

## Quark confinement, rising trajectories, and asymptotic behavior of form factors\*

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Several long-range force models of quark confinement are considered, with particular emphasis on gluon exchange analogous to linear and harmonic potentials. First, the nonrelativistic case is discussed and the connection between the rise of the Regge trajectories and the power of the potential is derived. Then semirelativistic and relativistic equations are considered. It is shown that in each case the rising trajectories result from large quark-gluon coupling constants. It is also shown that an asymptotic power decrease of bound-state form factors follows only if the interaction contains an additional Coulomb- or Yukawa-type part, and only in the spacelike or the timelike domain. The significance of infrared cutoffs is examined, violations of unitarity are pointed out, and the behavior in the static limit is discussed.

### I. INTRODUCTION

Considerable experimental evidence has accumulated in recent years in support of the idea that hadrons are composite. Also the most popular model of composite hadrons, the quark model, has been successful in explaining several high-energy phenomena. Yet all attempts to isolate and observe the quarks have failed to date. Consequently Johnson's suggestion<sup>1</sup> that the elementary constituents possibly never appear singly and instead are confined permanently in bounded regions of space has aroused widespread interest. The reason is that if quarks are somehow confined, then an understanding of their physics necessitates an understanding of the implications of the confining mechanism for observable quantities.

Numerous approaches to the problem of [permanent or temporary (high threshold)] quark confinement have been developed recently. All of these approaches assume that quarks may be classified in triplets carrying SU(3) color, that hadrons are color singlets, and that the sea of quark-antiquark pairs carries no quantum numbers. Many models assume that the quarks are effectively light, others that they are heavy, and that their relative motion is practically nonrelativistic. Roughly speaking, there are three types of models: (a)<sup>2</sup> those based on the Nambu mechanism<sup>3</sup> in which the unbinding of color nonsinglets is due to the repulsive nature of the quark interaction mediated by gauge vector bosons, whereas the binding of the color singlets is due to a scalar field<sup>4</sup> (and is unaffected by the gauge vector bosons); (b)<sup>5</sup> those employing classical field theory supplemented by suitable boundary conditions such that the quarks are confined to finite regions of space called bags; and (c)<sup>6-13</sup> those employing a long-range force, such as is provided by the classical simple harmonic oscillator or similar potentials in the static

limit. [Note that type (c) models can be related to those of type (a) if the energy needed to remove a quark from a color singlet is proportional to a power of its distance of separation.]

In the following we are concerned with models of type (c). The aim of our investigation is to understand—in the context of various models—the compatibility between those phenomenological features which (at the present stage) a dynamical model of elementary particles is expected to exhibit. These are particles classified as in the quark model, rising Regge trajectories (equivalently a fundamental length related to their slope), the nonobservability of free quarks, power-law falloff of form factors, and (presumably) scaling in the scaling limit. Our investigation is mainly concerned with confinement of quarks, rising Regge trajectories, and the asymptotic behavior of form factors. Incorporating unitary spin<sup>14</sup> and color is assumed to be a technical problem which does not destroy the general features of the models, but is convenient to be ignored here in order to allow the equations to be solved with relative ease. Also scaling will not be considered because it is related to the built-in compositeness of hadrons and so follows along the lines investigated by Drell and Lee<sup>15</sup> (in the context of Bethe-Salpeter-type bound-state models).

It is generally believed that if physical (i.e., dressed) quarks exist, then they would have a mass of several GeV because otherwise they would (presumably) have been seen. But if they are heavy, their binding has to be appropriately strong in order to yield masses typical of hadrons. If, in addition, this force is assumed to be not too singular and to increase with increasing separation (thus making it difficult for a single quark to escape) their relative motion is essentially nonrelativistic.<sup>16</sup> The interaction is also expected to be infrared divergent. However, if the confined

quarks are not extremely massive, which is an open possibility, the nonrelativistic model can no longer be trusted. It is therefore not *a priori* clear that a nonrelativistic or semirelativistic approximation to the equation governing their relative motion is to be preferred. For this reason we shall consider both relativistic and semirelativistic models.

It is common knowledge that the nonrelativistic harmonic oscillator leads to rising trajectories. An oscillator type of interaction also implies quark confinement because it is infinitely attractive and its spectrum of discrete eigenvalues is complete. However, as a model it is obviously too naive. Numerous theories which have been discussed recently suggest quark-quark interactions equivalent to a linear potential,<sup>7, 10-12</sup> which may be associated with a dipole gluon propagator. In order to obtain a better understanding of the type of interaction required we consider in the following various classes of power potentials, with particular emphasis on the relativistic generalization of the harmonic oscillator because of its mathematical tractability.

Few models of quark confinement have (so far) been investigated with regard to the asymptotic behavior of form factors. In view of the considerable experimental evidence favoring a power decrease, it seems essential that this behavior should also be exhibited by realistic models. In the following we concentrate particularly on the compatibility between the rising nature of Regge trajectories and the asymptotic power decrease of the form factors. We show that the latter requires an additional Coulomb- or Yukawa-type interaction and even then leads to violations of unitarity in either the spacelike or the timelike region. Of course, writing the quark-quark interaction as the sum of two terms, one representing the (mysterious) gluon exchange, the other a customary Yukawa force, is an arbitrary and so unsatisfactory procedure. Ideally one would prefer a single interaction which exists in two phases<sup>17</sup> and which can then be approximated by these two terms in certain limits. In the following we find that the rising trajectories are invariably related to large values of the quark-gluon coupling constant (which therefore cannot have a small value around zero), whereas the Yukawa force may be treated as a perturbation. If the latter (or equivalently the anomalous dimension of a renormalizable interaction) is allowed to vanish, the asymptotic power decrease of the form factor is destroyed. These observations strongly support a two-phase model such as that discussed by Wilson<sup>17</sup> in which the two phases correspond to strong coupling and weak coupling with a critical region in

between. The similarity of such models with magnetostatics and the BCS theory of superconductivity is particularly striking. In magnetostatics no free magnetic poles exist; the force binding them into pairs (corresponding to gluon exchange) is the internal force of the magnets. In the BCS theory of superconductivity the superconducting phase is due to a dominance of the quasiparticle-phonon interaction (phonon exchange corresponding to gluon exchange) over the screened Coulomb repulsion to give a net attraction for quasiparticles near the Fermi surface. The reason why the BCS theory works so well is that in real metals pair-pair correlations are almost entirely due to Pauli principle restrictions, rather than true dynamical interactions between pairs. Consequently the system can in lowest order be treated as if dynamical interactions exist only between the mates of a pair.

This article is organized as follows. In Sec. II we consider briefly the nonrelativistic case of infinitely attractive potentials and derive in particular the dependence of the rise of the trajectories on the power of the potential. This case demonstrates explicitly the connection between the power of the potential and the resulting mass spectrum. It also serves a better understanding of the physics underlying the relativistic cases discussed later in the so-called static limit. In Sec. III we consider semirelativistic and relativistic models in the limit of infinite target mass. Here the equations are of the type used by Feynman *et al.*,<sup>18</sup> Montvay,<sup>19</sup> and Rivers.<sup>8</sup> We derive the Regge trajectories and show that the Coulomb and Yukawa interaction leads to a distortion of the lower part of the rising trajectories obtained for the harmonic gluon interaction. We then investigate the asymptotic behavior of the form factor of the ground state of the two-body bound state. We show that only if the Coulomb or Yukawa coupling is nonzero will the form factor exhibit an asymptotic power decrease in the spacelike region, but not in the timelike region where it diverges exponentially, thus violating unitarity. This result is then verified by an examination of the momentum space equation. In Sec. IV we consider fully relativistic models and arrive at roughly similar though not identical results for the relativistic harmonic interaction. We also consider various types of interaction kernels and derive their counterparts in four-dimensional Euclidean configuration space and in the static limit. Finally, in Sec. V, we summarize our conclusions.

## II. THE NONRELATIVISTIC ANALOG

It is often instructive in particle physics to keep in mind nonrelativistic analogs. For this reason

we shall first consider briefly ordinary spinless two-body nonrelativistic potential scattering. In this case the motion of the center of mass may be separated off and for spherically symmetric interactions  $V(r)$  the relative motion of the two particles is described by the radial Schrödinger equation. It is convenient for our purposes to consider this equation in the normal form

$$\frac{d^2\psi}{dr^2} + \left[ k^2 - \frac{l(l+1)}{r^2} - V(r) \right] \psi = 0, \quad (2.1)$$

where, as usual,

$$\Psi = \frac{1}{r} \psi(r) P_l^m(\cos\theta) e^{im\phi}$$

and  $\bar{n} = c = 1$ , reduced mass  $M = \frac{1}{2}$ . The type of potential we wish to consider first is

$$V(r) = g^2 |r - r_0|^s, \quad 1 \leq s \quad (2.2a)$$

or more specifically the cases

$$V_1(r) = g^2 |r - r_0|, \quad (2.2b)$$

$$V_2(r) = g^2 |r - r_0|^2. \quad (2.2c)$$

We consider  $r_0$  as being a separation of the quarks such that for  $r \ll r_0$  they can—to a reasonable approximation—be considered as moving freely, i.e., independently of each other [as far as interactions of type (2.2a) are concerned]. Thus, for  $r \ll r_0$

$$V(r) \simeq g^2 |r - r_0|^s \left[ 1 - \frac{r}{r_0} \right],$$

and the force acting between the quarks is roughly constant. For quark separations  $r$  of the order of  $r_0$  the potential is approximately zero. For  $s > 1$  the force is also approximately zero for  $r$  of the order of  $r_0$ , whereas for  $s = 1$  the force is constant as a function of distance ( $r > r_0$ ). For quark separations  $r \gg r_0$  we have

$$V(r) \simeq g^2 r^s.$$

For  $s > 1$  the force acting between the quarks then increases with the separation, whereas for  $s = 1$  it stays constant independent of their separation. Thus for  $s = 1$  the quarks in the two-quark system are necessarily confined permanently (independent of the magnitude of the coupling  $g^2$ ). Permanent quark confinement (for finite coupling  $g^2$ ) therefore requires a potential having  $s \geq 1$ . A linear potential of the type (2.2b) has several phenomenologically appealing consequences as we shall show. This applies also to its relativistic analog, which we discuss later on. We might add here that a linear total energy and so potential energy<sup>17</sup> may be regarded as a natural consequence of locality for isolated quarks of infinite mass as was argued by Wilson<sup>7</sup> (see also Kogut<sup>20</sup>). Briefly, the argu-

ment is that in a theory with states having infinite energy, the Hamiltonian  $H$  is not a well-defined operator. Using instead  $e^{-Ht}$ , the expectation value for a quark-antiquark state with large separation  $r$  may be written

$$\langle q\bar{q} | e^{-Ht} | q\bar{q} \rangle \simeq \langle q | e^{-Ht} | q \rangle \langle \bar{q} | e^{-Ht} | \bar{q} \rangle + O(e^{-b(t)r}).$$

For mass  $m$  of the quarks we then have

$$e^{-E(r)t} = e^{-2mt} + O(e^{-b(t)r}).$$

If  $m \rightarrow \infty$  this relation yields for the energy  $E(r)$ , consisting of kinetic and potential energy, the relation

$$E(r)t = b(t)r, \text{ i.e., } E \propto r.$$

Our next objective is to calculate the Regge trajectories for potentials of the type (2.2a). We shall use a simplified WKB method and set

$$\begin{aligned} z(r) &= \int_0^r [k^2 - V(r)]^{1/2} dr \\ &= [k^2 - V(r)]^{1/2} r + \frac{1}{2} \int_0^r \frac{r(dV/dr)dr}{[k^2 - V(r)]^{1/2}} \\ &= [k^2 - V(r)]^{1/2} r + O\left(\frac{1}{[k^2 - V(r)]^{1/2}}\right). \end{aligned} \quad (2.3)$$

The radial wave equation may then be written

$$\frac{d^2\psi}{dz^2} + \left[ 1 - \frac{l(l+1)}{z^2} \right] \psi = O\left(\frac{1}{[k^2 - V(r)]^{1/2}}\right). \quad (2.4)$$

For sufficiently large  $|k^2 - V(r)|$  the right-hand side of this equation may be neglected to a first approximation. The solutions of the zero-order equation

$$\frac{d^2\psi^{(0)}}{dz^2} + \left[ 1 - \frac{l(l+1)}{z^2} \right] \psi^{(0)} = 0 \quad (2.5)$$

are  $\psi^{(0)}(z) = \sqrt{z} Z_{l+1/2}(z)$ , where  $Z_\nu$  is a Bessel function. A solution satisfying the boundary condition of regularity at  $r = 0$  is given by  $Z_\nu = J_\nu$ . For  $l = 0$  the appropriate solution of (2.5) is  $\text{sinz}$ . Now, the eigenvalues of the problem—and thus the Regge trajectories—are determined largely by the oscillatory behavior of the solution  $\psi$  and so by its behavior in the region where  $k^2 - V(r)$  is positive, as will be seen below.

