Propagation of photons in homogeneous magnetic fields: Index of refraction*

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The index of refraction associated with the vacuum polarization induced by homogeneous magnetic fields is calculated for two cases: (i) high-energy photons traversing fields weak compared with the critical field, $H_{cr} = m^2c^3/e\hbar \cong 4.41\times 10^{13}$ G; and (ii) low-energy photons in fields of arbitrary intensity. Some implications for the physical optics of intense magnetic fields are briefly discussed.

I. INTRODUCTION

The index of refraction of a homogeneous magnetic field was first calculated by Toll from the photon-pair creation rate via a dispersion relation.[1] Subsequent field-theoretical derivations displayed both the absorptive as well as the dispersive effects of vacuum polarization in a unified fashion.[2-5] Nevertheless, an asymmetry persisted in the sense that although practically useful results for pair creation became available—and were widely utilized for pulsar physics—the only explicit representations of the index of refraction were based on empirical approximations.[6] Recently the index of refraction due to vacuum polarization and the corresponding absorption coefficient for pair creation in homogeneous magnetic fields were obtained directly from the photon mass operator in terms of two-parameter integral representations.[7] These were reduced still further in two particular cases: the index of refraction for low-frequency photons in weak fields, and the photon absorption coefficient for high-energy photons in weak fields.[7] The purpose of the present paper is to extend the range of these results for the index of refraction. Starting from the exact mass operator expressions given in paper I, we will demonstrate that the index of refraction for high-energy photons in weak fields can be expressed in terms of a single-parameter integral form analogous to the absorption coefficient. The index of refraction for very-low-frequency photons in fields of arbitary intensity can be integrated explicitly. This constitutes one of the few vacuum polarization effects known in closed form.

II. HIGH-ENERGY PHOTONS; WEAK MAGNETIC FIELDS

We shall suppose that the orientations of the magnetic field $\vec{H}$, the photon propagation vector $\vec{k}$, and the polarization vector of the photon $\vec{\epsilon}$ conform to the conventions adopted in paper I: Specifically, $\vec{H}$ coincides with the $+z$ direction and $\vec{k}$ lies in the x-z plane. The polarization vector of the photon can be resolved into parallel ($\vec{\epsilon}_\parallel$) and perpendicular ($\vec{\epsilon}_\perp$) components corresponding to the directions parallel and perpendicular to the plane containing $\vec{k}$ and $\vec{H}$. The actual photon propagation direction is specified by intoducing θ to denote the angle between $\vec{k}$ and $\vec{H}$. The index of refraction is essentially determined by the real part of the photon mass operator. In particular, for high-energy photons ($\omega/m \gg 1$) propagating in weak magnetic fields ($eH/m^2 \ll 1$), simplifications paralleling those carried out in Eqs. (I50–I55) lead to the representation

$$n_{\parallel,\perp} = 1 - \frac{\alpha}{4\pi}\left(\frac{eH}{m^2}\sin\theta\right)^2 I_{\parallel,\perp}, \tag{1}$$

where

$$I_{\parallel,\perp} = \frac{18}{\lambda^2}\int_0^\infty dy\, y \times \int_0^1 \frac{dv}{1-v^2}\left[\left(1-\frac{v^2}{3}\right)_\parallel, \left(\frac{1}{2}+\frac{v^2}{6}\right)_\perp\right]\cos\Theta. \tag{2}$$

The auxiliary variables are given by

$$\lambda = \frac{3}{2}\frac{\omega}{m}\frac{eH}{m^2}\sin\theta, \quad \xi = \frac{4}{\lambda}\frac{1}{1-v^2}, \quad \Theta = \frac{3}{2}\xi\left(y+\frac{y^3}{3}\right). \tag{3}$$

The y integration can be carried out explicitly in terms of generalized Airy functions[8]:

$$n_{\parallel,\perp} = 1 - \frac{\alpha}{4\pi}\left(\frac{eH}{m^2}\sin\theta\right)^2 2^{1/3}\left(\frac{3}{\lambda}\right)^{4/3} \times \int_0^1 dv\,(1-v^2)^{-1/3} \times \left[\left(1-\frac{v^2}{3}\right)_\parallel, \left(\frac{1}{2}+\frac{v^2}{6}\right)_\perp\right]\tilde{e}_0'[-(\tfrac{3}{2}\xi)^{2/3}]. \tag{4}$$

