PHYSICAL REVIEW D VOLUME 12, NUMBER 4 15 AUGUST 1975

Perturbation expansion around extended-particle states in quantum field theory*

J.-L. Gervais
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris

A. Jevicki and B. Sakita
The City College of The City University of New York, New York, New York 10031
(Received 21 April 1975)

The quantum mechanics of solitary-wave classical solutions of nonlinear wave equations is discussed in detail for the kink solution of two-dimensional ϕ^4 field theory. The formalism provides a natural interpretation of an extended particle, the *soliton*, for the classical kink. The perturbation theory around the extended particle is developed and used to calculate the radiative corrections for the mass of soliton up to one loop. The mass renormalization is discussed in detail to show that the mass counterterm to the nonsoliton sector also does the job for the present case, i.e. one-soliton sector. Although our formalism is not manifestly Lorentz-covariant, the Lorentz covariance is shown explicitly by calculating the soliton energy for a fixed momentum. The paper also contains the perturbation calculation of matrix element of ϕ fields between one-soliton states.

I. INTRODUCTION

Recently we proposed a method which deals with the quantum mechanics of classical solitary wave solutions of nonlinear field theories.[1] The method is a generalization of the collective coordinate method of many-body theory to quantum field theories,[2] and it has previously been applied to the study of strong-coupling theory in static models.[3] In the two-dimensional Φ^4 theory with the wrong sign of the mass term, a classical solution is the kink solution and the position of the kink is treated as a collective coordinate. The system is then considered as an interacting system of a particle with a field with constraint. Thus, the theory naturally provides the interpretation of the kink as an extended particle,[4,5] which we call the soliton.

The quantization of classical solutions of nonlinear field theories has also been discussed by Dashen, Hasslacher, and Neveu,[6] and by Goldstone and Jackiw.[7] The former authors developed an elegant field-theoretic generalization of the WKB method to this problem. As in the ordinary quantum mechanics, however, the precise correction to the approximation is hard to estimate by this method. The latter authors, on the other hand, developed a method which consists of a set of self-consistent equations for the matrix elements of the field between physical states. In this method it is in principle possible to calculate all the matrix elements as a power series of coupling constant by successive iterations. But it is difficult to obtain systematic expansion rules.

As has been emphasized in Ref. 1, it is straightforward in our formalism to develop a perturbation theory in terms of the method used in Ref. 3. We discuss this problem in detail in the present paper.

In order to make the paper self-contained and also to make the method more transparent, in Sec. II we briefly describe the method using the Hamiltonian formalism and rederive the main results of the previous paper.[8]

In Sec. III we will present a systematic perturbation expansion in the coupling constant. It is essentially the same as the strong-coupling expansion of Ref. 3 and our definition of the coupling constant as $1/\lambda^2$ is actually made in order to emphasize the similarity with the strong-coupling case. We derive the Feynman rules which can be used to perform perturbative calculations in the one-soliton sector to arbitrary orders in the coupling constant.

As the first example of such calculations we compute the corrections to the soliton energy, which is analogous to the computation of the corrections to the isobar energy levels in the static strong-coupling model. In the lowest order this energy is equal to $M_0 + P^2/2M_0$ and, since our approach is not manifestly relativistic, the important question is whether the Lorentz invariance is restored with the higher-order contributions. We show in Sec. IV that all tree-graph corrections sum up to give the correct Lorentz-invariant form for the soliton energy $E = (P^2 + M_0^2)^{1/2}$. Thus we see that the coupling-constant expansion is in the same time a nonrelativistic expansion and so the formalism is indeed Lorentz-invariant.

In Sec. V we perform one-loop computations and show that all ultraviolet divergences can be removed by renormalization. The remarkable fact is that the one-soliton sector is made finite by the same mass counterterms which renormalize the meson sector of our theory. The first two quantum corrections to the soliton energy which we calculated are of the form $\Delta M - (P^2/2M_0^2)\Delta M$, with ΔM being the same as that given by Dashen, Hasslacher, and Neveu. From this we see that one can consistently interpret ΔM as the first quantum correction to the soliton mass.

In Sec. VI we will make perturbative calculations of the ϕ field matrix elements and Green's functions in the one-soliton sector.

II. DESCRIPTION OF THE METHOD

The Lagrangian of the two-dimensional field theory we consider is

$$L=\int \pi\dot{\phi}\,dx - H(\pi,\phi),$$
$$H(\pi,\phi)=\frac{1}{2}\int\left(\pi^2+\phi'^2-\phi^2+\frac{1}{2\lambda^2}\phi^4+\frac{1}{2}\lambda^2\right)dx\ , \tag{2.1}$$

where $\phi'=\partial\phi/\partial x$. This field theory possesses the so-called kink solutions

$$\phi_0(x-X)\equiv\lambda\tanh\left(\frac{x-X}{\sqrt{2}}\right). \tag{2.2}$$

The parameter X indicates the kink position. The transition amplitude between initial and final states described by the wave functions Ψ_1 and Ψ_f is given by the following phase space functional integration:

$$S_{fi}=\int\cdots\int \mathfrak{D}\phi\,\mathfrak{D}\pi\, e^{i\int L dt}\,\Psi_f^*[\phi(+\infty,x)]\Psi_i[\phi(-\infty,x)]. \tag{2.3}$$

We then regard X as a dynamical variable, accordingly a function of t, and introduce a corresponding conjugate momentum $\tilde{P}$ through the following change of variables in the integration[9]:

$$\phi(t,x)=\phi_0(x-X(t))+\eta(t,x)\ ,$$
$$\pi(t,x)=\pi_0(x;\tilde{P}(t),X(t))+\zeta(t,x). \tag{2.4}$$

We extract X and $\tilde{P}$ out of ϕ and π by inserting the identity

$$\int\int \mathfrak{D}X\,\mathfrak{D}\tilde{P}\,\delta(F_1[X;\phi])\delta(F_2[\tilde{P};\pi,\phi])\frac{\partial F_1}{\partial X}\frac{\partial F_2}{\partial \tilde{P}}=1 \tag{2.5}$$

into (2.3) and find the forms of π_0, F_1, and F_2 such that

$$\int \pi\dot{\phi}\,dx=\tilde{P}\dot{X}+\int dx\,\zeta\dot{\eta} \tag{2.6}$$

so that ζ and η as well as $\tilde{P}$ and X are canonical conjugates to each other. The solution is

$$\pi_0=-\frac{\tilde{P}}{M_0[1+(1/M_0)\xi]}\frac{\partial\phi_0(x-X(t))}{\partial x}\ , \tag{2.7}$$

$$F_1=\int dx\frac{\partial\phi_0(x-X(t))}{\partial x}\phi(t,x)$$
$$=\int dx\frac{\partial\phi_0(x-X(t))}{\partial x}\eta(t,x)\ ,$$
$$F_2=-\frac{\tilde{P}}{[1+(1/M_0)\xi]}+\int dx\frac{\partial\phi_0(x-X(t))}{\partial x}\pi(t,x)$$
$$=\int dx\frac{\partial\phi_0(x-X(t))}{\partial x}\zeta(t,x)\ , \tag{2.8}$$

where M_0 is the bare mass of the kink,

$$M_0=\frac{2\sqrt{2}}{3}\lambda^2, \tag{2.9}$$

and

$$\xi(t)=\int dx\frac{\partial\phi_0(x-X(t))}{\partial x}\frac{\partial\eta(t,x)}{\partial x}. \tag{2.10}$$