Since  $\text{sinz}$  has zeros at  $z = n\pi$ ,  $n = 0, 1, \dots$ , the number of zeros in the interval  $(0, R)$ , where  $R$  is given by  $k^2 - V(R) = 0$ , is the integral part of  $z(R)/\pi$ . In the case  $l \neq 0$  the zeros of  $\psi^{(0)}$  are easiest to determine in the region of large  $|z|$ , which implies large values of  $n$ . The solution  $\psi(r)$  satisfying the boundary condition  $\psi(0) = 0$  is obtained for

$$\psi^{(0)}(z) = \sqrt{z} J_{l+1/2}(z).$$

For large  $|z|$  the Bessel function has the asymp-

otic behavior

$$J_{l+1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left[ \cos\left(z - \frac{1}{2}l\pi - \frac{1}{2}\pi\right) + O\left(\frac{1}{|z|}\right) \right],$$

which is periodic for real values of  $z$ . Clearly, the number of zeros in an interval  $(0, R)$  and hence the eigenvalues are given by

$$z - \frac{1}{2}l\pi - \frac{1}{2}\pi = (2n+1)\frac{\pi}{2},$$

i.e.,  $z = (n + \frac{1}{2}l + 1)\pi$ , where  $n = 0, 1, 2, \dots$ . Using (2.3), we may rewrite this expression in the form

$$(n + \frac{1}{2}l + 1) \approx \frac{1}{\pi} \int_0^R \{k^2 - V(r)\}^{1/2} dr, \quad (2.6)$$

where correction terms are of  $O(0)$  in  $k^2$  or  $n$ . It is known that a more detailed analysis<sup>21</sup> taking into account terms of  $O(0)$  yields the corrected version

$$(n + \frac{1}{2}l + \frac{3}{4}) = \frac{1}{\pi} \int_0^R [k^2 - V(r)]^{1/2} dr + O\left(\frac{1}{n}\right). \quad (2.7)$$

We now insert in (2.7) the potential (2.2a). Then taking  $r_0 = 0$ , for simplicity, we have

$$\begin{aligned} (n + \frac{1}{2}l + \frac{3}{4}) &= \frac{1}{\pi} \frac{k^{1+2/s}}{sg^{2/s}} \int_0^1 (1-t)^{1/2} t^{1/s-1} dt \\ &\quad + O\left(\frac{1}{n}\right) \\ &= \frac{k^{1+2/s}}{sg^{2/s}\pi} \frac{\Gamma(\frac{3}{2})\Gamma(1/s)}{\Gamma(\frac{3}{2}+1/s)} + O\left(\frac{1}{n}\right). \end{aligned}$$

Thus

$$\begin{aligned} l &= \alpha_n(k) \\ &= -2n - \frac{3}{2} + \frac{\Gamma(\frac{1}{2})\Gamma(1/s)}{\Gamma(\frac{3}{2}+1/s)} \frac{k^{1+2/s}}{sg^{2/s}\pi} + O\left(\frac{1}{n}\right). \end{aligned} \quad (2.8)$$

For the linear potential (2.2b) this becomes

$$\alpha_n(k) = -2n - \frac{3}{2} + \frac{4}{3} \frac{k^3}{g^2\pi} + O\left(\frac{1}{n}\right) \quad (r_0 = 0), \quad (2.9)$$

and for the oscillator potential (2.2c)

$$\alpha_n(k) = -2n - \frac{3}{2} + \frac{k^2}{2g} + O\left(\frac{1}{n}\right) \quad (r_0 = 0). \quad (2.10)$$

In the latter case our result agrees in fact with the exact expression (which can be derived from the complete solution of the differential equation as in the next section).

We now interpret the results (2.8), (2.9), and (2.10) as being applicable over the entire range of

the energy  $k^2$ . From (2.8) we see that the linear potential yields the most rapidly rising trajectory (considering only integer power potentials  $r^s$ ,  $s > 0$ ). The oscillator potential (2.2c), of course, yields the linearly rising trajectory, as is well known. Regge trajectories for the harmonic potential and related versions have been discussed by several authors and so need not be considered in further detail here.<sup>22</sup> Its spectrum of infinitely many pure bound states for  $k^2 > 0$  is, of course, unphysical. In order to make it physically meaningful it is necessary to imagine the introduction of a deviation of the potential from the pure  $r^s$  behavior which introduces sufficient nonanalyticity so that the "bound states" become resonances (i.e., all except possibly the lowest for energies  $< 2M$ ), a continuous spectrum thereby being introduced. This problem is related to the infrared divergence and thus to the nonexistence of the Fourier transform of the power potential. It can presumably be cured by taking into account vacuum polarization effects which have a screening effect on the bare power potential. This aspect will be discussed in more detail later on.

Regge trajectories for Yukawa potentials have been discussed in considerable detail in the literature. It is well known that they do not rise linearly but fall off rapidly with increasing energy; this applies also in the strong-coupling limit.<sup>23</sup> However, as pointed out in the Introduction, a Yukawa-type force (i.e., particle exchange) seems to be responsible for the power-law falloff of form factors and so vertex functions. It is therefore of interest to understand the behavior of wave functions and Regge trajectories when the potential contains a harmonic as well as a Yukawa-type part. Unfortunately, however, the radial wave equation is difficult to solve except when one of the potentials may be neglected. We therefore do not pursue this case further except later in the context of relativistic models.

There are several other types of potentials which are of interest in this connection. One which has recently attracted some interest is a potential with a finite range singularity, i.e., a potential  $V_\lambda(r)$  which is singular at a point  $r = \lambda$ , where  $\lambda \neq 0$  or  $\infty$ . The example discussed by Filippov<sup>13</sup> is

$$V_\lambda(r) = -\frac{g^2}{r^2 - \lambda^2} \quad (2.11a)$$

(with the distribution or principal value prescription when the Fourier transform is calculated). Such a potential can be obtained by taking the static limit of the configuration space representation of the superpropagator of a nonpolynomial field theory. Thus, for  $r \neq \lambda$ ,

$$V_\lambda(r) = +\frac{g^2}{\lambda^2} \left[ 1 + \frac{r^2}{\lambda^2} + \left( \frac{r^2}{\lambda^2} \right)^2 + \dots \right] \theta(\lambda - r) \\ - \frac{g^2}{r^2} \left[ 1 + \frac{\lambda^2}{r^2} + \left( \frac{\lambda^2}{r^2} \right)^2 + \dots \right] \theta(r - \lambda). \quad (2.11b)$$

We observe that for  $r \ll \lambda$  the potential behaves like a modified harmonic oscillator. Its eigenvalues and Regge trajectories (which can be easily calculated perturbation theoretically to any desired order in  $1/g$ ) follow from the condition of continuity of the wave function at  $r = \lambda$  or, approximately, from (2.10):

$$\alpha_n(k) = -2n - \frac{3}{2} + \frac{k^2 \lambda^2 - g^2}{2g} + O(g^0). \quad (2.12)$$

The quark is now trapped inside the range determined by the finite distance singularity. We observe that for  $\lambda$  approaching zero the linear rise of the trajectory is lost, the potential thereby reducing to one of centrifugal type.

Another potential we might mention is

$$V_\mu(r) = +g^2 r^2 e^{-\mu^2 r^2} \\ = \frac{\partial}{\partial \mu^2} (-g^2 e^{-\mu^2 r^2}). \quad (2.13)$$

For large values of  $g^2$  and small values of  $\mu^2$  the Schrödinger eigenvalues for this potential are easily calculated by the methods used in Ref. 24. One finds an expression similar to (2.12), which, of course, reduces to (2.10) in the limit  $\mu \rightarrow 0$ . We return to a discussion of these potentials in the relativistic context later on.

### III. SEMIRELATIVISTIC AND RELATIVISTIC MODELS IN THE LIMIT OF INFINITE TARGET MASS

#### A. A semirelativistic model with rising trajectories

Guided by the nonrelativistic considerations of the preceding section we now consider a dynamical

$$\left[ -\frac{1}{4} P^2 - g^2 V^2 + \partial^2 - m^2 + g P \cdot V - i P \cdot \partial + 2i g V \cdot \partial - i g \gamma^\mu (\partial_\mu V_\nu) \gamma^\nu \right] \chi(\xi) = 0. \quad (3.4)$$

In general this equation is difficult to solve. We therefore make some plausible approximations which are analogous to those used in the discussion of the nonrelativistic case. In fact, we will assume that both  $|P_\mu|$  and  $g$  are considerably larger than 1. In this domain Eq. (3.4) may be written

$$\left[ -\frac{1}{4} P^2 - g^2 V^2 + \partial^2 - m^2 + g P \cdot V \right] \chi(\xi) = O(|P_\mu|, g), \quad (3.5)$$

where  $\chi$  is now proportional to the unit matrix; the left-hand side of (3.5) represents in effect the

model in which quarks interact via a neutral vector-gluon field  $V_\mu(x)$ . We assume that the quarks have a bare (unrenormalized) mass  $m$  and leave open (for the moment) the question as to whether the physical mass of a quark is meaningful or not, i.e., whether quarks do arise as asymptotic states (and so are only temporarily confined) or not. The equation for the spin- $\frac{1}{2}$  quark field  $q(x)$  may then be written<sup>25</sup> (using the metric  $g_{00} = -1$ ,  $g_{ii} = +1$  for  $i = 1, 2, 3$ )

$$(i\gamma^\mu \partial_\mu + m) q(x) = g\gamma^\mu V_\mu q(x), \quad (3.1)$$

where  $g$  is the coupling of the gluon to the quark. The equation, describing the motion of one quark in the limit of infinite mass of another quark is, of course, analogous to the equation describing the motion of a charged spin- $\frac{1}{2}$  particle in the field of a force. Like the Bethe-Salpeter amplitude, the wave function for a quark-antiquark pair of total momentum  $P_\mu$  is given by

$$\langle 0 | T(\bar{q}_\beta(x') q_\alpha(x)) | \phi_P \rangle = \chi_{\alpha\beta}(\xi) e^{-iP \cdot \eta}, \quad (3.2)$$

where  $\alpha$  and  $\beta$  are spinor indices and the expression on the right-hand side shows the separation of relative and center-of-mass coordinates  $\xi = x - x'$ ,  $\eta = \frac{1}{2}(x + x')$ . We assume now that the gluon field may be replaced by a  $c$ -number potential which depends only on the relative coordinate  $\xi_\mu$ . Then rewriting (3.1) in terms of  $\xi$  and  $\eta$  and using (3.2) we obtain (ignoring a  $\delta$  function)

$$\left[ +\frac{1}{2} P^\mu \gamma_\mu + i\gamma^\mu \partial_\mu - g\gamma^\mu V_\mu(\xi) + m \right] \chi(\xi) = 0 \quad (3.3)$$

as the equation describing the relative motion of the quark and antiquark. Multiplying the equation from the left by the differential operator with  $m$  replaced by  $-m$ , it becomes

equation for scalar instead of spinor quarks. It is this equation which we shall study in detail in the following.

In solving (3.5) we could proceed in two (effectively equivalent) ways. In a three-dimensional configuration of the hadrons we could consider  $V$  as a function of the three-dimensional radial distance  $r$  given by

$$r^2 = +\xi^2 - (P \cdot \xi)^2 / P^2$$

in the rest frame of the hadron (considered as the

bound state of a quark-antiquark pair). In this case the time dependence of  $\chi(\xi)$  is first separated from the configuration space dependence, and then the resulting three-dimensional equation is solved. Alternatively one may use the four-dimensional treatment familiar from its application to the Bethe-Salpeter equation. Here we shall use the former method because it is more appropriate for the treatment of the nonrelativistic potential which we shall use. Thus, we go to the rest frame of the hadron ( $\vec{P}=0$ ) and write

$$V_0 = -\alpha \frac{e^{-\mu r}}{r} + \beta r, \quad \vec{V} = 0. \quad (3.6)$$

Here the first part represents the (nonrelativistic) exchange of a spin-zero particle of mass  $\mu$ , which we will subsequently assume to be zero (although it is not difficult—in the context of the equation used below—to derive a perturbation expansion in rising powers of  $\mu$ ).<sup>26</sup> The second term again is the linear potential representing the gluon exchange between the quarks. Inserting this linear potential as the fourth component of the four-vector potential into the Dirac equation we are confronted with the problem of the Klein paradox for large distance. The crux of this difficulty is that as the potential and thus the energy become larger and larger, creation and annihilation processes (vacuum polarization effects) become increasingly important. These effects are, however, not included in our simple treatment of the Dirac equation. We shall indicate later that (much as in quantum electrodynamics) vacuum polarization effects screen the bare propagator potential, so that the inconsistency of the Klein paradox is removed when these effects are included.