According to the conventions adopted by Nosova and Tumarkin,[8]

$$\bar{e}_0(t) = \int_0^\infty dx \sin\left(tx - \frac{x^3}{3}\right), \tag{5}$$

and the primes denote derivatives with respect to the argument, i.e.,

$$\bar{e}_0'(t) = \frac{d}{dt}\bar{e}_0(t). \tag{6}$$

We note that the representation (4), in terms of a one-parameter integral over Bessel functions, is more compact and much simpler than the form obtained by using the dispersion relations,[1,4] and has a form analogous to that which occurs in calculations of synchrotron radiation and the absorption coefficient.[7] Since the properties of the $\bar{e}_\nu(t)$ functions are well known and extensive numerical tabulations are available,[8] all essential features of the index of refraction can easily be derived from Eq. (4). In particular, from the asymptotic forms

$$\bar{e}_0'(t) = \begin{cases} -t^{-2}, & t \to -\infty \quad (7) \\ \frac{3}{2}^{-1/3}\Gamma(\frac{2}{3}), & t \to 0, \quad (8) \end{cases}$$

we can compute the limiting cases corresponding to $\lambda \ll 1$,

$$n_{\parallel,\perp} = 1 + \frac{\alpha}{4\pi}\left(\frac{eH}{m^2}\sin\theta\right)^2 [(\tfrac{14}{45})_\parallel, (\tfrac{8}{45})_\perp], \tag{9}$$

and to $\lambda \gg 1$,

$$n_{\parallel,\perp} = 1 - \frac{\alpha}{4\pi}\left(\frac{eH}{m^2}\sin\theta\right)^2 \{\tfrac{9}{7}\pi^{1/2} 2^{1/3}\lambda^{-4/3}[\Gamma(\tfrac{2}{3})]^2 \times [\Gamma(\tfrac{1}{6})]^{-1}[(3)_\parallel, (2)_\perp]\}. \tag{10}$$

These asymptotic results furnish a check on the indirect prior computations.[6,9]

It is apparent from Eqs. (4)–(10) that the index of refraction essentially depends on the dimensionless energy-field product $\lambda = \frac{3}{2}[\hbar\omega/(mc^2)](H/H_{cr})$, where $H_{cr} = (m^2c^3)/(e\hbar)$, and therefore it is plausible that the auxiliary hypothesis $\omega/m \gg 1$ is actually superfluous. This is consistent with the fact that the small-λ limit, Eq. (9), is identical to the low-frequency weak-field result (I53), as well as the low-frequency arbitary-field expressions (37a) and (37b). We also note that both components of the index of refraction increase monotonically in the range $0 \leq \lambda \lesssim 1.2$, decrease monotonically for $1.2 \lesssim \lambda \lesssim 24$, and, in fact, decrease below unity at $\lambda \sim 24$. For $\lambda > 24$, the indices approach unity from below in accordance with Eq. (10). Detailed numerical information on the variation of $n_{\parallel,\perp}$ is given in Refs. 1, 6, and 10.

III. LOW-ENERGY PHOTONS; ARBITRARY MAGNETIC FIELDS

Next we consider the case of very-low-energy photons propagating in magnetic fields of arbitrary strength. The representation appropriate for this limit follows from (I45), (I47), and the constraint that, below the pair creation threshold ($\omega < 2m$), the mass operator $M_{\parallel,\perp}$ must be real. Specifically, we have

$$n_{\parallel,\perp} = 1 + \frac{\alpha}{4\pi}\sin^2\theta\, J_{\parallel,\perp}, \tag{11}$$

where

$$J_{\parallel,\perp} = -\int_0^\infty \frac{dz}{z} e^{-izm^2/eH}\int_0^1 dv\, N_{\parallel,\perp}(z,v), \tag{12}$$

and

$$N_\parallel(z,v) = -z\cot z\left(1 - v^2 + \frac{v\sin vz}{\sin z} + \frac{z\cos vz}{\sin z}\right), \tag{13}$$

$$N_\perp(z,v) = -\frac{z\cos vz}{\sin z} + \frac{vz\cot z\sin vz}{\sin z} + \frac{2z(\cos vz - \cos z)}{\sin^3 z}. \tag{14}$$