The subsidiary conditions, $F_1=0$ and $F_2=0$, still contain the kink coordinate $X(t)$. We eliminate it from the subsidiary conditions using the kink fixed-coordinate system

$$\rho=x-X(t). \tag{2.11}$$

We define the canonical meson fields in this system by

$$\chi(t,\rho)=\eta(t,x),$$
$$\pi(t,\rho)=\zeta(t,x) \tag{2.12}$$

and the momentum by

$$P=\tilde{P}-\int\pi\chi'\,d\rho, \tag{2.13}$$

where $\chi'=\partial\chi/\partial\rho$. Although we used π in (2.12) we like to note that this π is not identical to the π appearing in the previous expressions [e.g., (2.1)].

After some straightforward calculations we obtain

$$S_{fi}=\int\cdots\int\mathfrak{D}X\,\mathfrak{D}P\,\mathfrak{D}\chi\,\mathfrak{D}\pi\,\delta\left(\int\phi_0'\chi\,d\rho\right)\delta\left(\int\phi_0'\pi\,d\rho\right)\exp\left(i\int L\,dt\right)\Psi_f^*[X(\infty),\chi]\,\Psi_i[X(-\infty),\chi] \tag{2.14}$$

and

$$L=P\dot{X}+\int\pi\dot{\chi}\,d\rho-H,$$
$$H=M_0+\frac{(P+\int\pi\chi'\,d\rho)^2}{2M_0[1+(1/M_0)\xi]^2}+\frac{1}{2}\int\left[\pi^2+\chi'^2+\left(\frac{3\phi_0^2}{\lambda^2}-1\right)\chi^2+\frac{2}{\lambda^2}\phi_0\chi^3+\frac{1}{2\lambda^2}\chi^4\right]d\rho, \tag{2.15}$$

which is the main result of Ref. 1.

A soliton wave function with momentum p should be a function of X only and is described by

$$\psi_p[X] = e^{ipX} = \langle X|p\rangle . \tag{2.16}$$

If one restricts the initial and final state to a soliton state, the corresponding operator expression of (2.14) would be $\langle p|S|p\rangle$. One would also be interested in the time-ordered Green's function such as

$$\langle p'|T(\phi(x)\phi(y)\cdots)|p\rangle .$$

This can be obtained from (2.14) by inserting corresponding ϕ's into the integrand and using (2.4), (2.11), and (2.12). The integration over χ and π can be done by a standard perturbation method which will be discussed in the following section. The resulting expression is then a functional integration over X and P, for which we shall use a corresponding operator method.

III. PERTURBATION THEORY AND FEYNMAN RULES

Based on the formalism described in Sec. II, one can formulate a systematic perturbative theory in powers of the coupling constant. In our notation, the expansion parameter is actually $1/\lambda^2$, and since it is a dimensional parameter, the expansion is valid for small ratio m^2/λ^2. We will present in this section the derivation of Feynman rules. Using these rules, one can then make perturbative computations of energy, matrix elements, and Green's functions to arbitrary orders in the coupling constant.

Let us consider the generating functional

$$Z[J,K] = \int \mathfrak{D}\chi\, \mathfrak{D}\pi\, \delta\Big(\int \phi_0'\chi\Big)\delta\Big(\int \phi_0'\pi\Big) \times \exp\Big\{i\int dt\Big[\int d\rho(\pi\dot\chi + J\chi + K\pi) - H\Big]\Big\}, \tag{3.1}$$

where H is the total Hamiltonian given by Eq. (2.15) and J and K are external sources. We separate the Hamiltonian into a free χ-field part and an interaction part given by

$$H_0 = \int d\rho\Big[\frac{1}{2}\pi^2 + \frac{1}{2}\chi'^2 - \frac{1}{2}\Big(1-\frac{3\phi_0^2}{\lambda^2}\Big)\chi^2\Big], \tag{3.2}$$

$$H' = \frac{(P+\int d\rho\, \pi\chi')^2}{2M_0(1+\xi/M_0)^2} - \int d\rho\Big(\frac{\phi_0}{\lambda^2}\chi^3 + \frac{1}{4\lambda^2}\chi^4\Big), \tag{3.3}$$

where $\xi = \int d\rho\, \phi_0'\chi'$. Then the generating functional can be written in the form

$$Z[J,K] = \exp\Big[-i\int dt\, H'\Big(\frac{1}{i}\frac{\delta}{\delta J}, \frac{1}{i}\frac{\delta}{\delta K}\Big)\Big] Z_0[J,K], \tag{3.4}$$

where $Z_0[J,K]$ is the free generating functional

$$Z_0[J,K] = \int \mathfrak{D}\chi\, \mathfrak{D}\pi\, \delta\Big(\int \phi_0'\chi\Big)\delta\Big(\int \phi_0'\pi\Big) \times \exp\Big\{i\int dt\, d\rho\Big[\pi\dot\chi - \frac{1}{2}\pi^2 - \frac{1}{2}\chi'^2 - \frac{1}{2}\Big(1-\frac{3\phi_0^2}{\lambda^2}\Big)\chi^2 + J\chi + K\pi\Big]\Big\}. \tag{3.5}$$

This quadratic functional integral can easily be evaluated by expanding the fields χ and π in terms of eigenfunctions ψ_n which are solutions of the eigenequation

$$\Omega^2\psi_n \equiv \Big(-\frac{d^2}{d\rho^2} - 1 + \frac{3\phi_0^2}{\lambda^2}\Big)\psi_n = \omega_n^2\psi_n . \tag{3.6}$$