Substituting (3.6) into (3.5) and separating off the angular part of the wave function we obtain for the radial part of  $\chi$ , i.e.,  $\psi(r)$ , the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} + \left[ -\frac{l(l+1)}{r^2} + (\mathcal{E} - gV_0)^2 - m^2 \right] \psi = 0, \quad (3.7)$$

within our approximation of ignoring terms of  $O(|P_\mu|, \mathcal{E})$  for quarks of spin  $\frac{1}{2}$ . Also, we have set  $\mathcal{E} = P_0/2$ . Substituting into (3.7) the potential (3.6), we have (for  $\mu=0$ )

$$\frac{d^2\psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} + \left[ Ar^2 + B + \frac{C}{r^2} \right] \psi = 0, \quad (3.8)$$

where

$$\begin{aligned} A &= g^2 \beta^2, \\ B &= \mathcal{E}^2 - m^2 - 2\alpha\beta g^2, \\ C &= g^2 \alpha^2 - l(l+1) \equiv -L(L+1). \end{aligned} \quad (3.9)$$

Also we have neglected on the left-hand side the

term

$$2\mathcal{E}g \left( \frac{\alpha}{r} - \beta r \right),$$

which contributes significantly only in the transition region, and so does not control the behavior of the solutions in the regions around  $r=0$  and  $r=\infty$  in which we are primarily interested. In principle this term could be dealt with perturbation theoretically (in a suitable range of  $\alpha$  and  $\beta$ ) like the Yukawa terms in  $\mu$ .

Our next step is to solve Eq. (3.8). Its general solution in any finite region of  $r$ ,  $\mathcal{E}$ , and  $g$  which satisfies the condition of regularity at the origin is found to be (apart from an over-all constant)

$$\psi(r) = z^L e^{-z^2/4} \Phi(a, b; \frac{1}{2}z^2), \quad (3.10)$$

where we have set

$$z = (2iA^{1/2})^{1/2} r \quad (3.11)$$

and

$$a = \frac{1}{2} \left( L + \frac{3}{2} \right) - \frac{B}{4iA^{1/2}}, \quad b = L + \frac{3}{2}. \quad (3.12)$$

The function  $\Phi(a, b; s)$  is the confluent hypergeometric function which is defined as any solution of the equation

$$s \frac{d^2\Phi}{ds^2} + (b-s) \frac{d\Phi}{ds} - a\Phi = 0.$$

In analogy to the familiar treatment of the hydrogen atom, the eigenvalues and thus the Regge trajectories are determined by those values of the energy for which

$$a = -n, \quad n = 0, 1, 2, \dots$$

i.e.,

$$L = -2n - \frac{3}{2} + \frac{B}{2iA^{1/2}}. \quad (3.13)$$

We observe that since  $B$  is proportional to the square of the total energy  $\mathcal{E}^2$ , the trajectory  $\alpha_n(\mathcal{E}) = l_n$ , which is obtained by solving Eq. (3.13) for  $l$ , also rises with the energy. It is clear from the relation between  $l$  and  $L$  that sufficiently moderate values of the coupling constant  $\alpha$  distort the Regge trajectories generated by the pure gluon (i.e.,  $\alpha=0$ ) interaction in an appropriately moderate way in the region of small  $l$ . Hence the Coulomb (or more generally Yukawa) part of the interaction does not destroy the rising behavior of the trajectories. However, from (3.13) we also deduce that the gluon coupling  $A$  cannot be zero; in fact it must be large enough in order not to represent simply a perturbation on the Coulomb potential, and so to destroy the linearly rising behavior of the trajectories.

We note here in passing that in the context of a

relativistic wave equation the Coulomb potential gives rise to anomalous dimensions. This can be seen by looking at the behavior of the wave function near the origin which is

$$\psi(r) \sim r^L = r^{-1/2 + [l(l+1) + 1/4 - g^2 \alpha^2]^{1/2}},$$

and is seen to depend on the coupling constant. These anomalous dimensions appear in general whenever the interaction has the same dimension as the kinetic energy term, i.e., for renormalizable interactions.

Of course, a self-consistent scheme requires in addition to the quark equation with gluon and meson sources, also gluon and meson equations with sources involving the quark current. However, since such a system of simultaneous equations is vastly more difficult to solve than our equation above, we will not consider these here. Still, for the discussion in later sections, it is useful to keep in mind that the linear potential (2.2b) satisfies the equation

$$\nabla^2 |\vec{r}| = \frac{2}{|\vec{r}|}.$$

Then, since

$$\nabla^2 \left( -\frac{1}{4\pi |\vec{r}|} \right) = \delta(\vec{r}),$$

we have

$$\nabla^2 \nabla^2 |\vec{r}| = -8\pi \delta(\vec{r}).$$

In three dimensions the equation of the free gluon  $\chi$  (corresponding to the linear potential) is thus

$$\nabla^2 \nabla^2 \chi = 0.$$

One may therefore speculate that in four dimensions the equation of the free gluon is given by ( $\square^2 \equiv \partial_\mu \partial^\mu$ )

$$\square^2 \square^2 \chi = 0.$$

The propagator of the free gluon is then

$$(1/k^2)^2;$$

however, its configuration space representation is not the four-dimensional  $|\vec{r}|$ , as will be seen later, although  $|\vec{r}|$  may be regarded as its static limit in the sense of the above equations (corresponding to the relationship between the spinless particle propagator and the Yukawa potential). In the case of the harmonic potential (2.2c) it is more difficult to calculate the propagator. We return to this problem in Sec. IV.

#### B. Another model with rising trajectories

We consider a second semirelativistic example. We ignore the spin of the quarks and assume a  $\phi^3$ -

like interaction (i.e., the scalar quarks of bare mass  $m$  interact through exchange of scalar particles of mass  $\mu$ ) in addition to their interaction with gluons, which we assume to have the form of a four-dimensional scalar harmonic potential in configuration space. Then the equation of motion of a quark is given by

$$(-\partial^2 + m^2)q(x) = v(x)q(x), \quad (3.14)$$

where  $v(x)$  represents its relativistic interaction with the scalars and gluons.

The wave function of the bound state of two scalar quarks of total four-momentum  $P_\mu$  is given by

$$\begin{aligned} \Psi(x_1, x_2) &= \langle 0 | T(q(x)q(x')) | \phi_P \rangle \\ &= \psi(\xi) e^{-iP \cdot \eta}, \end{aligned} \quad (3.15)$$

where the last expression shows the separation of relative and center-of-mass coordinates. Proceeding as before, i.e., writing

$$\left( \frac{\partial}{\partial x} \right)^2 = \left( \frac{\partial}{\partial \xi} + \frac{1}{2} \frac{\partial}{\partial \eta} \right)^2$$

and acting on  $\Psi(x_1, x_2)$ , we obtain the equation in  $\xi_\mu$ , i.e.,

$$(-\partial^2 + \frac{1}{4}P^2 + m^2 + iP \cdot \partial)\psi(\xi) = v(\xi)\psi(\xi), \quad (3.16)$$

assuming that  $v(x) = v(\xi)$ . A slightly different model equation for the bound-state wave function is obtained by writing it as a two-particle wave equation containing a coupling potential

$$\begin{aligned} [(-\partial_x^2 + m^2) + (-\partial_{x'}^2 + m^2)]\Psi(x_1, x_2) \\ = 2v(x_1 - x_2)\Psi(x_1, x_2). \end{aligned} \quad (3.17)$$

This equation is sometimes called the Goldstein equation and the bilocal field  $\Psi(x_1, x_2)$  the Goldstein field. The equation was used by Feynman *et al.*<sup>18</sup> and Rivers.<sup>8</sup> For the bound-state solution of the form (3.15) we then have the equation in the relative coordinates  $\xi_\mu$

$$(-2\partial^2 + \frac{1}{2}P^2 + 2m^2)\psi(\xi) = 2v(\xi)\psi(\xi), \quad (3.18)$$

which is identical with (3.16) except for the term  $iP \cdot \partial \psi$  in that equation. We shall ignore this term in the following, and so work effectively with (3.18).

We assume now, as mentioned earlier, that the interaction  $v(\xi)$  is the sum of a  $\phi^3$ -like scalar exchange and a harmonic-oscillator-like gluon interaction. Also, we work in terms of a four-dimensional Euclidean metric. Then

$$v(\xi) = g^2 V(\xi) + f^2 \xi^2, \quad (3.19a)$$

and in the Wick-rotated four-dimensional Euclidean space, with  $r = (\sum_{i=1}^4 \xi_i^2)^{1/2}$ ,  $\xi_4 = -i\xi_0$ ,

$$V(\xi) - V(r) = \frac{\mu K_1(\mu r)}{4\pi^2 r} \text{ for } \mu \neq 0$$

and

$$V(\xi) - V(r) = \frac{1}{4\pi^2 r^2} \text{ for } \mu = 0 .$$

In view of the  $O(4)$  symmetry of Eq. (3.18) we can expand the solutions  $\psi(r)$  in terms of four-dimensional spherical harmonics  $H_{Llm}$ , which we write

$$H_{Llm}(\psi, \theta, \phi) = A_{Ll} (\sin\psi)^l C_{L-l}^{l+1}(\cos\psi) Y_{lm}(\theta, \phi) , \quad (3.20)$$

where

$$|A_{Ll}|^2 = \frac{1}{\pi(L+l+1)!} 2^{2l+1} (L+1) (L-l)! (l!)^2 ,$$

$$|m| \leq l \leq L$$

so that

$$\int_0^\pi \sin^2\psi d\psi \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi |H|^2 = 1 .$$

Separating off the angular part of the wave function,<sup>27</sup> we are left with the equation for the four-dimensional radial wave function  $\psi_L(r)$ , i.e.,

$$\left[ \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} - \frac{L(L+2)}{r^2} + g^2 V(r) + f^2 r^2 - \frac{1}{4} P^2 - m^2 \right] \psi_L(r) = 0 , \quad (3.21)$$

where physically  $+if = |f|$ , i.e.,  $f$  is pure imaginary.

For simplicity we consider explicitly again only the case  $\mu=0$  in (3.19). Thus, setting

$$L'(L'+2) = L(L+2) - \frac{g^2}{4\pi^2} , \quad (3.22)$$

we may rewrite Eq. (3.21)

$$\left[ \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} - \frac{L'(L'+2)}{r^2} - \frac{1}{4} P^2 - m^2 + f^2 r^2 \right] \psi_L(r) = 0 . \quad (3.23)$$

The regular solution of this equation may be written down from our knowledge of the solution (3.10) of Eq. (3.8). Thus, taking into account the different coefficient of the first derivative, we have (apart from an over-all multiplicative constant)

$$\psi_L(z) = z^{L'} e^{-(1/4)z^2} \Phi(a, b; \frac{1}{2}z^2) , \quad (3.24)$$

where

$$z = (2if)^{1/2} r \quad (3.25)$$

and

$$a = \frac{1}{2}(L'+2) + \frac{m^2 + \frac{1}{4}P^2}{4if} , \quad (3.26)$$

$$b = L'+2 .$$

Of course, the Regge trajectories are again determined by the condition of normalizability of the bound-state wave function, i.e., by

$$a = -n, \quad n=0, 1, 2, \dots$$

or

$$L' = -2n - 2 - \frac{1}{2if}(m^2 + \frac{1}{4}P^2) , \quad (3.27)$$

for which the confluent hypergeometric function becomes a Laguerre polynomial. Again the Coulomb interaction or, equivalently, the anomalous dimension, leads to a moderate distortion of the Toller poles  $L = \alpha_n(P^2)$  and so of the Regge parent ( $m=0$ ) and daughter ( $m=1, 2, 3, \dots$ ) trajectories  $l = \alpha_{nm}(P^2) = L - m$  in the region of small  $l$ . And again the trajectories rise linearly with the square of the energy, i.e.,  $t = -P^2$ .

### C. Form factor for two-body bound states

Our next objective is to obtain the asymptotic behavior of the form factor shown in Fig. 1 of the two-body bound state at large momentum transfer. We consider only the spinless case and assume that only one particle is charged. In this case the charge form factor  $F_2(q^2)$  is the convolution integral of the ingoing and outgoing bound-state wave functions in momentum space,<sup>28</sup> i.e.,

$$F_2(q^2) = \int d^4k \tilde{\psi}^*(k) \tilde{\psi}(k-q) , \quad (3.28)$$

where

$$(k^2 + \frac{1}{4}P^2 + m^2) \tilde{\psi}(k) = \int \tilde{v}(k-k') \tilde{\psi}(k') d^4k' , \quad (3.29)$$

and

$$\tilde{v}(k) = \frac{1}{(2\pi)^4} \int v(\xi) e^{-ik\xi} d^4\xi$$

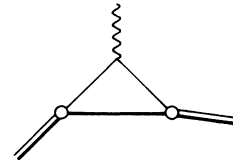
[and correspondingly for  $\tilde{\psi}(k)$ ]. We observe that

FIG. 1. Triangle diagram for the electromagnetic vertex. The heavy line corresponds to the infinite-mass limit for one constituent.