It is convenient to begin the evaluation of (12) with an integration by parts:

$$-\int_0^\infty dz\, e^{-im^2z/eH}\frac{v\sin vz\cot z}{\sin z} = -v^2 + \int_0^\infty dz \frac{e^{-im^2z/eH}}{\sin z}\left(-v^2\cos vz + i\frac{m^2}{eH}v\sin vz\right); \tag{15}$$

$$2\int_0^\infty dz \frac{e^{-im^2z/eH}}{\sin^3 z}(\cos vz - \cos z) = \tfrac{1}{2}(1 - v^2) + \int_0^\infty dz \frac{e^{-im^2z/eH}}{\sin z}\left(-v\sin vz\cot z + \cos vz - i\frac{m^2}{eH}\cot z(\cos vz - \cos z)\right); \tag{16}$$

and

$$\int_0^\infty dz\, e^{-im^2z/eH}\frac{\cot z}{\sin z}(\cos vz - \cos z) = \int_0^\infty dz\, e^{-im^2z/eH}\left(1 - \frac{v\sin vz}{\sin z} - i\frac{m^2}{eH}\frac{\cos vz - \cos z}{\sin z}\right). \tag{17}$$

The basic expressions $J_{\parallel,\perp}$ can then be rewritten as

$$J_{\parallel} = \tfrac{1}{3} - \mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3, \tag{18}$$

$$J_{\perp} = \tfrac{2}{3} - \mathcal{J}_2 + 6h^2 \mathcal{J}_1, \tag{19}$$

where we have introduced the dimensionless parameter

$$h = \frac{H_{cr}}{2H}, \tag{20}$$

and the auxiliary integrals are given by

$$\mathcal{J}_1 = -\tfrac{2}{3}\int_0^\infty dz\, e^{-2ihz}(\cot z - z^{-1}), \tag{21}$$

$$\mathcal{J}_2 = 2ih\int_0^\infty dz\, e^{-2ihz}\int_0^1 dv \frac{v \sin vz}{\sin z}, \tag{22}$$

and

$$\mathcal{J}_3 = \int_0^\infty dz\, e^{-2ihz}\int_0^1 dv(1-v^2)\left(\frac{\cos vz}{\sin z} - z^{-1}\right). \tag{23}$$

These quadratures can easily be carried out by rotating the path of integration to the negative imaginary axis ($z \to -ix$); in virtue of the structure of $M_{\parallel,\perp}$ the residues at the poles of $\sin z$ do not obstruct this deformation. Finally, by shifting the variables to

$$t = 2x, \quad u = \tfrac{1}{2}(1+v), \tag{24}$$

the remaining integrals can be recast into the standard forms of various Γ functions. The results are

$$\mathcal{J}_1 = \tfrac{2}{3}[\psi(1+h) - \ln h - (2h)^{-1}], \tag{25}$$

$$\mathcal{J}_2 = 2hK_1(h), \tag{26}$$

$$\mathcal{J}_3 = \tfrac{2}{3}\ln h - 4K_2(h), \tag{27}$$

where

$$K_1(h) = \int_0^1 du(2u-1)\psi(u+h)$$
$$= 2\ln\Gamma(1+h) - (1+2h)\ln h - \ln(2\pi) + 2h, \tag{28}$$

and

$$K_2(h) = \int_0^1 du\, u(1-u)\psi(u+h)$$
$$= 2\ln\Gamma_1(1+h) - 2L_1 - h(1+h)\ln h + \frac{h^2}{2}. \tag{29}$$

As usual, ψ denotes the logarithmic derivative of the Γ function,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x). \tag{30}$$

The generalized Γ function $\Gamma_1(x)$ appears by virtue of the integral[11]

$$\ln\Gamma_1(x) = \int_0^x dt\ln\Gamma(t) + \frac{x}{2}(x-1) - \frac{x}{2}\ln(2\pi). \tag{31a}$$

It satisfies the functional equation

$$\Gamma_1(1+x) = x^x\Gamma_1(x), \tag{31b}$$

with the constraints $\Gamma_1(0) = \Gamma_1(1) = \Gamma_1(2) = 1$, and for integer values of the argument reduces to

$$\Gamma_1(1+n) = 1^1 \times 2^2 \times \cdots \times n^n, \quad n \geq 0. \tag{31c}$$

The constant L_1 which appears in Eq. (29) can be obtained from the Raabe integral

$$L_1 = \tfrac{1}{3} + \int_0^1 dx \ln\Gamma_1(1+x)$$
$$\cong 0.248\,754\,477. \tag{32}$$