There are two discreet eigenvalues for $n=0$ and $n=1$, and the $\omega_0=0$ eigenfunction is just $(1/\sqrt{M_0})\phi_0'$. There is also a continuous spectrum for $\omega_k^2 = k^2+2$ and the normalized scattering eigenfunctions are

$$\psi_k = \frac{1}{N_k} e^{ik\rho} \times [3\tanh^2(\rho/\sqrt{2}) - 3ik\sqrt{2}\tanh(\rho/\sqrt{2}) - 1 - 2k^2], \tag{3.7}$$

$$N_k^2 = 2L(k^2+2)(2k^2+1) - 12\sqrt{2}(k^2+1).$$

Here L is the length of the box, since we use the box normalization and periodic boundary conditions. Introducing the notation

$$\begin{aligned}\chi_n(t) &= \int d\rho\, \chi(t,\rho)\psi_n(\rho) \equiv (\chi, \psi_n),\\ \pi_n(t) &= \int d\rho\, \pi(t,\rho)\psi_n(\rho) \equiv (\pi, \psi_n)\end{aligned} \tag{3.8}$$

we have

$$\begin{aligned}\chi(t,\rho) &= \sum_n \chi_n(t)\psi_n^*(\rho),\\ \pi(t,\rho) &= \sum_n \pi_n(t)\psi_n^*(\rho)\end{aligned} \tag{3.9}$$

and the π integral is now

$$\int \prod_n \mathfrak{D}\pi_n\, \delta(\pi_0)\exp\Big\{i\int dt \sum_n [-\tfrac{1}{2}\pi_n^*\pi_n + \pi_n^*(\dot\chi_n + K_n)]\Big\} = \exp\Big[i\int dt\, \tfrac{1}{2}{\sum_n}'(\dot\chi_n^* + K_n^*)(\dot\chi_n + K_n)\Big]. \tag{3.10}$$

Here the sum $\sum'$ is such that the zero-frequency mode $(n=0)$ is omitted.

Next, one has the χ integral

$$\int \prod_n \mathfrak{D}\chi_n \,\delta(\chi_0) \exp\left\{ i \int \sum_n \left[\tfrac{1}{2}\chi_n^* i G_n^{-1}\chi_n + (J_n - \dot{K}_n)\chi_n^*\right]\right\}, \tag{3.11}$$

where

$$iG_n^{-1} = (-\partial_t^2 - \omega_n^2 + i\epsilon)\delta(t-t'), \tag{3.12}$$

and the answer is

$$\exp\left[i \int \tfrac{1}{2} \sum_n{}' (J_n^* - \dot{K}_n^*) i G_n (J_n - \dot{K}_n)\right]. \tag{3.13}$$

Now in the (ρ, t) representation the Green's function is given by

$$G(t-t';\rho\rho') = \sum_n{}' \psi_n(\rho) \int \frac{d\omega}{2\pi} e^{i\omega(t-t')} \frac{i}{\omega^2 - \omega_n^2 + i\epsilon} \psi_n^*(\rho'), \tag{3.14}$$

defining

$$\Delta(t-t';\rho\rho') = -i\delta(t-t') \sum_n{}' \psi_n(\rho)\psi_n^*(\rho'). \tag{3.15}$$

We can write the final form for the generating functional

$$Z_0[J,K] = \exp\Big(-\int dt\, d\rho \int dt'\, d\rho' \,\{\tfrac{1}{2}[J(t,\rho) - \dot{K}(t,\rho)]G(t-t';\rho\rho')[J(t',\rho') - \dot{K}(t',\rho')] + \tfrac{1}{2}K(t,\rho)\Delta(t-t';\rho\rho')K(t',\rho')\}\Big). \tag{3.16}$$

From this free generating functional one can deduce the Feynman propagators. We see that there are three types of propagators, i.e., by differentiating with respect to sources J and K we get the χ-χ, χ-π, and the π-π propagators, respectively:

$$\frac{1}{i}\frac{\delta}{\delta J(t,\rho)}\frac{1}{i}\frac{\delta}{\delta J(t',\rho')} Z_0[J,K]\bigg|_{J=K=0} = G(t-t';\rho\rho'), \tag{3.17}$$

$$\frac{1}{i}\frac{\delta}{\delta J(t,\rho)}\frac{1}{i}\frac{\delta}{\delta K(t',\rho')} Z_0[J,K]\bigg|_{J=K=0} = \partial_{t'} G(t-t';\rho\rho'), \tag{3.18}$$

$$\frac{1}{i}\frac{\delta}{\delta K(t,\rho)}\frac{1}{i}\frac{\delta}{\delta K(t',\rho')} Z_0[J,K]\bigg|_{J=K=0} = \partial_t \partial_{t'} G(t-t';\rho\rho') + \Delta(t-t';\rho\rho'). \tag{3.19}$$

Their graphical representation is given in Fig. 1.

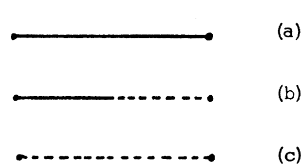


FIG. 1. Meson propagators in the one-soliton sector: (a) χ-χ, (b) χ-π, and (c) π-π.

It is important to note that since in (3.14) and (3.15) the zero frequency mode is excluded, these propagators avoid the infrared divergences associated with it. This is the consequence of the subsidiary conditions, i.e., δ-function conditions in (3.1).

The vertices of our perturbation theory are determined by the interaction part H'. Besides the ordinary vertices $(\phi_0/\lambda^2)\chi^3$ and $(1/4\lambda^2)\chi^4$ which are represented by Fig. 2(a) and Fig. 2(b), respectively, we have an infinite series of vertices coming from the first term in H':

$$\frac{(P + \int \pi\chi')^2}{2M_0(1+\xi/M_0)^2}\,.$$

Since ξ/M_0 is of the order $1/\lambda$ our perturbation expansion will be in the powers of $1/\lambda$. Expanding $1/(1+\xi/M_0)^2$ one gets the first set of vertices pro-

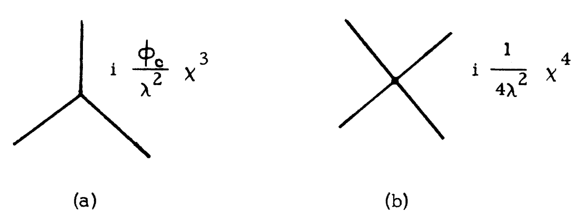


FIG. 2. Meson field vertices.

portional to P^2,

$$-\frac{P^2}{2M_0}\left(-2\frac{\xi}{M_0}+3\frac{\xi^2}{M_0^2}-4\frac{\xi^3}{M_0^3}+\cdots\right), \qquad (3.20)$$

which are of the order of $1/\lambda^3, 1/\lambda^4, \ldots$ successively and are graphically represented by Fig. 3(a).

The second set of vertices is proportional to P and is given by

$$-\frac{P}{M_0}\int \pi\chi'\left(1-2\frac{\xi}{M_0}+3\frac{\xi^2}{M_0^2}-4\frac{\xi^4}{M_0^3}+\cdots\right). \qquad (3.21)$$

The first two terms are represented by the graphs in Fig. 3(b).

It is important to observe here that these vertices are local in time but nonlocal in the space variable.