Eq. (3.28) is simply the Fourier transform of the product of the configuration-space representations of the respective wave functions, i.e.,

$$F_2(q^2) = \frac{1}{(2\pi)^4} \int \psi^*(\xi) \psi(\xi) e^{+i a \xi} d^4 \xi. \quad (3.30)$$

Since we know the solutions  $\psi(\xi)$  for the models described above, we can calculate the form factor. We have

$$\psi(\xi) = \sum_{Llm} H_{Llm}(\psi, \theta, \phi) \psi_L(r).$$

For simplicity we consider in the following only the ground state of the spinless composite particle. Thus  $n=0$  and  $L=l=m=0$ . Then

$$\begin{aligned} \tilde{\psi}(k) &= \sum_{Llm} \frac{i}{(2\pi)^4} \int \prod_{j=1}^4 d\xi_j e^{-i k \cdot \xi} H_{Llm}(\psi, \theta, \phi) \psi_L(r) \\ &= \frac{A_{00}}{(2\pi)^4} \int r^3 dr \sin^2 \psi d\psi \\ &\quad \times \sin \theta d\theta d\phi e^{-i k r \cos \psi} \psi_{L=0}(r), \end{aligned} \quad (3.31)$$

where

$$\xi_0 = i \xi_4, \quad r^2 = \sum_{i=1}^4 \xi_i^2, \quad \text{and } k = \sqrt{k^2}.$$

Also, in writing down this expression we have chosen the direction of the Euclidean vector  $k_\mu$  parallel to the 4-axis, so that the angle between  $k_\mu$  and  $\xi_\mu$  is  $\psi$ . Performing the angle integrations in (3.31) with the help of the relation

$$\begin{aligned} J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\pi e^{+i z \cos \psi} \sin^{2\nu} \psi d\psi \\ &[\text{Re}(\nu + \frac{1}{2}) > 0], \end{aligned} \quad (3.32)$$

we obtain

$$\tilde{\psi}(k) = \frac{A_{00}}{(2\pi)^2 k} \int_0^\infty r^2 dr J_1(kr) \psi_{L=0}(r). \quad (3.33)$$

Inserting here the solution (3.24) for  $\psi_L$  (note that this solution incorporates the boundary condition of regularity in configuration space), we have

$$\begin{aligned} \tilde{\psi}(k) &= A_{00} \frac{(2if)^{(1/2)L'}}{(2\pi)^2 k} \int_0^\infty r^{L'+2} dr J_1(kr) \\ &\quad \times e^{-(1/2)if r^2} \Phi(a, b; if r^2). \end{aligned}$$

Since we are considering the ground state for which  $a = -n = 0$ , the confluent hypergeometric function under the integral is simply 1. Then, using the formula

$$\begin{aligned} \int_0^\infty t^{\mu-1} dt J_\nu(at) e^{-p^2 t^2} \\ = \frac{(a/2p)^\nu \Gamma((\nu + \mu)/2)}{2p^\mu \Gamma(\nu + 1)} \Phi\left(\frac{\nu + \mu}{2}, \nu + 1; -\frac{a^2}{4p^2}\right), \end{aligned}$$

we find

$$\tilde{\psi}(k) = \frac{A_{00} 2^{L'} \Gamma((L'+4)/2)}{(2\pi)^2 (if)^2} \Phi\left(\frac{L'+4}{2}, 2; \frac{k^2}{2if}\right). \quad (3.34)$$

From (3.22) we have

$$\begin{aligned} L' &= -1 \begin{matrix} + \\ - \end{matrix} \left(1 - \frac{g^2}{4\pi^2}\right)^{1/2} \\ &\approx -g^2/8\pi^2 \text{ for } |g^2| \ll 4\pi^2. \end{aligned} \quad (3.35)$$

In order to ensure the regularity of the bound-state wave function at the origin in configuration space we must choose the upper sign in (3.35) (implying  $L'=0$  for  $g^2=0$ ).

The large  $k^2$  asymptotic behavior of the wave function now follows from that of the confluent hypergeometric function. We quote the latter explicitly in order to exhibit the source of the different asymptotic behavior of  $\tilde{\psi}(k)$  in the spacelike and timelike regions. Thus

$$\begin{aligned} \Phi(a, b; z) &\approx \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left[1 + O\left(\frac{1}{z}\right)\right] \\ &\text{for } \text{Re}(z) \rightarrow +\infty \end{aligned} \quad (3.36)$$

and

$$\begin{aligned} \Phi(a, b; z) &\approx \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \left[1 + O\left(\frac{1}{z}\right)\right] \\ &\text{for } \text{Re}(z) \rightarrow -\infty. \end{aligned}$$

Hence for  $k^2 \rightarrow +\infty$  (i.e., spacelike with the metric we are using) and since  $if = |f|$  [cf. (3.21)]

$$\begin{aligned} \tilde{\psi}(k) &\approx \frac{A_{00} 2^{L'} \Gamma((L'+4)/2)}{(2\pi)^2 (if)^2 \Gamma(-L'/2)} \left(\frac{k^2}{2if}\right)^{-2-(1/2)L'} \\ &\quad \times \left[1 + O\left(\frac{1}{k^2}\right)\right], \end{aligned} \quad (3.37a)$$

whereas for  $k^2 < 0$  (i.e., timelike)

$$\begin{aligned} \tilde{\psi}(k) &\approx \frac{A_{00} 2^{L'/2} (-k^2)^{L'/2}}{(2\pi)^2 (if)^{2+(1/2)L'}} e^{-k^2/2if} \left[1 + O\left(\frac{1}{k^2}\right)\right]. \end{aligned} \quad (3.37b)$$

Thus in the spacelike region of  $k^2$ , the wave function falls off asymptotically like a power, whereas in the timelike region it diverges exponentially. We also observe that if we set the Coulomb coupling  $g$  (or the anomalous dimension) equal to zero, we have [cf. (3.35)]  $L'=0$  and the confluent hyper-

geometric function in (3.34) reduces to an exponential. Hence for  $g \rightarrow 0$

$$\bar{\psi}(k) = \frac{A_{00}}{(2\pi)^2 (if)^2} e^{-k^2/2if}. \quad (3.38)$$

Thus (since  $if = |f|$ ) the wave function now falls off exponentially in the spacelike region, the diverging behavior in the timelike region remaining unaffected. The discussion of the momentum space equations given in the next section will verify these findings.

Finally we consider the form factor itself. It is clear from its definition (3.28) that its asymptotic behavior in  $q^2$  follows immediately from that of  $\bar{\psi}(q)$ , i.e., for  $q^2 \rightarrow +\infty$

$$F_2(q^2) \sim (q^2)^{-2-(1/2)L'}, \quad (3.39a)$$

and for  $q^2 \rightarrow -\infty$

$$F_2(q^2) \sim (-q^2)^{L'/2} \exp(-q^2/2if). \quad (3.39b)$$

The Coulomb or Yukawa interaction thus serves to ensure the (physically plausible) asymptotic power falloff of the form factor in the spacelike region, but not in the timelike region where it diverges exponentially. Of course, this power falloff in the spacelike region is simply a reflection of the regular power behavior of the bound-state wave function near the origin in configuration space, and this is determined by the Coulomb or Yukawa interaction. However, it is not completely trivial to see that this power behavior is not swamped by an exponential due to the harmonic gluon interaction. On the other hand, the divergent and thus unitarity violating behavior in the timelike region is related to the infinite rise of the Regge trajectories. The model shows that this infinite rise is unphysical.

#### D. Relativistic wave equations in momentum space

The Bethe-Salpeter equation is mostly treated in the momentum space representation. Considerable insight into its tractability and into the plausibility of approximations used in solving it is often gained by comparing it with its counterpart when one of the particles has infinite mass.

Thus here (also in order to make the discussion of the Bethe-Salpeter equation in the next section more transparent) we will discuss briefly the equation considered above in its momentum space representation together with some related problems.

We consider Eq. (3.29). The momentum-space representation of the interaction defined by Eq. (3.19) is

$$\begin{aligned} \bar{v}(k) &= \frac{1}{(2\pi)^4} \int \left( \frac{g^2}{4\pi^2 \xi^2} + f^2 \xi^2 \right) e^{-ik \cdot \xi} d^4 \xi \\ &= \frac{g^2}{(2\pi)^4 k^2} - f^2 \square_k^2 \delta^4(k). \end{aligned} \quad (3.40)$$

Equation (3.29) may then be written

$$\begin{aligned} (k^2 + \frac{1}{4} P^2 + m^2) \bar{\psi}(k) &= \int \left[ \frac{g^2}{(2\pi)^4 (k-k')^2} \right. \\ &\quad \left. - f^2 \square_k^2 \delta^4(k-k') \right] \bar{\psi}(k') d^4 k' \end{aligned} \quad (3.41)$$

or

$$(k^2 + \frac{1}{4} P^2 + m^2) \bar{\psi}(k) = -f^2 \square_k^2 \bar{\psi}(k) + \frac{g^2}{(2\pi)^4} \int \frac{\bar{\psi}(k') d^4 k'}{(k-k')^2}.$$

Using the relation<sup>29</sup>

$$\square_k^2 \frac{1}{(k-k')^2} = -4\pi^2 \delta^4(k-k'), \quad (3.42)$$

we can rewrite the equation as

$$\begin{aligned} \square_k^2 (k^2 + \frac{1}{4} P^2 + m^2) \bar{\psi}(k) &= - \left( \frac{g^2}{4\pi^2} + f^2 \square_k^2 \square_k^2 \right) \\ &\quad \times \bar{\psi}(k). \end{aligned} \quad (3.43)$$

We investigate the equation a little further in the form (3.41). After the Wick rotation the equation possesses O(4) symmetry in the four-dimensional Euclidean space of  $k_\mu$ . The solutions therefore transform like the four-dimensional spherical harmonics  $H_{Llm}(\psi, \theta, \phi)$ , which have been defined earlier. Thus, setting

$$\bar{\psi}(k) = \sum_{Llm} \bar{\psi}_L(|k|) H_{Llm}(\psi, \theta, \phi)$$

and using the relations

$$\begin{aligned} \frac{1}{(k-k')^2} &= \sum_{L=0}^{\infty} C_L^1 \left( \frac{k \cdot k'}{|k| |k'|} \right) R_L(|k|, |k'|), \\ R_L(|k|, |k'|) &= \left[ \frac{|k|^L}{|k'|^{L+2}} \theta(|k'| - |k|) + \frac{|k'|^L}{|k|^{L+2}} \theta(|k| - |k'|) \right], \\ C_L^1 \left( \frac{k \cdot k'}{|k| |k'|} \right) &= \frac{2\pi^2}{L+1} \sum_{l=0}^L \sum_{m=-l}^l H_{Llm}(\psi, \theta, \phi) H_{Llm}^*(\psi', \theta', \phi'), \end{aligned} \quad (3.44)$$

we obtain

$$(k^2 + \frac{1}{4}P^2 + m^2)\tilde{\psi}_L(|k|) = -f^2 D_L(|k|)\tilde{\psi}_L(|k|) + \frac{g^2}{8\pi^2(L+1)} \int R_L(|k|, |k'|)k'^3 dk' \tilde{\psi}_L(|k'|), \quad (3.45)$$

where

$$D_L(|k|) \equiv \left[ \frac{1}{|k|^3} \frac{\partial}{\partial |k|} \left( |k|^3 \frac{\partial}{\partial |k|} \right) - \frac{L(L+2)}{k^2} \right]. \quad (3.46)$$

Using the relation

$$D_L(|k|)R_L(|k|, |k'|) = -\frac{2(L+1)\delta(|k|-|k'|)}{|k|^3}, \quad (3.47)$$

we obtain the radial form of Eq. (3.43):

$$D_L(|k|)(k^2 + \frac{1}{4}P^2 + m^2)\tilde{\psi}_L(|k|) = -f^2 D_L^2(|k|)\tilde{\psi}_L(|k|) - \frac{g^2}{4\pi^2}\tilde{\psi}_L(|k|). \quad (3.48)$$

We consider Eq. (3.45). Dividing by  $f^2$  (the gluon coupling has to be large, as pointed out earlier) we may write the equation

$$\left[ \frac{d^2}{dk^2} + \frac{3}{|k|} \frac{d}{d|k|} - \frac{L(L+2)}{k^2} - \frac{k^2 + \frac{1}{4}P^2 + m^2}{(if)^2} \right] \tilde{\psi}_L(|k|) = O\left(\frac{g^2}{f^2}\right). \quad (3.49)$$

The solution  $\tilde{\psi}_L^{(0)}$  of this equation with the right-hand side zero may be read off Eqs. (3.23) to (3.26). Thus

$$\tilde{\psi}_L^{(0)}(|k|) = \bar{z}^L e^{-(1/4)\bar{z}^2} \Phi(\bar{a}, \bar{b}; \frac{1}{2}\bar{z}^2), \quad (3.50)$$

where

$$\begin{aligned} \bar{z} &= (2/if)^{1/2} \sqrt{k^2}, \\ \bar{a} &= \frac{1}{2}(L+2) + \frac{m^2 + \frac{1}{4}P^2}{4if}, \\ \bar{b} &= L+2. \end{aligned} \quad (3.51)$$

The eigenvalues are again given by the condition that the infinite Kummer series in (3.50) be broken off after a finite number of terms, i.e.,

$$\bar{a} = -n, \quad n = 0, 1, 2, \dots$$

provided  $\bar{z}^2 > 0$ , i.e.,  $k^2 > 0$  for  $if = |f| > 0$  to ensure normalizability of the wave function. We observe that this condition is identical with (3.27) in the limit  $g \rightarrow 0$  (the case under discussion here). We also observe that for  $k^2 \rightarrow -\infty$  the bound-state wave function  $\tilde{\psi}_L(k)$  diverges exponentially as seen in the preceding section. The next step in solving (3.49) is clearly to develop perturbation expansions in rising powers of  $g^2/f^2$  for both the solutions and eigenvalues. The zero-order solution

(3.50) leaves uncompensated on the right-hand side of (3.49) the contribution

$$R_n^{(0)} = \frac{1}{8\pi^2(L+1)} \frac{g^2}{f^2} \int_0^\infty R_L(|k|, |k'|)k'^3 \times dk' \tilde{\psi}_L^{(0)}(|k'|).$$

This integral may be evaluated in terms of incomplete  $\Gamma$  functions which can be reexpressed in terms of confluent hypergeometric functions. However, the next step of calculating the first-order perturbation corrections to  $\tilde{\psi}_L^{(0)}$  and its eigenvalue are complicated. We therefore do not go into further details here. Of course the fourth-order differential equation (3.48) has the same solution, as is easily verified.