Finally, assembling the results given in Eqs. (25)–(32), we obtain

$$n_{\parallel}(H,\theta) = 1 + \frac{\alpha}{4\pi}\sin^2\theta\,\eta_{\parallel}(h), \tag{33a}$$

where $h = m^2/(2eH)$, and

$$\eta_{\parallel}(h) = 8\ln\Gamma_1(1+h) - 4h\ln\Gamma(1+h) - \tfrac{2}{3}\psi(1+h)$$
$$- 2h\ln h - 2h^2 + 2h\ln(2\pi) + (3h)^{-1} + [\tfrac{1}{3} - 8L_1]; \tag{33b}$$

furthermore,

$$n_{\perp}(H,\theta) = 1 + \frac{\alpha}{4\pi}\sin^2\theta\,\eta_{\perp}(h), \tag{34a}$$

where

$$\eta_{\perp}(h) = -4h\ln\Gamma(1+h) + 4h^2\psi(1+h) + 2h\ln h$$
$$+ 2h[\ln(2\pi) - 1] - 4h^2 + \tfrac{2}{3}. \tag{34b}$$

The asymptotic representations corresponding to $h \gg 1$ are

$$K_1(h) \cong \frac{1}{6h} - \frac{1}{180h^3} + \frac{1}{630h^5}, \tag{35a}$$

$$K_2(h) \cong \frac{\ln h}{6} + \frac{1}{360h^2} - \frac{1}{2520h^4}, \tag{35b}$$

$$\psi(1+h) \cong \ln h + \frac{1}{2h} - \frac{1}{12h^2} + \frac{1}{120h^4}. \tag{35c}$$

In case $h \ll 1$, we have the series expansions

$$K_1(h) = -(1+2h)\ln h - \ln(2\pi) + 2h(1-\gamma)$$
$$+ \frac{h^2}{2}\left(1 + \frac{\pi^2}{6}\right) - \frac{h^3}{3}[1 + \zeta(3)] + \cdots, \tag{36a}$$

$$K_2(h) = -h(1+h)\ln h - 2L_1 + h[1 - \ln(2\pi)]$$
$$+ h^2(\tfrac{3}{2} - \gamma) + h^3\frac{\pi^2}{18} + \cdots, \tag{36b}$$

$$\psi(1+h) = -\gamma + \frac{\pi^2}{6}h - \zeta(3)h^2 + \cdots, \tag{36c}$$

where γ is Euler's constant, ζ denotes the Riemann ζ function, and L_1 is given in (32). it is then a straightforward matter to compute the limiting forms of the indices of refraction corresponding to $H \ll H_{cr}$,

$$n_\parallel(H,\theta) \cong 1 + \frac{\alpha}{4\pi}\sin^2\theta\left[\frac{14}{45}\left(\frac{H}{H_{cr}}\right)^2 - \frac{13}{315}\left(\frac{H}{H_{cr}}\right)^4\right], \tag{37a}$$

$$n_\perp(H,\theta) \cong 1 + \frac{\alpha}{4\pi}\sin^2\theta\left[\frac{8}{45}\left(\frac{H}{H_{cr}}\right)^2 - \frac{379}{5040}\left(\frac{H}{H_{cr}}\right)^4\right], \tag{37b}$$

as well as those corresponding to $H \gg H_{cr}$,

$$n_\parallel(H,\theta) \cong 1 + \frac{\alpha}{4\pi}\sin^2\theta\left[\frac{2}{3}\left(\frac{H}{H_{cr}}\right) + \left(\frac{1}{3}+\frac{2\gamma}{3} - 8L_1\right) + \frac{H_{cr}}{H}\ln\left(\frac{2H}{H_{cr}}\right) + \frac{H_{cr}}{H}\left(2 - \ln(2\pi) - \frac{\pi^2}{18}\right) + \left(\frac{H_{cr}}{H}\right)^2\left(\frac{1}{2}+\frac{\zeta(3)}{6}\right)\right], \tag{38a}$$

$$n_\perp(H,\theta) \cong 1 + \frac{\alpha}{4\pi}\sin^2\theta\left[\frac{2}{3} - \frac{H_{cr}}{H}\ln\left(\frac{2H}{H_{cr}}\right) + \frac{H_{cr}}{H}[\ln(2\pi) - 1] - \left(\frac{H_{cr}}{H}\right)^2 + \left(\frac{H_{cr}}{2H}\right)^3\left(\frac{\pi^2}{2} - 1\right)\right]. \tag{38b}$$

The detailed numerical variation of the indices in the range $0.1 \leq H/H_{cr} \leq 10$ is displayed in Table I. Outside this interval the various limiting forms given in (37a)–(38b) are accurate to at least 0.05%. Finally, we note the technical point that the field ratios should actually be written as $|H/H_{cr}|$ since the mass operator essentially depends only on the combination $+(F_{\mu\nu}F^{\mu\nu})^{1/2}$ (see Ref. 12).