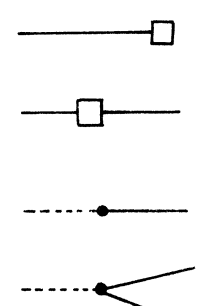
It is trivial to generalize these Feynman rules to arbitrary two-dimensional field theory described by the $\mathcal{L}=\frac{1}{2}(\partial\phi)^2-U(\phi)$, which has a classical solitary wave solution $\phi_0(x)$. Then the propagators have the same forms as those given by Eq. (3.14) and Eq. (3.15), but now with ψ_n and ω_n^2 obtained from the eigenequation

$$\left[-\frac{d^2}{d^2\rho}+U''(\phi_0)\right]\psi_n(\rho)=\omega_n^2\psi_n. \qquad (3.22)$$

The meson field vertices are given by the cubic and higher terms in the expansion of the potential $U(\phi)$,

$$U(\phi_0+\chi)=\sum_{l=0}\frac{1}{l!}\chi^l U^{(l)}(\phi_0), \qquad (3.23)$$

and depend on the specific form of the potential. Finally, we observe that the meson-soliton vertices remain the same as those given by Eqs. (3.20) and (3.21) and are thus independent of the form of $U(\phi)$.



(a) $-i\frac{P^2}{M_0^2}\int(\phi_0'',\chi)dt$

$-i\frac{3}{2}\frac{P^2}{M_0^3}\int dt(\phi_0'',\chi)^2$

(b) $-i\frac{P}{M_0}\int(\pi,\chi')dt$

$-2i\frac{P}{M_0^2}\int dt(\pi,\chi')(\phi_0'',\chi)$

FIG. 3. Meson-soliton vertices.

IV. TREE DIAGRAMS AND LORENTZ INVARIANCE

The separation of soliton degrees of freedom described in the Introduction is obviously not Lorentz-invariant. The free soliton Hamiltonian has a Galilei-invariant form so that in the leading orders the soliton energy is $E(P)=M_0+P^2/2M_0$. These are just the first two terms of the relativistic expansion:

$$E(P)=(P^2+M_0^2)^{1/2}$$
$$=M_0+\frac{1}{2}\frac{P^2}{M_0}-\frac{1}{4}\frac{P^4}{M_0^3}+\frac{1}{16}\frac{P^6}{M_0^5}+\cdots. \qquad (4.1)$$

This expansion in P^2 is at the same time an expansion in $1/\lambda^2$ so it is obvious that the nonrelativistic treatment of the soliton is connected with the nature of our weak-coupling perturbation expansion. Thus it looks very appealing to us that the Lorentz-invariant form for the energy is recovered with the higher-order calculation.

One defines an effective soliton Hamiltonian given by

$$\exp\left[-i\int dt\, H_{\rm eff}(P)\right]=\int \mathfrak{D}\chi\,\mathfrak{D}\pi\,\delta\left(\int\phi_0'\chi\right)\delta\left(\int\phi_0'\pi\right)\times\exp\left\{i\int\left[\int\pi\dot\chi\,d\rho-H\right]\right\}, \qquad (4.2)$$

where

$$H=M_0+\frac{(P+\int\pi\chi'\,d\rho)^2}{2M_0[1+(1/M_0)\xi]^2}+\frac{1}{2}\int d\rho\left[\pi^2+\chi'^2-\left(1-\frac{3\phi_0^2}{\lambda^2}\right)\chi^2+\frac{2}{\lambda^2}\phi_0\chi^3+\frac{1}{2\lambda^2}\chi^4\right], \qquad (4.3)$$

which can then be computed perturbatively. If we are only interested in calculations where the soliton momentum is conserved as in the case of soliton energy given by

$$\langle p|S|p\rangle=\exp\left[-i\int dt\,E(P)\right], \qquad (4.4)$$

it is enough to treat P as a constant.

In this section we calculate the corrections to the soliton energy $M_0+P^2/2M_0$, keeping only the leading term in $1/\lambda^2$ expansion for a given power of P^2. It is easily seen that these corrections come from the connected tree graphs only. We will prove that summing all these contributions, one indeed recovers the Lorentz-invariant form. Besides these, the computations are to demonstrate how one can perform explicit calculations using the complicated Feynman rules derived in the previous section.

We start with the first tree-diagram contribution which is proportional to P^4 and is represented graphically in Fig. 4. It gives the following expression for the energy:

$$-i\Delta E_4 = \frac{1}{2}\left(-i\,\frac{P^2}{M_0^2}\right)^2 \times \int d\tau \int d\rho\, d\rho'\, \phi_0''(\rho) G(\tau;\rho\rho')\phi_0''(\rho'). \tag{4.5}$$

Direct calculation of this expression, using the explicit forms for ψ_n and ϕ_0, would lead to a complicated integral, although it still can be evaluated analytically. Here, instead, we use the identity $d/d\rho = \frac{1}{2}[\rho, \Omega^2]$, with $\Omega^2 = -\partial^2 + U''(\phi_0)$, to show that

$$(\phi_0'', \psi_n) = -\tfrac{1}{2}(\phi_0', \rho\,\psi_n)\omega_n^2 \tag{4.6}$$

and, accordingly ($\tilde{G}$ is the Fourier transform of G),

$$\int d\rho\, \phi_0''(\rho)\tilde{G}(0;\rho\rho') = \tfrac{1}{2} i\rho'\phi_0'(\rho'). \tag{4.7}$$

FIG. 4. P^4 power tree graph contributing to the soliton energy.

Thus one obtains

$$\Delta E_4 = \frac{1}{4}\,\frac{P^4}{M_0^4}\int d\rho\, \phi_0'\rho\phi_0'' = -\frac{1}{8}\,\frac{P^4}{M_0^3} \tag{4.8}$$

because of $\int \phi_0'^2\, d\rho = M_0$.

This result encourages us to go on to the next contribution which is proportional to P^6. There are four diagrams shown in Fig. 5 and the expressions are more complicated. We start with the first one [Fig. 5(a)], which gives

$$-i\Delta E_a = (-i)^4\left(\frac{P^2}{M_0^2}\right)^3 \int d\rho_1\, \phi_0''(\rho_1)\int d\rho_2\, \phi_0''(\rho_2)\int d\rho_3\, \phi_0''(\rho_3)\int d\rho\, \frac{\phi_0(\rho)}{\lambda^2} \times \int d\tau_1 G(\tau_1;\rho_1\rho)\int d\tau_2 G(\tau_2;\rho_2\rho)\int d\tau_3 G(\tau_3;\rho_3\rho). \tag{4.9}$$

Using the relation (4.7) we obtain

$$\Delta E_a = \frac{1}{8}\left(\frac{P^2}{M_0^2}\right)^3 \frac{1}{\lambda^2}\int (\rho\phi_0')^3\phi_0\, d\rho. \tag{4.10}$$

After rearranging the integral, we end up with

$$\Delta E_a = \frac{1}{16}\,\frac{P^6}{M_0^5}\left[2(\psi_0', \rho^2\psi_0') - 1\right]. \tag{4.11}$$