Another example of interest is the potential

$$v(\xi) = \frac{g^2}{4\pi^2} + f^2 \xi^2, \quad (3.52)$$

which corresponds to a near harmonic interaction.  $-g^2$ , of course, has the meaning of a potential well parameter. The Fourier transform of (3.52) is

$$\bar{v}(k) = \left( \frac{g^2}{4\pi^2} - f^2 \square_k^2 \right) \delta^4(k), \quad (3.53)$$

and the equation describing the relative motion of the quark-antiquark system is

$$(k^2 + \frac{1}{4}P^2 + m^2)\tilde{\psi}(k) = \frac{g^2}{4\pi^2}\tilde{\psi}(k) - f^2 \square_k^2 \tilde{\psi}(k). \quad (3.54)$$

Thus in this case one obtains a simple equation, the solutions of which can be read off Eqs. (3.49) and (3.50). We note in particular that the potential well does not lead to an asymptotic power behavior of the electromagnetic form factor of the bound state.

#### IV. BETHE-SALPETER MODELS

##### A. Tractable, approximate form of the Wick-Cutkosky Bethe-Salpeter equation

For the discussion which follows it is useful to have for reference a standard equation which is known to be easy to solve. It is well known that the Bethe-Salpeter equation of the Wick-Cutkosky model (in which the exchanged scalar particle is massless) can be solved explicitly in the ladder approximation, even in the case of unequal masses. Explicit perturbation solutions have, for instance, been given in Ref. 30. However, for in-

vestigating certain principal properties of vertex functions it is much more convenient to use simple, but approximate forms rather than complicated complete and explicit solutions. For this reason we derive first a more tractable approximate form of the Wick-Cutkosky Bethe-Salpeter equation, which in standard notation reads

$$\Gamma(p, q) = \frac{\lambda}{4\pi^2 i} \times \int \frac{\Gamma(p', q) d^4 p'}{[(p' + \frac{1}{2}q)^2 + m^2][(p' - \frac{1}{2}q)^2 + m^2](p - p')^2}, \quad (4.1)$$

where  $p$  and  $q$  are defined in terms of the momenta  $p_1, p_2$  of the external scalar particles of mass  $m$ :  $p = \frac{1}{2}(p_1 - p_2)$ ,  $q = p_1 + p_2$ . The equation may be rewritten

$$\tilde{\Gamma}(p, q)[(p + \frac{1}{2}q)^2 + m^2][(p - \frac{1}{2}q)^2 + m^2] = \frac{\lambda}{4\pi^2 i} \int \frac{\tilde{\Gamma}(p', q) d^4 p'}{(p - p')^2}, \quad (4.2)$$

where

$$\Gamma(p, q) = \tilde{\Gamma}(p, q)[(p + \frac{1}{2}q)^2 + m^2][(p - \frac{1}{2}q)^2 + m^2].$$

Here

$$[(p + \frac{1}{2}q)^2 + m^2][(p - \frac{1}{2}q)^2 + m^2] \simeq (p^2 + m^2 + \frac{1}{4}q^2)^2,$$

provided

$$\frac{(p \cdot q)^2}{(p^2 + m^2 + \frac{1}{4}q^2)^2} \ll 1. \quad (4.3)$$

This approximation is valid if  $q^2 \ll p^2$  or  $p^2 \ll q^2$ .

On Wick-rotating Eq. (4.2) we obtain

$$\tilde{\Gamma}(p, q)(p^2 + m^2 + \frac{1}{4}q^2)^2 \simeq \frac{\lambda}{4\pi^2} \int \frac{\tilde{\Gamma}(p', q) d^4 p'}{(p - p')^2} \quad (4.4)$$

(Euclidean metric understood). Our approximation has also made the equation invariant under rotations in the four-dimensional Euclidean space of  $p$ . Hence expanding  $\tilde{\Gamma}(p, q)$  in terms of four-dimensional spherical harmonics, we obtain the radial equation in momentum space, i.e.,

$$\left[ D_L(|p|) + \frac{\lambda}{(p^2 + m^2 + \frac{1}{4}q^2)^2} \right] \Gamma_L(|p|) = 0. \quad (4.5)$$

The close connection between this equation and (3.48) is obvious. In fact, for  $f=0$  the latter follows from (4.5) if we replace  $\tilde{\Gamma}_L$  by  $\Gamma_L$  and go to the limit in which the mass of one of the external scalar particles is allowed to approach infinity together with  $\lambda/m^2$  remaining finite and nonzero. The solution of Eq. (4.5) is given in terms of the hypergeometric function

$$\Gamma_L(|p|) = |p|^{L-1} F\left(\bar{\alpha}, 1 - \bar{\alpha}; L + 2; \frac{p^2}{p^2 + m^2 + \frac{1}{4}q^2}\right), \quad (4.6)$$

where

$$\bar{\alpha} = \frac{1}{2}[1 + (1 + \lambda)^{1/2}].$$

The Regge parent and daughter poles  $\alpha_{n\mu}$ ,  $n, \mu = 0, 1, 2, \dots$  are given by

$$\alpha_{n\mu} = \alpha_n - \mu, \quad \alpha_n + 1 + \bar{\alpha} = -n + O(q^2) \quad (4.7)$$

where  $\alpha_n$  are the Toller poles. [Note that the trajectories (4.7) rise linearly with  $q^2$  only for  $|q^2| < 4m^2$ ; see Ref. 30.] These eigenvalues (determined by the vanishing of  $\tilde{\Gamma}_L$  in the limit  $p^2 \rightarrow \infty$ , and so for  $q^2 \ll p^2$ ) are in fact identical with those calculated from the exact equation for small difference between the masses. The differential form of Eq. (4.4) follows immediately with the help of (3.42):

$$\square_p^2 \tilde{\Gamma}(p, q)(p^2 + m^2 + \frac{1}{2}q^2)^2 + \lambda \tilde{\Gamma}(p, q) = 0. \quad (4.8)$$

The radial part of this equation is, of course, (4.5).

In the following it is also useful to have for comparison the configuration space equations obtained by Fourier transformation of the above. We have

$$(-\square_\xi^2 + \frac{1}{4}q^2 + m^2)^2 \tilde{\Gamma}(\xi) = \frac{\lambda}{\xi^2} \tilde{\Gamma}(\xi), \quad (4.9)$$

where  $\lambda = g^2/4\pi^2$  in our earlier notation and

$$\tilde{\Gamma}(p, q) = \frac{1}{(2\pi)^4} \int \tilde{\Gamma}(\xi) e^{-i p \cdot \xi} d^4 \xi.$$

Expanding the solutions of (4.9) in terms of four-dimensional spherical harmonics one obtains the radial equation

$$[D_L(r) - m^2 - \frac{1}{4}q^2]^2 \psi_L(r) = \frac{\lambda}{r^2} \psi_L(r). \quad (4.10)$$

In the limit of infinite mass of one of the external particles (as considered earlier) the kinetic energy represented by the operator  $D_L(r)$  may be considered to be negligible compared to its mass, and so the fourth-order differential equation reduces to a second-order differential equation. It is easy to convince oneself that the extreme simplicity of the equations of the Wick-Cutkosky model in momentum space is due to the relation (3.42), which says that the Wick-Cutkosky potential is the Green's function of the D'Alembertian.

## B. Interaction kernels

We have seen that the three-dimensional linear potential suggests the gluon propagator  $(1/k^2)^2$ . The propagator corresponding to the three-dimensional harmonic potential does not follow from a similar simple reasoning. It is therefore suggestive to consider four-dimensional generalizations which simulate rising three-dimensional potentials, e.g., in the approximation in which retardation effects are neglected. Alternatively one

TABLE I. Four-dimensional Euclidean Fourier transforms  $FU_\mu(r)$  of interaction kernels  $U_\mu(r)$ .

$U_\mu(r)$	$FU_\mu(r)$
(a) $\frac{\mu K_1(\mu r)}{4\pi^2 r}$	$\frac{1}{(2\pi)^4(k^2 + \mu^2)} \quad (\text{Re}\mu > 0, k > 0)$
(b) $\mu^\nu r^\nu K_\nu(\mu r)$	$\frac{(\sqrt{2}\mu)^{2\nu} \Gamma(\nu+2)}{2\pi^2(k^2 + \mu^2)^{\nu+2}}$ $(\text{Re}\mu >  \text{Re}\nu  - 1, \text{Re}\mu >  \text{Im}k )$
(c) $e^{-\mu^2 r^2}$	$\frac{1}{(4\pi)^2 \mu^4} e^{-k^2/4\mu^2} \quad (\mu \neq 0,  \arg k  < \frac{1}{4}\pi)$
(d) $e^{-\mu r}$	$\frac{3}{2\pi^2(\mu^2 + k^2)^{5/2}} \quad (\text{Re}\mu >  \text{Im}k )$
(e) $\frac{e^{-\mu r}}{r}$	$\frac{1}{4\pi^2(\mu^2 + k^2)^{3/2}} \quad (\text{Re}\mu >  \text{Im}k )$
(f) $r^2(\mu r)K_1(\mu r)$	$\frac{48\mu^2}{\pi^2} \frac{(\mu^2 - k^2)}{(k^2 + \mu^2)^5} = \frac{1}{4\pi^2} [D_{L=0}( k )]^2 \frac{1}{k^2 + \mu^2}$ $( k /\mu < 1, \mu \neq 0)$
(g) $r(\mu r)K_1(\mu r)$	$\frac{45}{16\pi\mu^5} \left\{ 1 - \frac{15}{4} \frac{k}{\mu} + \frac{3}{2} \frac{k}{\mu} \left[ J_0\left(\frac{2k}{\mu}\right) - J_2\left(\frac{2k}{\mu}\right) \right] \right.$ $\left. + \left( \frac{9}{4} - \frac{k^2}{\mu^2} \right) J_1\left(\frac{2k}{\mu}\right) \right\} \quad (\mu \neq 0)$
(h) $\frac{1}{4\pi^2 r(\mu'^2 - \mu^2)} [\mu K_1(\mu r) - \mu' K_1(\mu' r)]$	$\frac{1}{(2\pi)^4(k^2 + \mu^2)(k^2 + \mu'^2)} \quad (\mu \neq \mu')$
(i) $(\ln r)\theta(\mu - r)$	$\frac{1}{4\pi^2 k} \left[ \frac{1}{k} \mu^2 (\ln \mu) J_2(\mu k) + k \mu J_1(\mu k) \right.$ $\left. + 2J_0(\mu k) - 2 \right] \quad (0 < \mu < \infty)$

may consider classes of four-dimensional Euclidean potentials which can then be related to the problems of quark confinement and rising trajectories. This is the approach we shall follow here.

Now, in finding the momentum space representation of the four-dimensional linear or harmonic gluon interaction, one is immediately confronted with the fact that the four-dimensional Euclidean Fourier transform of  $r$  or  $r^2$  does not exist. It is therefore necessary to use a method of infrared regularization, or the well-known artifice of differentiation with respect to a parameter which (if permissible and desired) is allowed to approach zero at the end of the integration or, in some cases, the trick of absorbing a vanishing mass into an infinitely large coupling constant. From the study of the Veneziano and other dual models it is well known that infinitely rising linear Regge trajectories (due to potentials increasing without bound at infinite separations) do not lead to amplitudes satisfying unitarity. Thus the introduction of a mass parameter which has a damping effect on the potential and wave functions and which is

not strictly zero may in fact be desirable in order to restore unitarity (which then would also have to imply a bending over of the Regge trajectories at highest energies).