IV. DISCUSSION

The physical optics of the polarized vacuum is essentially determined by the indices of refraction $n_{\parallel,\perp}(H,\theta,\omega)$, and the attenuation coefficients $\alpha_{\parallel,\perp}(H,\theta,\omega)$, corresponding to pair production.[13] Although these functions have a complicated analytic structure in H, they do satisfy dispersion relations in the ω plane.[1] In the following, we will discuss some physical implications deduced from the dispersion relations and the results obtained here and in paper I (see Ref. 7).

(i) Suppose we consider

$$n_{\parallel,\perp}(\omega) - 1 = \frac{c}{\pi}\int_0^\infty d\overline{\omega}\,\frac{\alpha_{\parallel,\perp}(\overline{\omega})}{\overline{\omega}^2 - \omega^2}, \tag{39}$$

and specialize to the limit $\omega \to 0$. Furthermore, if we introduce the superscripts w and A to distinguish "weak" ($H \ll H_{cr}$) and "arbitrary" field intensities, then the direction of the inequality $n_\parallel^{(w)}(0) > n_\perp^{(w)}(0)$ [compare Eq. (9)] is consistent with $\alpha_\parallel^{(w)}(\omega) > \alpha_\perp^{(w)}(\omega)$ [compare Eqs. (I59a) and (I59b)]; the symmetry in the pair production rates is directly linked with the preferential overlap of the $e^\pm$ wave functions in the plane perpendicular to $\vec{k}$ and $\vec{H}$.[14] Since $n_\parallel^{(A)}(0) > n_\perp^{(A)}(0)$ [compare Eqs. (33a)–(38b)] it is plausible that similar constraints prevail for $\alpha_\parallel^{(A)}(\omega)$ and $\alpha_\perp^{(A)}(\omega)$. Furthermore, the "formal" inequalities $n_{\parallel,\perp}^{(w)}(0) > n_{\parallel,\perp}^{(A)}(0)$ imply that for some range of ω the weak-field attenuation coefficient $\alpha_{\parallel,\perp}^{(w)}$ must actually exceed the arbitrary-field attenuation $\alpha_{\parallel,\perp}^{(A)}$.

(ii) It can easily be verified that the magnetic "oscillator strength" given by the sum rule $\int_0^\infty d\overline{\omega}\,\alpha^{(w)}(\overline{\omega})$ diverges. However, the next moment exists and may be used to assign an effective frequency to the field: We find

$$\langle\hbar\omega\rangle_{\text{eff}} = \int_0^\infty \frac{d\overline{\omega}}{\overline{\omega}}\,\alpha^{(w)}(\overline{\omega}) = \left(\frac{13\,\alpha}{4\sqrt{3}}\,\frac{H}{H_{cr}}\right)mc^2, \tag{40}$$

where $\alpha^{(w)}$ is given by (I60).

(iii) The dispersion relation conjugate to Eq. (39) is

$$\alpha_{\parallel,\perp}(\omega) = -\frac{4\omega^2}{\pi c}\int_0^\infty d\overline{\omega}\,\frac{n_{\parallel,\perp}(\overline{\omega}) - 1}{\overline{\omega}^2 - \omega^2}. \tag{41a}$$

This is useful since it follows directly from Eqs. (4) and (10) that $n_{\parallel,\perp}^{(w)} - 1$ satisfies the "superconvergence" criteria

$$|n_{\parallel,\perp}^{(w)}(\omega) - 1| \leq O((\omega\ln^{1+\nu}\omega)^{-1}), \quad \nu > 0;$$

$$|n_{\parallel,\perp}^{(w)}(\omega_1) - n_{\parallel,\perp}^{(w)}(\omega_2)| \leq O\left((\omega_2\ln^{1+\nu}\omega_2)^{-1}\left|\frac{\omega_1-\omega_2}{\omega_2}\right|^\beta\right), \quad \omega_1 > \omega_2,$$

with Hölder index $\beta > 0$. Under these circumstances the asymptotic evaluation of the Hilbert transform (41a) leads to the "inertial" sum rule[15]

$$\int_0^\infty d\overline{\omega}[n_{\parallel,\perp}^{(w)}(\overline{\omega}) - 1] = \frac{\pi c}{4}\lim_{\omega\to\infty}[\alpha_{\parallel,\perp}^{(w)}(\omega)] = 0. \tag{41b}$$