The contribution of Fig. 5(b) is given by

$$-i\Delta E_b = (-i)^3\frac{3}{2}\,\frac{P^2}{M_0^3}\left[\frac{P^2}{M_0^2}\int \phi_0''(\rho)\int G(\tau;\rho\rho')\phi_0''(\rho')\, d\tau\, d\rho\, d\rho'\right]^2, \tag{4.12}$$

which can be calculated in a similar manner to that used before to find

$$\Delta E_b = \frac{1}{16}\,\frac{3}{2}\,\frac{P^6}{M_0^5}. \tag{4.13}$$

The contribution from Fig. 5(c) is given by

$$-i\Delta E_c = \frac{1}{2}\,\frac{P^6}{M_0^6}\int d\tau\, d\rho\, \phi_0''(\rho)\int d\rho_1\, d\rho_2\, \partial_{\rho_1} G(\tau;\rho\rho_1)\int dt\, \Delta(t;\rho_1\rho_2)\int d\tau'\, d\rho'\, \phi_0''(\rho')\partial_{\rho_2} G(\tau';\rho_2\rho') \tag{4.14}$$

since the $\partial_t\partial_{t'}G(t-t';\rho\rho')$ does not contribute. After a similar calculation to those performed before we obtain

$$\Delta E_c = \frac{1}{16}\,\frac{P^6}{M_0^5}\left[2(\psi_0, \rho\partial^2\rho\,\psi_0) + \tfrac{1}{2}\right]. \tag{4.15}$$

The last graph [Fig. 5(d)] is equal to zero. Therefore, the total contribution to the energy proportional to P^6 is

$$\Delta E_6 = \Delta E_a + \Delta E_b + \Delta E_c = \frac{1}{16}\,\frac{P^6}{M_0^5}\left[2(\psi_0', \rho^2\psi_0') + 2(\psi_0, \rho\partial^2\rho\,\psi_0) + 1\right] = \frac{1}{16}\,\frac{P^6}{M_0^5}, \tag{4.16}$$

which is just the fourth term in the expansion (4.1).

These calculations lead us to expect that by summing all tree graphs, one would recover the relativistic form for the energy. It is more and more difficult to compute further terms in these series and we will now use a more powerful method to sum all tree diagrams contributing to the soliton energy. Since the total energy is given by the path integral form

$$e^{-i\int dt\, E(P)}=\int \mathfrak{D}\chi\, \mathfrak{D}\pi\, \delta\Big(\int \phi_0{}'\chi\Big)\delta\Big(\int \phi_0{}'\pi\Big)\times e^{iS[\chi,\pi,P]}, \tag{4.17}$$

it is equal to the sum of all connected vacuum diagrams[10] with an external source P present. It has an expansion in number of loops $E(P)=E_0(P)+E_1(P)+\cdots$. Then the sum of all tree diagrams $E_0(P)$ can be evaluated by the stationary phase method and it is equal to the value of the action $S[\chi,\pi,P]$ evaluated with π and χ being the solutions of the equations of motion. Here, a little more care is needed because of the δ-function constraints in Eq. (4.17). We can rewrite the constraints using Lagrange multipliers $\bar{\lambda}(t)$ and $\bar{\nu}(t)$,

$$\prod \delta\Big(\int \phi_0{}'(\rho)\chi(t,\rho)\,d\rho\Big)=\int \mathfrak{D}\bar{\lambda}\exp\Big[i\int dt\,d\rho\,\bar{\lambda}(t)\phi_0{}'(\rho)\chi(t,\rho)\Big], \tag{4.18}$$

$$\prod_t \delta\Big(\int \phi_0{}'(\rho)\pi(t,\rho)\,d\rho\Big)=\int \mathfrak{D}\bar{\nu}\exp\Big[i\int dt\,d\rho\,\bar{\nu}(t)\phi_0{}'(\rho)\pi(t,\rho)\Big],$$

so that the total action is given by

$$S_{\rm tot}[\pi,\chi,P,\bar{\lambda},\bar{\nu}]=S[\chi,\pi,P]+\int[\bar{\lambda}\phi_0{}'\chi+\bar{\nu}\phi_0{}'\pi]\,dt\,d\rho. \tag{4.19}$$

The corresponding equations of motion are

$$\int d\rho\,\phi_0{}'(\rho)\chi(t,\rho)=0,\qquad \int d\rho\,\phi_0{}'(\rho)\pi(t,\rho)=0, \tag{4.20}$$

$$\pi+\chi'\frac{P+\int\pi\chi'}{M_0(1+\xi/M_0)^2}+\bar{\nu}\phi_0{}'=0, \tag{4.21}$$

$$-\chi''-\Big(1-\frac{3\phi_0{}^2}{\lambda^2}\Big)\chi+\frac{3\phi_0}{\lambda^2}\chi^2+\frac{1}{\lambda^2}\chi^3-\pi'\frac{P+\int\pi\chi'}{M_0(1+\xi/M_0)^2}+\phi_0{}''\frac{(P+\int\pi\chi')^2}{M_0{}^2(1+\xi/M_0)^3}+\bar{\lambda}\phi_0{}'=0. \tag{4.22}$$

In these equations we omitted the terms which contain time derivatives of fields since we are looking for the time-independent solutions for a constant P. One can eliminate $\bar{\nu}$ from (4.21) and one can obtain

$$\pi+\chi'\frac{P+\int\pi\chi'}{M_0(1+\xi/M_0)^2}-\phi_0{}'\frac{\xi}{M_0}\,\frac{P+\int\pi\chi'}{M_0(1+\xi/M_0)^2}=0. \tag{4.23}$$

Next, multiplying this by χ' and integrating, we find that

$$\frac{P}{M_0+2\xi+\int\chi'^2}=\frac{P+\int\pi\chi'}{M_0(1+\xi/M_0)^2}\,. \tag{4.24}$$

Now, using this relation and eliminating π from (4.22), we end up with a single nonlocal integro-differential equation

$$-\chi''-\Big(1-\frac{3\phi_0{}^2}{\lambda^2}\Big)\chi+\frac{3\phi_0}{\lambda^2}\chi^2+\frac{1}{\lambda^2}\chi^3+(\chi''+\phi_0{}'')\frac{P^2}{(M_0+2\xi+\int\chi'^2)^2}+\bar{\lambda}\phi_0{}'=0. \tag{4.25}$$

This looks very complicated to solve, but, after the shift $\chi=-\phi_0+\varphi_0$ we get a simpler equation,

$$-\varphi_0{}''-\varphi_0+\frac{1}{\lambda^2}\varphi_0{}^3+\frac{P^2}{(\int\varphi_0{}'^2\,d\rho)^2}\varphi_0{}''+\bar{\lambda}\phi_0{}'=0. \tag{4.26}$$

Denoting

$$a^2=\frac{P^2}{(\int\varphi_0{}'^2\,d\rho)^2}\,, \tag{4.27}$$

we have the solution of this equation in the form

$$\varphi_0(\rho)=\phi_0(\rho/(1-a^2)^{1/2}),\qquad \bar{\lambda}=0. \tag{4.28}$$