In the following  $\mu$  is the mass or reciprocal length parameter which is in some way a measure of the finite extent of the quark-confining region of space. Of course, if one attributes to  $\mu$  such a significance, then the meaning of the limit  $\mu \rightarrow 0$  is anything but clear. It may be argued, however, that this difficulty is related to the fact that  $\mu$  itself (i.e., the damping) should follow from the underlying dynamics of gluon confinement (by vacuum polarization). Unless one imposes further constraints there is, of course, in general, an unlimited number of possible  $\mu$ -dependent momentum space representations of the interaction which are such that their Fourier transforms reduce to or behave like  $r$  or  $r^2$  in the limit of vanishing  $\mu$ . This nonuniqueness of the momentum space representation is inherent in our phenomenological approach (as also in the analogous work of Bender *et al.*<sup>8</sup>), which does not include an equivalent and

over-all self-consistent treatment of the gluon wave equations together with the corresponding problem of gluon confinement. Such an over-all consistent treatment— as, for instance, the field-theoretic model discussed by Blaha<sup>10</sup>— would lead naturally to a damping of the (e.g.) three dimensional linear potential at large distances due to important vacuum polarization effects (i.e., gluon propagator self-energy insertions) which result in a powerlike asymptotic decrease of the potential thereby avoiding a violation of unitarity.

In Table I we list several interaction kernels  $g^2 U_\mu(r)$  together with their four-dimensional Euclidean Fourier transforms  $g^2 F U_\mu(r)$  defined by

$$\begin{aligned} F U_\mu(r) &= \frac{1}{(2\pi)^4} \int d^4 \xi e^{-i k \cdot \xi} U_\mu(\xi) \\ &= \frac{1}{4\pi^2 k} \int_0^\infty r^2 dr J_1(kr) U_\mu(r) \end{aligned} \quad (4.11)$$

(here  $d^4 \xi = \prod_{i=1}^4 d \xi_i$ ), where we have used Eq. (3.32). For orientation purposes we include in Table I also the well-known case of the Yukawa potential (exchange of spinless mesons).

For  $\mu(\mu')$  positive and nonzero each of the kernels given in Table I has a decreasing behavior for  $r \rightarrow \infty$  [recall that  $K_n(x) \simeq (\pi/2x)^{1/2} e^{-x}$  for  $x \rightarrow +\infty$ ]. The conditions of validity stated in Table I are those which result from the integration; in several cases they may be relaxed to allow the limit  $\mu \rightarrow 0$  to be taken. The results in Table I allow the following observations. Constructing potentials along the lines suggested by the well-known case of the Yukawa potential, we see that the linear potential (g) (i.e., linear for  $\mu \rightarrow 0$ ) has a most unpleasant momentum space representation, whereas the Fourier transform of the harmonic potential (f) (i.e., harmonic for  $\mu \rightarrow 0$ ) looks surprisingly simple (our expression has the same form as that quoted by Sundaresan and Watson<sup>31</sup>). The propagator  $(1/k^2)^4$  implies, of course, a strong infrared divergence which is made worse by a multiplicative factor  $1/\mu^2$  in the configuration space potential which cannot be absorbed in the coupling constant. Various forms of potentials which are constant in the limit  $\mu \rightarrow 0$  are given by (b), again accompanied by factors of  $\mu$ . The potentials (c), (d), and (e) are included because they lead to power-behaved interactions when differentiated with respect to  $\mu$  or  $\mu^2$ . For instance, from (d) we have

$$r^2 = \left[ \left( \frac{\partial}{\partial \mu} \right)^2 e^{-\mu r} \right]_{\mu=0},$$

with the momentum space representation

$$\begin{aligned} \left[ \left( \frac{\partial}{\partial \mu} \right)^2 \frac{3}{2\pi^2 (k^2 + \mu^2)^{5/2}} \right]_{\mu=0} &= \left[ -\frac{15(k^2 - 6\mu^2)}{2\pi^2 (k^2 + \mu^2)^{9/2}} \right]_{\mu=0} \\ &= -\frac{15}{2\pi^2 (k^2)^{7/2}}, \end{aligned} \quad (4.12)$$

which is independent of  $\mu$ .

Quark and gluon confinement models involving the free gluon propagator  $(1/k^2)^2$  have been discussed by several authors.<sup>10-12</sup> From (h) we see that this propagator would arise in the limits  $\mu, \mu' \rightarrow 0$ . Can we take this limit in the configuration space representation? For  $\mu, \mu' \neq 0$  but  $r \rightarrow 0$  one finds from the expansion defining the modified Bessel function  $K_1(\mu r)$  the following behavior of  $U_{\mu, \mu'}(r)$ :

$$\begin{aligned} U_{\mu, \mu'}(r) &\simeq -\frac{1}{8\pi^2} \left[ \ln \left( \frac{\gamma}{2} \right) + \gamma + \left( \frac{\mu^2 \ln \mu - \mu'^2 \ln \mu'}{\mu^2 - \mu'^2} \right) \right. \\ &\quad \left. + O(r^2; \mu, \mu') \right], \end{aligned} \quad (4.13)$$

where  $\gamma$  is Euler's constant,  $\simeq 0.577$ . This result agrees with that discussed by Kiskis<sup>12</sup> for "the case  $\mu = \mu' = 0$ ." We observe that not both  $\mu$  and  $\mu'$  can be zero, otherwise the constant in (4.13) would be undefined. We point out here that a propagator behaving like  $\ln r$  is also obtained in one-space-one-time dimensional quantum electrodynamics<sup>32</sup> (there, of course, with the normal type of— but two-dimensional— propagator) and in Wilson's four-dimensional lattice gauge field theory.<sup>7</sup> However, from entry (i) in Table I it can be seen that the (singular) logarithmic interaction with infrared cutoff does not correspond to a propagator behaving like  $(1/k^2)^2$ . Extreme caution, therefore, seems advisable in the use of such an interaction. We observe that the Fourier transform depends on the method used to regularize the integral. This is particularly clear for the case under discussion because the propagator  $(1/k^2)^2$  may also be extracted from entry (b) in Table I, which for  $\nu = 0$  and

$$F U_\mu(r) = \frac{1}{2\pi^2} \frac{1}{(k^2 + \mu^2)^2}$$

gives

$$U_\mu(r) = K_0(\mu r) \simeq -\left( \gamma + \ln \frac{\mu r}{2} \right)$$

for  $|\mu r| \rightarrow 0$ , in agreement with Eq. (4.13) provided  $\mu' = 0$ . Nonetheless it is interesting to observe that the propagator  $(1/k^2)^2$  may be related to the superposition of an attractive and a repulsive Yukawa potential with the same coupling (and so looks similar to a bubble in  $\phi^4$  theory).

In the literature reference is frequently made to the so-called static limit of a relativistic propagator. What is meant is the case  $k^2 = \vec{k}^2(1 - k_0^2/\vec{k}^2)$

$\simeq \bar{k}^2$  so that the off-shell propagator  $1/(k^2 + \mu^2) \simeq 1/(\bar{k}^2 + \mu^2)$ , i.e.,  $|k_0^2/(\bar{k}^2 + \mu^2)| < 1$ , and its four-dimensional Minkowskian Fourier transform is the Yukawa potential, i.e.,

$$\frac{1}{(2\pi)^4} \int \frac{e^{i\bar{k}x} d^4k}{\bar{k}^2 + \mu^2} = \frac{1}{4\pi} \delta(x_0) \frac{e^{-\mu|\bar{x}|}}{|\bar{x}|}$$

or

$$\frac{1}{(2\pi)^4} \frac{1}{\bar{k}^2 + \mu^2} = \frac{1}{(2\pi)^4} \int e^{-i\bar{k}\cdot\bar{x}} d^3x \frac{e^{-\mu|\bar{x}|}}{4\pi|\bar{x}|}.$$

Thus, in searching for suitable representations of the gluon propagator, we could look at the spatial Fourier transform of a power-behaved potential  $|\bar{x}|^s$  and therein replace  $\bar{k}^2$  by  $k^2$  so that the propagator is  $\delta(x_0)|\bar{x}|^s$  in the static limit. Then, for integral values of  $s \geq 0$  and  $\mu > 0$  we have

$$\begin{aligned} \frac{1}{(2\pi)^4} \int e^{-i\bar{k}\cdot\bar{x}} d^3x |\bar{x}|^s e^{-\mu|\bar{x}|} \\ = \frac{2(s+1)! \operatorname{Im}(i|\bar{k}| + \mu)^{s+2}}{(2\pi)^3 |\bar{k}| (\bar{k}^2 + \mu^2)^{s+2}}, \end{aligned}$$

where we have inserted an exponential in the infrared cutoff  $\mu$  in order to ensure the existence of the Fourier integral.<sup>33</sup> Replacing  $\bar{k}^2$  by  $k^2$  we have (in conformity with Table I) the gluon propagator

$$FU_\mu^{(s)}(r) = \frac{2(s+1)! \operatorname{Im}(i\sqrt{k^2} + \mu)^{s+2}}{(2\pi)^3 \sqrt{k^2} (k^2 + \mu^2)^{s+2}},$$

which in the static limit and for  $\mu \rightarrow 0$  corresponds to  $\delta(x_0)|\bar{x}|^s$ . Particular cases are

$$FU_\mu^{(0)}(r) = \frac{4\mu}{(2\pi)^3 (k^2 + \mu^2)^2},$$

$$FU_\mu^{(1)}(r) = \frac{-4(k^2 + 3\mu^2)}{(2\pi)^3 (k^2 + \mu^2)^3},$$

$$FU_\mu^{(2)}(r) = \frac{48\mu(\mu^2 - k^2)}{(2\pi)^3 (k^2 + \mu^2)^4},$$

$$FU_\mu^{(3)}(r) = \frac{48(k^4 - 10k^2\mu^2 + 5\mu^4)}{(2\pi)^3 (k^2 + \mu^2)^5}.$$

We see from  $FU_\mu^{(1)}(r)$  that if the potential is to be linear in the static limit, then the gluon propagator is (in the limit  $\mu \rightarrow 0$ )  $(1/k^2)$  apart from a multiplicative constant (ignoring vacuum polarization effects). We have observed earlier that in the four-dimensional Euclidean configuration space the corresponding potential behaves like  $\ln(\mu r)$ . (Aspects of this case as well as  $FU_\mu^{(3)}$  have also been discussed by Blaha.<sup>10</sup>)

In the following we consider in detail the four-dimensional Euclidean harmonic interaction given by entry (f) in Table I. In order to obtain the corresponding three-dimensional potential in the static limit, we observe that

$$\frac{48\mu(\mu^2 - \bar{k}^2)}{(\bar{k}^2 + \mu^2)^5} = F_3 \left( \frac{e^{-\mu|\bar{x}|}}{4\pi|\bar{x}|} \right) F_3(\bar{x}^2 e^{-\mu|\bar{x}|}),$$

where  $F_3$  means "three-dimensional Fourier transform." Using now the well-known theorem which states that the Fourier transform of a product of two functions is the convolution of the Fourier transforms of those functions, we find that for  $\mu \rightarrow 0$  but still  $\neq 0$

$$F_3 \left( \frac{96\pi\mu(\mu^2 - \bar{k}^2)}{(\bar{k}^2 + \mu^2)^5} \right) = \frac{a}{\mu^4} + \frac{b}{\mu^2} \bar{x}^2 + O(\mu),$$

where  $a$  and  $b$  are numbers. Of course, this expression is still to be multiplied by a coupling constant  $\bar{g}^2$ . The potential is meaningful if  $\bar{g}^2/\mu^4 = f^2$  is finite. It behaves like a constant for  $|\bar{x}| \sim 1/\mu^\alpha$  having  $0 < \alpha \leq 1$ , and like  $|\bar{x}|^{2-(2/\alpha)}$  (apart from an additive constant) for  $|\bar{x}| \sim 1/\mu^\alpha$  having  $1 < \alpha < \infty$ . Thus, if  $|\bar{x}|$  is sufficiently large in relation to  $1/\mu$  the static potential is again  $\bar{x}^2$ ; for smaller values of  $|\bar{x}|$  it behaves like a lower power of  $|\bar{x}|$  or even like a constant (corresponding to a zero binding force).

### C. Bethe-Salpeter models

From the table of phenomenological gluon interaction kernels discussed above it appears that one of the simpler candidates to deal with is  $U_\mu(r) = r^2(\mu r)K_1(\mu r)$  near the limit  $\mu \rightarrow 0$ . In order to ensure that both its momentum space and configuration space representations are physically meaningful it will be essential to assume that  $\mu$  is small but nonzero, thereby providing also the otherwise essential infrared cutoff. Further, the product of  $\mu^2$  and the quark-gluon coupling must be assumed to be finite, but nonzero; in fact, in the following it turns out that this coupling will again have to be large.

The momentum space representation of the Bethe-Salpeter equation is in standard notation (Euclidean metric understood)

$$\Gamma(p, q) = \int \frac{\Gamma(p', q) d^4p' \bar{v}(p - p')}{[(p' + \frac{1}{2}q)^2 + m^2][(p' - \frac{1}{2}q)^2 + m^2]}$$

or

$$\begin{aligned} [(p + \frac{1}{2}q)^2 + m^2][(p - \frac{1}{2}q)^2 + m^2] \bar{\Gamma}(p, q) \\ = \int \bar{v}(p - p') \bar{\Gamma}(p', q) d^4p', \quad (4.14) \end{aligned}$$

where

$$\bar{\Gamma}(p, q) = \Gamma(p, q) / \{[(p + \frac{1}{2}q)^2 + m^2][(p - \frac{1}{2}q)^2 + m^2]\}$$

is the Fourier transform of the amplitude

$$\langle 0 | T(\bar{q}(x)q(0)) | \phi_q \rangle,$$

and  $\bar{v}$  is the interaction kernel.