(iv) Since $n_{\parallel,\perp}^{(w)}(0) - 1 > 0$, it follows from (41b) that there must be some range of frequencies where the phase velocity exceeds c. This is, of course, consistent with the explicit results given in Eqs. (4) and (10). If the inertial sum rule (41b) retains its validity for arbitrary field strengths, then Eq. (38a) makes it plausible that $n_\parallel^{(A)}(\omega)$ could become negative. This is subject to the caution that when $H/H_{cr} \gtrsim \pi/\alpha \sim 430$, the mass operator itself is liable to significant $O(\alpha^2)$ radiative corrections.[5]

TABLE I. Numerical values for the indices of refraction: $n_{\parallel,\perp}(H,\theta)=1+(\alpha/4\pi)\sin^2\theta\,\eta_{\parallel,\perp}(h)$; $h=(H_{cr}/2H)$.

H/H_{cr}	h	$\eta_{\parallel}(h)$	$\eta_{\perp}(h)$
50	0.01	32.146	0.591
10	0.05	5.663	0.441
8.33	0.06	4.585	0.416
7.14	0.07	3.821	0.393
6.25	0.08	3.254	0.373
5.56	0.09	2.817	0.354
5.00	0.1	2.471	0.337
2.50	0.2	0.983	0.220
1.67	0.3	0.541	0.156
1.25	0.4	0.345	0.116
1.00	0.5	0.239	0.0894
0.833	0.6	0.175	0.0708
0.714	0.7	0.134	0.0573
0.625	0.8	0.106	0.0473
0.555	0.9	0.0858	0.0395
0.500	1.0	0.0705	0.0335
0.454	1.1	0.0591	0.0288
0.417	1.2	0.0502	0.0249
0.385	1.3	0.0432	0.0218
0.357	1.4	0.0375	0.0192
0.333	1.5	0.0329	0.0170
0.312	1.6	0.0291	0.0152
0.294	1.7	0.0258	0.0136
0.278	1.8	0.0231	0.0123
0.263	1.9	0.0208	0.0111
0.250	2.0	0.0189	0.0101
0.238	2.1	0.0171	0.00926
0.227	2.2	0.0157	0.00850
0.217	2.3	0.0148	0.00782
0.208	2.4	0.0132	0.00722
0.200	2.5	0.0122	0.00669
0.192	2.6	0.0113	0.00621
0.185	2.7	0.0104	0.00578
0.179	2.8	9.80×10^{-3}	5.39×10^{-3}
0.172	2.9	9.09×10^{-3}	5.04×10^{-3}
0.167	3.0	8.52×10^{-3}	4.72×10^{-3}
0.143	3.5	6.28×10^{-3}	3.50×10^{-3}
0.125	4.0	4.82×10^{-3}	2.70×10^{-3}
0.111	4.5	3.81×10^{-3}	2.15×10^{-3}
0.100	5.0	3.09×10^{-3}	1.75×10^{-3}

(v) The advent of pulsars has stimulated far-ranging speculations on observational consequences of magneto-optic effects in fields of the order of $H\sim H_{cr}\sim10^{13}$ G. In this respect Eqs. (33a)–(38b) are grist for the mill since they are among the few quantum electrodynamics results which actually ought to be valid for field intensities up to $H\leqslant10^{15}$ G. One effect which could be significant is transverse double refraction[16] (Cotton-Mouton effect): Specifically, if we consider light of wavelength λ traversing a path length l normal to a field H, then for 45° initial polarization, the angular rotation of the plane of polarization is given by

$$\theta\simeq2\pi(n_{\parallel}-n_{\perp})l/\lambda$$

$$\sim\frac{2\alpha}{3}\frac{l}{\lambda}\left|\frac{H}{H_{cr}}\right|,\quad 1\lesssim\left|\frac{H}{H_{cr}}\right|\lesssim430. \tag{42}$$

Another obvious possibility concerns variable time delays: From (38a) we infer that variations in the product (field intensity)$\times$(path length) of the order of 3×10^{14} G m can give rise to time delays of a nanosecond.

(vi) Finally, we note that high-energy electrons traversing intense fields ($H>H_{cr}$) can generate a hybrid Čerenkov bremsstrahlung. Earlier estimates of the threshold for this effect can now be revised on the basis of Eq. (38a). We find

$$\frac{E}{mc^2}\left(\frac{H}{H_{cr}}\right)^{1/2}\geqslant36, \tag{43}$$

which is less stringent, and more reliable, than Eq. (5.6a) of Ref. 10. A complete discussion of this hybrid radiation phenomenon will be given elsewhere.

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