Next, substituting this solution back into (4.26) we get the relation $1/(1-a^2)=1+P^2/M_0{}^2$. Now it is easy to find the solution for π:

$$\pi_0(\rho)=a\Big[\phi_0{}'(\rho)\frac{\int\phi_0{}'\varphi_0{}'}{M_0}-\varphi_0{}'(\rho)\Big]. \tag{4.29}$$

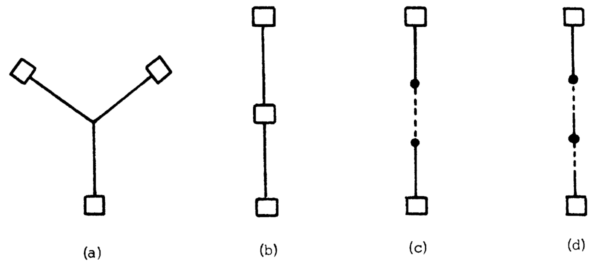


FIG. 5. P^6 power tree graphs contributing to the soliton energy.

Thus the sum of all connected tree diagrams contributing to the soliton energy is given by the value of action (4.19) evaluated with this time-independent solution for χ and π:

$$E(P) = -S[-\phi_0 + \varphi_0, \pi_0, P]\,. \tag{4.30}$$

After some computation we found $E(P) = (P^2 + M_0^{\,2})^{1/2}$.

V. ONE-LOOP CALCULATIONS AND RENORMALIZATION

In this section we discuss the one-loop contributions to the soliton energy. Because of the logarithmic divergences appearing in the calculations, one has to face the problem of renormalization in the one-soliton sector. It is well known that these two-dimensional field theories require only a mass renormalization, and thus in the Lagrangian one has the infinite bare mass $m_0^{\,2}$ instead of the finite parameter m^2. The remarkable fact is that the mass counterterm $\delta m^2 = m_0^{\,2} - m^2$, which makes the nonsoliton sector of our field theory finite, also renormalizes the one-soliton sector. Thus, one needs no new counterterm to cancel all the divergences which appear. This is similar to the situation with spontaneously broken field theories, where the divergence structure of a renormalizable theory is not affected by the occurrence of spontaneous symmetry breakdown, so that the same counterterms which renormalize the theory with the unbroken vacuum are enough to renormalize the corresponding theory with the spontaneously broken vacuum.[11]

Thus, if we are interested in, at most, one-loop calculations, then the Lagrangian which appears in the functional integral expression for the S-matrix element in the one-soliton sector contains the one-loop mass counterterm. To be specific, for the ϕ^4 theory the Lagrangian is given by

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}(m^2 + \delta m^2)(\phi^2 - \lambda^2) - (1/4\lambda^2)(\phi^4 - \lambda^4), \tag{5.1}$$

with δm^2 given by

$$\delta m^2 = \frac{3}{2\pi\lambda^2}\int_0^{\Lambda^2} \frac{dk}{(k^2+2)^{1/2}} + O\left(\frac{1}{\lambda^4}\right) \tag{5.2}$$

obtained from the one-loop renormalization of the nonsoliton sector. Next, one can perform exactly the same canonical transformation as that described in the Introduction, with ϕ_0 still being the solution of the equation of motion without the mass counterterm. Then we end up with an additional term in the Hamiltonian of the form

$$H_{\delta m^2} = -\tfrac{1}{2}\delta m^2[(\phi_0^{\,2} - \lambda^2) + 2\phi_0\chi + \chi^2]\,, \tag{5.3}$$

which will be sufficient to cancel out all the divergences which appear in the one-soliton–sector calculations. Thus, one should add the additional graphs, Figs. 6(a) and 6(b), due to (5.3) to the Feynman rules described in Sec. III.

First, let us write down, just for completeness, the order $1/\lambda^2$ correction to the energy, which is given by the bubble diagram and is equivalent to the sum of zero-point energies:

$$\sum_n{}' \tfrac{1}{2}\omega_n\,. \tag{5.4}$$

This expression is both linearly and logarithmically divergent, and the first divergence is cancelled by subtracting out the infinite vacuum energy. With the box normalization, this vacuum energy is proportional to the length of the box L. The second divergence is cancelled by the contribution coming from the mass counterterm $\frac{1}{2}\delta m^2(\phi_0^{\,2} - \lambda^2)$. The total expression

$$\Delta M = \sum_n{}' \tfrac{1}{2}\omega_n - \sum_n \tfrac{1}{2}\omega_n' - \tfrac{1}{2}\delta m^2 \int (\phi_0^{\,2} - \lambda^2)\,d\rho \tag{5.5}$$

is now finite and was evaluated in detail by Dashen, Hasslacher, and Neveu.[6] We simply quote here their derivation. The relation between ω_n and ω_n' for scattering states is given by

$$k_n + \frac{1}{L}\,\delta(k_n) = k_n' = \frac{2\pi n}{L} \tag{5.6}$$

and $\omega_n = (k_n^{\,2} + 2)^{1/2}$. $\delta(k_n)$ is the scattering phase shift of the wave equation (3.6) and its derivative is given by

$$\frac{d\delta(k)}{dk} = -\frac{12\sqrt{2}\,(k^2+1)}{2(k^2+2)(2k^2+1)}\,. \tag{5.7}$$

Using (5.6) and (5.7), one finds

$$\Delta M = \frac{1}{2}\,\omega_1 - \frac{3}{\pi\sqrt{2}} + \int \frac{dk}{2\pi}\,\frac{d\delta(k)}{dk}\,\frac{1}{2}\,\omega(k) - \frac{1}{2}\,\delta m^2 \int d\rho(\phi_0^{\,2} - \lambda^2). \tag{5.8}$$

We will now explicitly evaluate the next correction to the energy which is proportional to P^2 and of the order $1/\lambda^4$ in the coupling constant. Since $M_0 = \frac{2}{3}\sqrt{2}\,\lambda^2$, it is of the form $-(P^2/2M_0)(\Delta M'/M_0)$. If the interpretation of ΔM as the quantum correction to the soliton mass is correct, then $\Delta M'$ should be equal to ΔM. The loop diagrams which

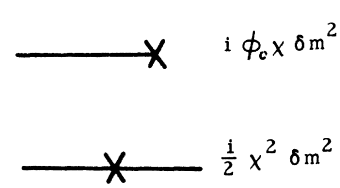


FIG. 6. Mass counterterm vertices.