In keeping with the motivation of our investigation we assume that  $\bar{v}$  consists of the sum of a gluon and a massless scalar exchange interaction. We assume therefore that  $\bar{v}(p)$  is given by (3.40), i.e.,

$$\bar{v}(p) = \frac{g^2}{(2\pi)^4 p^2} - f^2 \square_p^2 \delta^4(p). \quad (4.15)$$

In terms of the approximation discussed previously the Bethe-Salpeter equation can be written

$$(p^2 + \frac{1}{4}q^2 + m^2)^2 \bar{\Gamma}(p, q) = -f^2 \square_p^2 \bar{\Gamma}(p, q) + \frac{g^2}{(2\pi)^4} \int \frac{\bar{\Gamma}(p', q) d^4 p'}{(p-p')^2} \quad (4.16)$$

(Euclidean metric and infrared cutoff understood). In writing down (4.15) we have again made use of the fact that the (four-dimensional, "spherically symmetric") harmonic gluon interaction can be simulated by the action of the D'Alembertian on the four-dimensional delta function, i.e., we have

$$r^2 \simeq r^2(\mu r) K_1(\mu r) \text{ for } \mu \rightarrow 0$$

and

$$F r^2 = \frac{1}{(2\pi)^4} \int d^4 \xi e^{-i k \cdot \xi} \xi^2 = -\square_k^2 \delta^4(k).$$

Using (3.42), this becomes (for  $\mu$  small but non-zero)

$$F r^2 = \frac{1}{4\pi^2} \left\{ \square_k^2 \square_k^2 \frac{1}{k^2 + \mu^2} \right\},$$

which agrees with the Fourier transform calculated from (4.11) [see entry (f) in Table I].

We now observe the close analogy between Eqs. (4.16) and (3.41). In solving (4.16) we may therefore proceed in a similar way, and we obtain after separating off the angular dependence the following equation for the radial wave function  $\bar{\Gamma}_L(|p|)$  in momentum space

$$D_L(|p|)(p^2 + \frac{1}{4}q^2 + m^2)^2 \bar{\Gamma}_L(|p|) = -f^2 D_L^2(|p|) \bar{\Gamma}_L(|p|) - \frac{g^2}{4\pi^2} \bar{\Gamma}_L(|p|). \quad (4.17)$$

The corresponding radial equation in configuration space is

$$[D_L(r) - m^2 - \frac{1}{4}q^2] \psi_L(r) = \left( \frac{g^2}{4\pi^2 r^2} + f^2 r^2 \right) \psi_L(r). \quad (4.18)$$

(Note that here the dimensions of  $g$  and  $f$  are different from those in Sec. III.)

In general, when both  $f$  and  $g$  are nonzero, these equations are not easy to solve. For this reason we are compelled to consider various approximate cases separately which taken together give an understanding of the complete solution. We discuss first the eigenvalues, i.e., Regge trajectories. For  $f=0$  Eq. (4.17) reduces to the equation of the Wick-Cutkosky model for which the solutions are known. On the other hand, for  $g=0$  and  $f \neq 0$  the equation can be written

$$\left[ D_L(|p|) + \frac{(p^2 + \frac{1}{4}q^2 + m^2)^2}{f^2} \right] \bar{\Gamma}_L(|p|) = 0 \quad (g=0). \quad (4.19)$$

Setting

$$\bar{\Gamma}_L(|p|) = \bar{\gamma}_L(p)/p^{3/2}, \quad L' = L + \frac{1}{2},$$

Eq. (4.19) can be rewritten

$$\left[ \frac{d^2}{dp^2} - \frac{L'(L'+1)}{p^2} + \frac{(p^2 + m^2 + \frac{1}{4}q^2)^2}{f^2} \right] \bar{\gamma}_L(p) = 0. \quad (4.20)$$

As in Sec. II we can now use the WKB method for an approximate calculation of the Regge trajectories. For  $t \equiv -q^2 > 4m^2$  (and beyond by analytic continuation) one finds the Toller poles

$$L \simeq -2n - 2(\mp) \frac{4}{3\pi f} [-\frac{1}{4}q^2 + m^2]^{3/2} + O\left(\frac{1}{\sqrt{n}}\right). \quad (4.21)$$

(For the relation to Regge poles, see Sec. III B.) We observe that the larger number of dimensions compared to previous cases has no effect on the intercept term. Also,  $L$  is real in the timelike (i.e., resonance) region  $q^2 + 4m^2 < 0$  if  $f^2 > 0$ . But the fully relativistic treatment makes the trajectory rise more steeply with energy for  $f < 0$  and in the region of validity of the WKB approximation which is where  $n$  is large.

In order to obtain an estimate of the Regge trajectories when neither  $g$  nor  $f$  is zero, we use yet another modified WKB approximation but applied to the configuration space Eq. (4.18). We argue that for fixed values of the coupling constants  $g$  and  $f$  the maximum value of  $L(L+2)$  and so of  $L$  is obtained when the radial kinetic term

$$\mathcal{K}(r) \equiv - \left( \frac{d^2}{dr^2} + \frac{3}{r} \frac{d}{dr} \right)$$

in Eq. (4.18) becomes minimal. This may be justified as follows for the nonsingular Bethe-Salpeter potentials considered here. The eigenvalue problem  $\mathcal{K}\psi = \lambda^2\psi$  (with  $\psi$  square integrable) is equivalent to that of

$$\left( \frac{d^2}{dr^2} + \lambda^2 - \frac{3}{4r^2} \right) \left( \frac{1}{r^{3/2}} \psi \right) = 0.$$



Further, the eigenvalues of the operator  $d^2/dr^2$  are bounded from below by zero. Thus it is possible to expand the solutions of Eq. (4.18) in the neighborhood of that value of  $r$  for which  $\mathcal{K}(r)$  becomes minimal. Then (see below)  $r^3 \sim O(L(L+2))$  and the effect of the term  $3/4r^2$  is of  $O(1/L^{4/3})$ . (Note that interactions behaving like  $r^{-4}$  are of the same order as the kinetic part, whereas in the case of more singular interactions the kinetic part dominates in the region of large  $r$ .) Thus, ignoring the radial kinetic term, the four-dimensional angular momentum  $L$  for a classical circular orbit of radius  $r$  is given by

$$\left(-\frac{L(L+2)}{r^2} - m^2 - \frac{1}{4}q^2\right)^2 = -\frac{g^2}{4\pi^2 r^2} + f^2 r^2. \quad (4.22)$$

This equation defines  $L$  or  $N \equiv L(L+2)$  as a function of  $r$ . Next, we find that value of  $r$ , say  $r_0$ , for which  $N(r)$  is a maximum. We have

$$\begin{aligned} N(r) &= r^2 \left[ \left( f^2 r^2 + \frac{g^2}{4\pi^2 r^2} \right)^{1/2} - \left( m^2 + \frac{1}{4}q^2 \right) \right] \\ &\simeq f r^3 + \frac{g^2}{8\pi^2 f} \frac{1}{r} - \left( m^2 + \frac{1}{4}q^2 \right) r^2 \end{aligned}$$

for  $|g^2| < |4\pi^2 f^2 r^4|$ . Setting  $dN/dr = 0$  we obtain the equation

$$6f^2 r_0^4 - 4f \left( m^2 + \frac{1}{4}q^2 \right) r_0^3 = \frac{g^2}{4\pi^2}. \quad (4.23)$$

This equation may be solved for  $r_0$  by first ignoring the term on the right-hand side. Then, since we are looking for a maximum and  $f < 0$ ,  $m^2 + \frac{1}{4}q^2 < 0$  (as in the previous case)

$$r_0 \simeq \frac{2}{3} \frac{m^2 + \frac{1}{4}q^2}{f}.$$

Adding a correction term  $\epsilon$  to this expression, substituting  $r_0$  back into Eq. (4.23), and calculating  $\epsilon$  by ignoring terms which are nonlinear in  $\epsilon$ , we find

$$\begin{aligned} r_0 &\simeq \frac{2}{3} \frac{m^2 + \frac{1}{4}q^2}{f} \\ &+ \frac{9}{64\pi^2} \frac{g^2 f}{\left( m^2 + \frac{1}{4}q^2 \right)^3}. \end{aligned}$$

Finally we calculate  $L$  from the relation

$$L^2 \simeq N(r_0),$$

and find for the lowest ( $n=0$ ) eigenvalue

$$\begin{aligned} L^2 &\simeq + \frac{4}{27} \frac{\left( -m^2 - \frac{1}{4}q^2 \right)^3}{f^2} \\ &+ \frac{3g^2}{16\pi^2} \frac{1}{\left( m^2 + \frac{1}{4}q^2 \right)}. \end{aligned} \quad (4.24)$$

If we approximate  $4 \times 27 = 108$  by  $9\pi^2$  (implying an error of 18% in the coefficient), then the first term in Eq. (4.24) agrees with the corresponding term in Eq. (4.21). The expression (4.24) has the structure expected for large values of  $f^2$  and small values of  $g^2$ . Again we observe that the harmonic gluon interaction with large quark-gluon coupling constant leads to the rising trajectory.

Our next step is to obtain the large  $p^2$  asymptotic behavior of the momentum space representation of the wave function since this determines the corresponding asymptotic behavior of the form factor. We consider first the case  $g=0$ , i.e., pure harmonic gluon interaction. Changing the variable in Eq. (4.19) to  $x = p^2$ , this equation can be written

$$\begin{aligned} \left[ \frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{L(L+2)}{4x^2} + \frac{x}{4f^2} \right] \tilde{\Gamma}_L(|p|) \\ = - \frac{\left( \frac{1}{4}q^2 + m^2 \right)^2}{2f^2} \left( 1 + \frac{1}{2x} \right) \tilde{\Gamma}_L(|p|). \end{aligned} \quad (4.25)$$

For  $|x| \gg |\frac{1}{4}q^2 + m^2|$  and  $\infty > |f| \gg 1$ , the terms on the right-hand side are small compared to the last term on the left-hand side. The large- $x$  asymptotic behavior of  $\tilde{\Gamma}_L$  is therefore

$$\tilde{\Gamma}_L(|p|) \simeq \frac{1}{x^{1/2}} Z_{(1/9)\{1-L(L+2)\}} \left( \frac{x^{3/2}}{\pm 3f} \right),$$

where  $Z_\nu$  is a solution of Bessel's equation. The particular choice of  $Z_\nu$  to be made is dictated by the condition of regularity of the bound-state wave function, i.e.,

$$\begin{aligned} \tilde{\Gamma}_L(|p|) &\simeq \frac{1}{x^{1/2}} H_{(1/9)\{1-L(L+2)\}}^{(1)} \left( \frac{x^{3/2}}{3f} \right) \\ &\simeq \frac{c}{(p^2)^{5/4}} \exp\left( \frac{+i(p^2)^{3/2}}{3f} \right) \end{aligned} \quad (4.26)$$

for  $p^2 \rightarrow \pm \infty$  and for  $f < 0$ , where  $c$  is a (complex) constant. Thus, in the timelike region where  $p^2 < 0$ ,  $\sqrt{p^2} = +i|\sqrt{p^2}|$ , the wave function  $\tilde{\Gamma}_L(|p|)$  falls off exponentially, whereas in the spacelike region it falls off like  $1/(p^2)^{5/4}$  with a superimposed oscillatory behavior. Since the vertex function  $\Gamma_L(|p|)$  is related to the wave function  $\tilde{\Gamma}_L(|p|)$  by

$$\Gamma_L(|p|) \simeq (p^2 + m^2 + \frac{1}{4}q^2)^2 \tilde{\Gamma}_L(|p|),$$

we see that in the timelike region the vertex function falls off exponentially, but in the spacelike region it diverges like  $(p^2)^{11/4}$ .

We now consider the case  $g \neq 0$ . Then it is clear from Eq. (4.18) that the behavior of the wave function  $\psi_L(r)$  near  $r=0$  [and so of  $\tilde{\Gamma}_L(|p|)$  near  $p^2 \rightarrow \infty$ ] is determined by the singular Coulomb or Yukawa

interaction. From a detailed study of the Wick-Cutkosky model<sup>30</sup> we know that this interaction leads to a power-behaved asymptotic behavior of the vertex function, i.e.,

$$\Gamma_L(|p|) \sim O\left(\frac{1}{p^2}\right).$$

$$\langle P+q | j_\mu | P \rangle = 2i \int d^4p \Gamma_L(|p|) \frac{1}{(p + \frac{1}{2}P)^2 + m^2} (P+2p+q)_\mu \frac{1}{(p - \frac{1}{2}P)^2 + m^2} \frac{1}{(p + \frac{1}{2}P+q)^2 + m^2} \Gamma_L(|p+q/2|). \quad (4.27)$$

It is clear that if  $\Gamma_L(|p|) \approx O(p^2)^{-\epsilon}$  for  $p^2 \rightarrow -\infty$ , then  $F(q^2) \approx O(q^2)^{-1-\epsilon}$  for  $q^2 \rightarrow -\infty$ . Thus in our case  $F(q^2) \approx O(q^{-4})$  for  $q^2 \rightarrow -\infty$  whereas it diverges in the spacelike region.

