero at $\alpha = 0$ to $\frac{1}{2}$ at $\alpha = \alpha_0$ toward unity as $\alpha \to \infty$. Unlike the limits in the original approach (Secs. II C and III D), these limits arise only from $v^2 > 0$.

F. Generalized quasipotential equation for scalar interactions

On the basis of the classical covariant "Hamiltonian" for the generalized two-body problem given by Eq. (5.21) of Sec. V C we postulate the following generalization of the two-body homogeneous quasipotential equations for scalar interactions:

$$\left[-\nabla^2 + \frac{\beta_2(\beta_1^2 - m_1^2) + \beta_1(\beta_2^2 - m_2^2)}{B}\right]\psi_w(\vec{r}) = b^2\psi_w(\vec{r}). \tag{5.56}$$

This equation can be regarded as a generalization of (1.10), here written as

$$(-\nabla^2 + 2m_w \mathcal{V}_w)\psi_w(\vec{r}) = b^2\psi_w(\vec{r}), \tag{5.57}$$

for strong coupling. For the choice

$$\beta_1 = m_1 + \frac{\mathcal{V}_w m_2}{w}, \quad \beta_2 = m_2 + \frac{\mathcal{V}_w m_1}{w} \tag{5.58}$$

(5.56) becomes

$$\left[-\nabla^2 + \frac{2Mm_w\mathcal{V}_w + (2m_1m_2M + m_1^3 + m_2^3)\frac{\mathcal{V}_w^2}{w^2} + m_1m_2\frac{M\mathcal{V}_w^3}{w^3}}{M(1 + \mathcal{V}_w/w)}\right]\psi_w(\vec{r}) = b^2\psi_w(\vec{r}), \tag{5.59}$$

and for the choice

$$\beta_1 = m_1e^{m_2\mathcal{V}_w/m_1w}, \quad \beta_2 = m_2e^{m_1\mathcal{V}_w/m_2w}, \tag{5.60}$$

it takes the form

$$\left[-\nabla^2 + \frac{m_1^2m_2e^{m_1\mathcal{V}_w/m_2w}(e^{2m_2\mathcal{V}_w/m_1w} - 1) + m_2^2m_1e^{m_2\mathcal{V}_w/m_1w}(e^{2m_1\mathcal{V}_w/m_2w} - 1)}{m_1e^{m_2\mathcal{V}_w/m_1w} + m_2e^{m_1\mathcal{V}_w/m_2w}}\right]\psi_w(\vec{r}) = b^2\psi_w(\vec{r}). \tag{5.61}$$

Both generalized forms reduce to the original relativistic Schrödinger equation when $\mathcal{V}_w$ can be considered small. They also have the expected static limit.

G. The quantum spectrum: An example of deep binding for scalar interactions

As an illustration of the nature of the quantum spectrum in this case of the generalized quasi-potential equation, we choose the exactly soluble example of equal masses $(m_1 = m_2 \equiv m)$ with $\beta_1 = \beta_2 \equiv \beta = m(1 - \alpha/wr)$. With this substitution, (5.56) becomes

$$\left(-\nabla^2 - \frac{2m_w\alpha}{r} + \frac{m^2\alpha^2}{w^2r^2}\right)\psi_w(\vec{r}) = b^2\psi_w(\vec{r}). \qquad (5.62)$$

Solving this leads to the relation

$$w^2(w^2 - 4m^2) = -\frac{4m^4\alpha^2}{n'^2}, \qquad (5.63)$$

where

$$n' = n + \tfrac{1}{2}\left\{\left[(2l+1)^2 + \frac{4m^2}{w^2}\alpha^2\right]^{1/2} - 2l - 1\right\}. \qquad (5.64)$$

As with the classical case in solving (5.63), one takes

$$w^2 = 2m^2[1 \pm (1 - \alpha^2/n'^2)^{1/2}], \qquad (5.65)$$

with the positive root chosen for $0 \leq \alpha < \alpha_0$ and the negative root for $\alpha_0 \leq \alpha < \infty$, where α_0 is the solution to $\alpha = n'(w^2 = 2m^2, \alpha)$. As stated earlier, this choice follows from the fact that w' remains real and negative as w^2 decreases through $2m^2$. We demonstrate this explicitly for the case $n = 1$, $l = 0$. We can solve (5.63) rather easily in this case for α^2 in terms of w^2. We find

$$\alpha^2 = \frac{16m^2 - 4w^2}{w^2}. \qquad (5.66)$$

This is to be compared with

$$\alpha^2 = \frac{(4m^2 - w^2)}{4m^4}w^2 \qquad (5.67)$$

for the original equation with $n' = n = 1$. For $w^2 = 2m^2$, $\alpha^2 = 4$ from (5.66) and $\alpha^2 = 1$ from (5.67). The differences beyond this point are even more crucial. As pointed out before in the classical case w' is negative from (5.51) only for $w^2 > 2m^2$. This also holds for the quantum equation (5.67). On the other hand, the derivative of w as computed from (5.66) is

$$w' = \frac{-w^3}{32m^2}, \qquad (5.68)$$

which is negative for *all* w in the range $0 < w^2 \leq 4m^2$ for bound states. In the special case we have considered we switch roots when $\alpha = 2$, as beyond this w^2 decreases through $2m^2$.

VI. CONCLUSION

The generalization (5.56) of Todorov's relativistic Schrödinger equation can display, unlike the original equation (5.57), very deep binding in the strong coupling limit. We have shown the absence of complex energies for these generalized quantum equations for strong coupling to be related to the absence of orbital restrictions in the classical case. Although the generalization (5.56) is not unique, it does not require a modification of the basic assumptions behind the quasipotential approach. For example, the higher-order contributions to $\mathcal{V}_{\text{eff}}$ that arise from these generalizations could be subtracted from the higher-order corrections that come from including more Feynman diagrams.

In a future paper, now in preparation, we shall examine the trajectory $w(n, l)$ of bound states in the above models for arbitrary mass ratios as a function of the coupling constant.

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