contribute to this order are shown in Figs. 7(a), 7(b), and 7(c). Using the Feynman rules of Sec. III, one gets, corresponding respectively to the graphs 7(a), 7(b), and 7(c),

$$-3\int dt\,\frac{P^2}{M_0^2}\int d\rho\, d\rho'\,\phi_0''(\rho)\times\int d\tau\, G(\tau;\rho\rho')\,\frac{\phi_0(\rho')}{\lambda^2}\,G(0;\rho'\rho'),\tag{5.9}$$

$$-\frac{1}{2}\int dt\,\frac{P^2}{M_0^2}\int d\rho\, d\rho'\int d\tau[\Delta(\tau;\rho\rho')-2\partial_\tau^2 G(\tau;\rho\rho')]\times\partial\rho\,\partial\rho'\,G(\tau;\rho\rho'),\tag{5.10}$$

$$-i\,\frac{3}{2}\int dt\,\frac{P^2}{M_0^3}\int d\rho\, d\rho'\,\phi_0''(\rho)G(0;\rho\rho')\phi_0''(\rho')\,,\tag{5.11}$$

and the corresponding contributions to the mass $\Delta M'$ are

$$\Delta M_a=6{\sum_n}'(\phi_0'',\psi_n)\frac{1}{\omega_n^2}\int d\rho\,\psi_n^*(\rho)\,\frac{\phi_0(\rho)}{\lambda^2}\,G(0;\rho\rho),\tag{5.12}$$

$$\Delta M_b={\sum_{n,m}}'\frac{(\psi_n,\psi_m')(\psi_m,\psi_n')}{\omega_n+\omega_m}+{\sum_n}'\frac{(\psi_n',\psi_n')}{2\omega_n}+{\sum_n}'\frac{(\psi_n,\psi_0')(\psi_0,\psi_n')}{2\omega_n}\,,\tag{5.13}$$

$$\Delta M_c=3{\sum_n}'\frac{(\psi_n,\psi_0')(\psi_0,\psi_n')}{2\omega_n}\,.\tag{5.14}$$

As for the divergences in these expressions, one can see that only ΔM_a has a logarithmic divergence, while ΔM_b and ΔM_c are finite. There are no linear divergences, and that is encouraging, since we would have no way to remove them. The logarithmic divergence of ΔM_a is exactly cancelled by the contribution of graph Fig. 8, which is of the form

$$\int dt\,\frac{P^2}{M_0^2}\int d\rho\, d\rho'\,\phi_0''(\rho)\int d\tau\, G(\tau;\rho\rho')\delta m^2\phi_0(\rho').\tag{5.15}$$

Using the identity for $d/d\rho$ we can show that the corresponding contribution to the mass $\Delta M'$ is

$$\Delta M_r=-\tfrac{1}{2}\delta m^2\int d\rho\,(\phi_0^2-\lambda^2),\tag{5.16}$$

which is the same as that in relation (5.8).

Owing to the special nature of our Feynman rules, it is extremely difficult to evaluate these three expressions. If one writes down directly the corresponding integrals using the explicit form for the eigenfunctions, ψ_n, they are so complicated that it is not possible to evaluate them analytically.[12] Instead, we will first rearrange the expressions for ΔM_a and ΔM_b so that after evaluating some parts explicitly, we end up with the expression for $\Delta M'$ which is the same as that for ΔM.

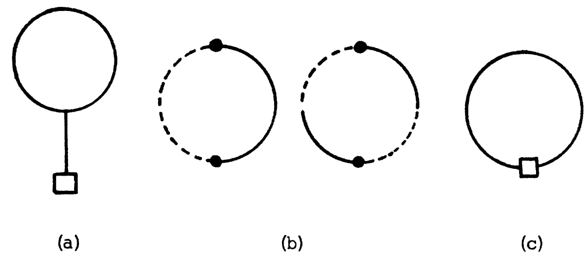


FIG. 7. One-loop graph proportional to P^2.

Let us start with ΔM_a, using the identity $d/d\rho=\frac{1}{2}[\rho,\Omega^2]$ and partially integrating to get

$$\Delta M_a=\frac{3}{2\lambda^2}\left\{-[\rho\phi_0 G(0;\rho\rho)]\Big|_{-L/2}^{L/2}+\int d\rho\,\phi_0^2[G(0;\rho\rho)+\rho G'(0;\rho\rho)]\right\}.\tag{5.17}$$

The first term is for large L, given by

$$-\frac{3}{\lambda^2}[\rho\phi_0^2(\rho)G(0;\rho\rho)]\Big|_{-L/2}^{L/2}={\sum_n}''\frac{-3}{2\omega_n}\,\frac{LR_n^2}{N_n^2}\,,\tag{5.18}$$

where

$$\frac{LR_n^2}{N_n^2}=L\psi_n(\tfrac{1}{2}L)\psi_n^*(\tfrac{1}{2}L)=\frac{2L(k_n^2+2)(2k_n^2+1)}{2L(k_n^2+2)(2k_n^2+1)-12\sqrt{2}(k_n^2+1)}\approx 1-\frac{1}{L}\,\frac{d\delta(k_n)}{dk_n}+O\!\left(\frac{1}{L^2}\right).\tag{5.19}$$

For the last step in (5.19) we used the explicit form of $d\delta(k)/dk$, i.e., Eq. (5.7). The states $n=0$ and $n=1$ do not contribute to the sum in (5.18) and the $\sum''$ denotes that the summation is over scattering states only. Next, using the identity

FIG. 8. Mass counterterm graph proportional to P^2.

$3\phi_0{}^2/\lambda^2 = \Omega^2 + \partial^2 + 1$, one obtains for the second term of (5.17)

$$\int d\rho \frac{3\phi_0{}^2}{\lambda^2} G(0;\rho\rho) = \sum_n{}' \frac{(\omega_n{}^2+1)}{2\omega_n} - \sum_n{}' \frac{(\psi_n{}', \psi_n{}')}{2\omega_n} \tag{5.20}$$

and after some calculations, the last term is found to be

$$\int d\rho \frac{3\phi_0{}^2}{\lambda^2} \rho G'(0;\rho\rho) = -\sum_n{}' \frac{\omega_n{}^2+1}{2\omega_n} + \sum_n{}'' \frac{2k_n{}^2+3}{2\omega_n}\frac{LR_n{}^2}{N_n{}^2} - \sum_n{}' \frac{(\psi_n{}', \psi_n{}')}{2\omega_n}. \tag{5.21}$$

Thus, the total expression for ΔM_a is given by

$$\Delta M_a = -\sum_n{}' \frac{(\psi_n{}', \psi_n{}')}{2\omega_n} + \sum_n{}'' \frac{k_n{}^2}{2\omega_n}\frac{LR_n{}^2}{N_n{}^2}. \tag{5.22}$$

Next, we will evaluate the complicated double sum which appears in the expression (5.13) for ΔM_b:

$$\sum_{n,m}{}' \frac{(\psi_n, \psi_m{}')(\psi_m, \psi_n{}')}{\omega_n+\omega_m}. \tag{5.23}$$