## V. CONCLUSION

In the foregoing we investigated various aspects of phenomenological models exhibiting quark confinement, rising Regge trajectories, and asymptotic power decrease of form factors. In view of some successes of the nonrelativistic quark model, we considered first the nonrelativistic case of infinitely attractive, quark-confining potentials and showed how the resulting mass spectrum depends on the power of the potential. The effect of screening is required in order to endow the trajectories with physically acceptable analyticity properties without destroying appreciably their linear behavior. This case illustrated also approximately the properties of semirelativistic or fully relativistic models in the static limit. We then considered semirelativistic models for interactions consisting of harmoniclike and a Coulomb- or Yukawa-type part and showed that both parts are necessary in order to ensure a linear rise of the trajectories as well as a power decrease of bound-state form factors in the spacelike region. Finally we considered the fully relativistic (though suitably approximated) Bethe-Salpeter equation for appropriately generalized interaction kernels. The trajectories were found to rise more steeply, and the power decrease of form factors was found to occur in the timelike region. Thus these semirelativistic and relativistic models possess the defect of violation of unitarity in either the timelike or the spacelike region [a well-known property of (unmodified) Veneziano models possessing

Thus, in the presence of both interactions one expects the ground-state vertex function to behave like  $O(1/p^2)$  in the region  $p^2 \rightarrow -\infty$  but to diverge in the region  $p^2 \rightarrow +\infty$ .

The form factor  $F(q^2)$  of the Bethe-Salpeter bound state<sup>34</sup> is given by the coefficient of  $(P+2p)_\mu$  in the current matrix element

linear trajectories]. A characteristic feature of all models is that the rising trajectories require strong quark-gluon coupling.

In Sec. IV we considered also classes of quark-confining interactions, i.e., phenomenological gluon exchange propagators, particularly the dipole propagator, and investigated their behavior in the four-dimensional Euclidean space and in the so-called static limit. Strong infrared divergences are evident. E.g., if we calculated the quark self-energy in perturbation theory, we would find a result which diverges as the infrared cutoff  $\mu$  is allowed to approach zero (in contrast to quantum electrodynamics where the result depends on an ultraviolet cutoff).

For simplicity we considered throughout only the equations of motion of the quarks with effective gluon sources. An over-all self-consistent treatment would have required the simultaneous consideration of appropriate gluon equations; also, we did not deal with such problems as acausality, indefinite metric in the Hilbert space of state vectors, and the violation of spectral conditions which are known to arise in Abelian quark-confining gluon field theories of the type considered here (see, e.g., Refs. 8, 10, and 12, where these problems are alluded to or discussed). It is possible that few or none of these difficulties are encountered in the context of pure Yang-Mills theories which, moreover, are known to be characterized by long-range forces screened by vacuum polarization (see, e.g., the last paper mentioned in Ref. 32). However, in such theories explicit calculations are formidable.

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<sup>1</sup>K. Johnson, Phys. Rev. D **6**, 1101 (1972).

<sup>2</sup>The most ambitious example is the "SLAC shell."

See W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, *Phys. Rev. D* **11**, 1094 (1975).

<sup>3</sup>For a detailed exposition see H. Lipkin, in Proceedings of the Summer Institute on Particle Physics, SLAC Report No. SLAC-167, Vol. I, p. 239 (unpublished).

<sup>4</sup>P. Vinciarelli, *Nuovo Cimento Lett.* **4**, 905 (1972); *Nucl. Phys.* **B89**, 463 (1975).

<sup>5</sup>The dominant example is the "MIT bag." See A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, *Phys. Rev. D* **9**, 3471 (1974).

<sup>6</sup>C. M. Bender, J. E. Mandula, and G. S. Guralnik, *Phys. Rev. D* **11**, 409 (1975); **11**, 429 (1975); **11**, 438 (1975); *Phys. Rev. Lett.* **32**, 1467 (1974); J. E. Mandula, in *Particle Interactions at Very High Energies*, proceedings of the 2nd Louvain Summer Institute on Particle Interactions at very High Energies: Duality in Elementary Particle Physics, 1973, edited by David Speiser *et al.* (Plenum, New York, 1974), Vol. 2, p. 131.

<sup>7</sup>K. G. Wilson, *Phys. Rev. D* **10**, 2445 (1974); also Cornell University Report No. CLNS-271, 1974 (unpublished).

<sup>8</sup>R. J. Rivers, *Nucl. Phys.* **B65**, 29 (1973); also Imperial College report, 1974 (unpublished).

<sup>9</sup>P. Ramond, in *Particles and Fields—1974*, proceedings of the Meeting of the Division of Particles and Fields of the A.P.S., Williamsburg, Va., edited by Carl E. Carlson (A.I.P., New York, 1975), p. 437.

<sup>10</sup>S. Blaha, Cornell University Report No. CLNS-272, 1974 (unpublished); also *Phys. Rev. D* **10**, 4268 (1974).

<sup>11</sup>S. K. Kauffmann, *Nucl. Phys.* **B87**, 133 (1975).

<sup>12</sup>J. E. Kiskis, *Phys. Rev. D* **11**, 2178 (1975).

<sup>13</sup>A. T. Filippov, *Dubna Reports Nos. E2-6936*, 1973 (unpublished), *E2-7563*, 1974 (unpublished), and *E2-7929*, 1974 (unpublished).

<sup>14</sup>This problem would be similar to that discussed by S. D. Drell and K. Johnson, *Phys. Rev. D* **6**, 3248 (1972).

<sup>15</sup>S. D. Drell and T.-D. Lee, *Phys. Rev. D* **5**, 1738 (1972); T.-D. Lee, *ibid.* **6**, 1110 (1972).

<sup>16</sup>This is the nonrelativistic quark model. See, e.g., G. Morpurgo, *Physics (N.Y.)* **2**, 95 (1965), and R. H. Dalitz, rapporteur's talk, in *Proceedings of the Thirteenth International Conference on High Energy Physics*, Berkeley, California, 1966 (University of California Press, Berkeley, 1967), p. 215. Considerations based on small quark masses have been given by H. Leutwyler, lectures given at Adriatic Summer Meeting, Rovinj, 1973, Bern University report (unpublished). Generally the effective mass of quarks inside hadrons is supposed to be small, and their mass outside large.

<sup>17</sup>This case has been discussed by Wilson (Ref. 7) in the context of his lattice theory of quark confinement. It may be noted that many field Hamiltonians can be regarded as describing a continuously infinite number of uncoupled nonlinear oscillators if the gradient terms are treated as nonprimary, perturbative contributions. Diagonalization of this system then requires replacement of the continuum space by a lattice space so that the infinite number of oscillators becomes countable. Although the covariance of the continuum theory is generally lost in such a procedure, the resulting equations are similar to those discussed here and so can be solved in the domain of large couplings. See, e.g., L. I. Schiff, *Phys. Rev.* **92**, 766 (1953).

<sup>18</sup>R. P. Feynman, M. Kislinger, and F. Ravndal, *Phys. Rev. D* **3**, 2706 (1971).

<sup>19</sup>I. Montvay, *Phys. Lett.* **40B**, 476 (1972).

<sup>20</sup>J. Kogut, in *Particles and Fields—1974*, proceedings of the 1974 Williamsburg meeting of the Division of Particles and Fields of the American Physical Society, edited by Carl E. Carlson (A.I.P., New York, 1975), p. 458.

<sup>21</sup>Note that in the literature the WKB method used here is frequently called the "modified WKB method" in order to distinguish it from the original WKB method in which Eq. (2.6) is replaced by  $n + 1 \approx (1/\pi) \int_0^R Q(r) dr$ , where

$$Q(r) = [k^2 - l(l+1)/r^2 - V(r)]^{1/2}$$

and  $R$  is determined by  $Q(r) = 0$ . See N. Fröman and P. O. Fröman, *The JWKB Approximation* (North-Holland, Amsterdam, 1965), p. 131; E. C. Titchmarsh, *Eigenfunction Expansions* (Clarendon Press, Oxford, 1962), Vol. I, p. 151.

<sup>22</sup>See, e.g., H. H. Aly, K. Schilcher, and H. J. W. Müller, *Nucl. Phys.* **B3**, 401 (1967).

<sup>23</sup>H. Cheng and T. T. Wu, *Phys. Rev. D* **5**, 3189 (1972); H. J. W. Müller-Kirsten and N. Vahedi-Faridi, *J. Math. Phys.* **14**, 1291 (1973).

<sup>24</sup>H. J. W. Müller, *J. Math. Phys.* **11**, 355 (1970).

<sup>25</sup>We consider the spin- $\frac{1}{2}$  case briefly in order to get an estimate of the deviations from the spinless case.

<sup>26</sup>For a similar calculation see H. J. W. Müller and K. Schilcher, *J. Math. Phys.* **9**, 255 (1968).

<sup>27</sup>Reexpressing the four-dimensional Euclidean D'Alembertian  $\square^2 = \partial_\mu \partial^\mu$  in spherical coordinates defined by

$$x_1 = r \sin \psi \sin \phi \sin \theta, \quad x_2 = r \sin \psi \sin \phi \cos \theta,$$

$$x_3 = r \sin \psi \cos \phi, \quad x_4 = r \cos \psi$$

$$(0 \leq \psi, \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq r < \infty),$$

one obtains

$$\begin{aligned} \square^2 = & \frac{1}{r^3} \frac{\partial}{\partial r} \left( r^3 \frac{\partial}{\partial r} \right) \\ & + \frac{1}{r^2 \sin^2 \psi} \left[ \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} \right] \\ & + \frac{1}{r^2 \sin^2 \psi} \frac{\partial}{\partial \psi} \left( \sin^2 \psi \frac{\partial}{\partial \psi} \right). \end{aligned}$$

Separation of the variables in the usual way then leads to the form of the angular functions in (3.20) and the radial equation (3.21).

<sup>28</sup>Here this is simply an expression of the triangle diagram shown in Fig. 1 in which the bound-state wave function is evaluated in the limit of infinite mass of one of the constituents. Also the photon is treated like a scalar so that the kinematical factor in the photon vertex does not arise.

<sup>29</sup>Note that this relation follows from

$$\int \frac{1}{\xi^2} e^{-ik \cdot \xi} d^4 \xi = \frac{4\pi^2}{k^2}$$

for  $k^2 \neq 0$ , and must therefore be understood in the distribution-theoretical sense at  $(k - k')^2 = 0$ . Physically one imagines  $k^2$  accompanied by a small but nonzero

quantity  $(k^2 - i\epsilon)$ ; then this difficulty does not arise.

<sup>30</sup>G. E. Hite and H. J. W. Müller-Kirsten, *Nuovo Cimento* **21A**, 351 (1974); *Phys. Rev. D* **9**, 1074 (1974).

<sup>31</sup>M. K. Sudaresan and P. J. S. Watson, *Ann. Phys.* (N.Y.) **59**, 375 (1970).

<sup>32</sup>J. Schwinger, *Phys. Rev.* **128**, 2425 (1962); A. Casher, J. Kogut, and L. Susskind, *Phys. Rev. Lett.* **31**, 792 (1973); J. Lowenstein and J. Swieca, *Ann. Phys.* (N.Y.) **68**, 172 (1971); J. Kogut and L. Susskind, *Phys. Rev. D* **9**, 501 (1974). Note that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} |x| e^{-\mu|x|} dx &= -\frac{1}{\pi} \frac{(k^2 - \mu^2)}{(k^2 + \mu^2)^2} \\ &\simeq -\frac{1}{\pi} \frac{1}{k^2} + O(\mu^2). \end{aligned}$$

<sup>33</sup>It seems important to observe the following. If  $\mu$  is taken equal to zero from the beginning, then the Fourier transform of, e.g.,  $|\vec{x}|$ , does not exist. Thus in order to ensure the existence of the propagator, the infrared cutoff  $\mu$  must be taken to be nonzero (at least until after integrations). This implies, of course, that the potential  $|\vec{x}|$  is damped at large distances by the factor

$e^{-\mu|\vec{x}|}$ . The question then arises: What is responsible for the screening (i.e., nonobservability) of single quarks? Is it the infrared cutoff or the modification of the potential  $|\vec{x}|$  due to vacuum polarization at large distances? In the dipole gluon model under discussion the total "charge"  $Q$  corresponding to "color" is proportional to

$$\int d\vec{s} \cdot \vec{\nabla} \square V(\vec{x})$$

(see, e.g., the second paper quoted in Ref. 10). Vacuum polarization (which has to be taken into account in a complete theory) is expected to have a damping effect on  $V$ , thus ensuring that  $Q=0$  when the surface of integration is allowed to go to infinity (implying that only neutral states exist). But this result is also obtained for the unmodified potential (i.e., without vacuum polarization effects) provided the limit  $\mu \rightarrow 0$  is taken after integration. Thus the answer to our question depends on whether one considers a complete field theory or only a phenomenological model.

<sup>34</sup>C. Alabiso and G. Schierholz, *Phys. Rev. D* **10**, 960 (1974).