Using the relation

$$(\psi_n, \psi_m{}') = \tfrac{1}{2}(\omega_m{}^2 - \omega_n{}^2)(\psi_n, \rho\,\psi_m) + S_{nm}, \tag{5.24}$$

where

$$S_{nm} = \tfrac{1}{2} i (k_n + k_m) L \psi_n^*(\tfrac{1}{2}L)\psi_m(\tfrac{1}{2}L) = \tfrac{1}{2} i (k_n+k_m)\frac{(-1)^{n+m}R_n R_m}{N_n N_m}, \tag{5.25}$$

we write this double sum as

$$\sum_{n,m}{}'' \frac{S_{nm}(\psi_m, \psi_n{}')}{\omega_n+\omega_m} + \sum_{n,m}{}' \tfrac{1}{2}(\omega_m - \omega_n)(\psi_n, \rho\,\psi_m)(\psi_m, \psi_n{}'). \tag{5.26}$$

Defining $\bar{\psi}_n = \psi_n{}' - ik_n\psi_n$, one can write the first term as

$$-\sum_n{}'' \frac{k_n{}^2}{2\omega_n}\frac{LR_n{}^2}{N_n{}^2} + \sum_{n,m}{}'' \frac{S_{nm}(\psi_m, \bar{\psi}_n)}{\omega_n+\omega_m}. \tag{5.27}$$

In the Appendix, we show that the second part of this expression is equal to zero. Since the second term in (5.26) is logarithmically divergent, one has to cut off the number of modes in the summation. Then, using the completeness relation for the eigenfunctions, ψ_n, we can reduce it to a form

$$-4\sum_n{}' \frac{(\psi_n, \psi_0{}')(\psi_0, \psi_n{}')}{2\omega_n} + \sum_n{}' \tfrac{1}{2}\omega_n - \sum{}'' \tfrac{1}{2}\omega_n \frac{LR_n{}^2}{N_n{}^2} + \sum_{\substack{|n|\le N \\ |m|\ge N+1}}{}'' \omega_n(\psi_n, \rho\,\psi_m)(\psi_m, \psi_n{}'). \tag{5.28}$$

The last sum in this expression is evaluated in the Appendix to be $-3/\pi\sqrt{2}$. Then, using (5.20), one obtains the following total expression for the double sum (5.23):

$$\sum_{n,m}{}' \frac{(\psi_n, \psi_m{}')(\psi_m, \psi_n{}')}{\omega_n+\omega_m} = -\sum_n{}'' \frac{k_n{}^2}{2\omega_n}\frac{LR_n{}^2}{N_n{}^2} - 4\sum_n \frac{(\psi_n, \psi_0{}')(\psi_0, \psi_n{}')}{2\omega_n} + \frac{1}{2}\omega_1 + \int \frac{dk}{2\pi}\frac{d\delta}{dk}\frac{1}{2}\omega(k) - \frac{3}{\pi\sqrt{2}}. \tag{5.29}$$

Now, summing all the contributions to $\Delta M'$, we get

$$\Delta M' = \Delta M_a + \Delta M_b + \Delta M_c + \Delta M_r = \frac{1}{2}\omega_1 - \frac{3}{\pi\sqrt{2}} + \int \frac{dk}{2\pi}\frac{d\delta}{dk}\frac{1}{2}\omega(k) - \frac{\delta m^2}{2}\int(\phi_0{}^2 - \lambda^2),$$

which is identical to (5.8), so that $\Delta M' = \Delta M$. Thus we found that the first two quantum corrections to the soliton energy are given by $\Delta E = \Delta M - (p^2/2M_0)(\Delta M/M_0)$.

One can calculate the total one-loop correction, extending the method used in Sec. IV. We have to shift the fields χ and π by the classical fields found there and then keep only the quadratic part in the quantum fields which gives all the one-loop graph contributions. Then, performing a similar calculation to the one in this section, we expect to obtain the result $\Delta M/(P^2+M_0{}^2)^{1/2}$ so that the total energy in the one-loop approximation is

$$(P^2+M_0{}^2)^{1/2} + \frac{\Delta M}{(P^2+M_0{}^2)^{1/2}}.$$

VI. MESON FIELD GREEN'S FUNCTIONS IN ONE-SOLITON SECTOR

With the systematic perturbation theory developed in Sec. III, one can make perturbative calculations of other quantities besides the soliton energy. Of special interest are the calculations of Green's functions in the one-soliton sector. The ϕ field matrix elements between the one-soliton and many-meson states $\langle p'\{k_i\}|\phi|p\{l_i\}\rangle$ were first considered by Goldstone and Jackiw[7] in their treatment of these two-dimensional extended particle theories. They used the method of Kerman and Klein, assuming that the connected matrix elements between m and n mesons have an expansion in powers of λ^{-1} and that the leading order is λ^{1-m-n}. These assumptions can now easily be justified.

Let us start with the matrix element $\langle p'|\phi(0,x)|p\rangle$. Performing the canonical transformation described in the Introduction, we get

$$\langle p'|\phi(0,x)|p\rangle = \langle p'|\phi_0(x-X(0))|p\rangle + \langle p'|\chi(0,x-X(0))|p\rangle\,, \tag{6.1}$$

where X is the coordinate operator. The first term is of the order λ, while the second term gives, at most, λ^0 contributions. Inserting the identity, we can easily evaluate the leading term:

$$\langle p'|\phi_0(x-X(0))|p\rangle = \int dy\, e^{i(p-p')y}\,\phi_0(x-y). \tag{6.2}$$

This essentially classical part of the matrix element was the initial ansatz of Goldstone and Jackiw. Next, one can compute the first quantum correction coming from the second term in (6.2). From the path-integral representation of $\langle p'|\chi(0,x-X)|p\rangle$ we see that one first has to find the "vacuum" expectation value of $\chi(0,\rho)$ and then make the substitution $\rho=x-X$ to evaluate this operator between the soliton states. Thus the first quantum correction is given by the tadpole graph of order $O(\lambda^{-1})$ and we have

$$\langle p'|\chi(0,x-X)|p\rangle = \left\langle p'\left|\int d\rho'\,\tilde{G}(0;x-X,\rho')\frac{3\phi_0(\rho')}{\lambda^2}G(0;\rho'\rho')\right|p\right\rangle. \tag{6.3}$$

This contribution is again of the form

$$\int dy\, e^{i(p-p')y}f(x-y), \tag{6.4}$$

with

$$f(x-y) = \int d\rho\,\tilde{G}(0;x-y,\rho)\frac{3\phi_0(\rho)}{\lambda^2}G(0;\rho\rho). \tag{6.5}$$

It is logarithmically divergent and the divergence is cancelled out by the contribution coming from the mass counterterm $\delta m^2\phi_0\chi$ which also gives a tadpole graph of the order $O(\lambda^{-1})$. Finally, in order to see the connection with the result of Ref. 7, we observe that $f(x)$ satisfies

$$\left[-\frac{d^2}{dx^2}+U''(\phi_0)\right]f(x) = \frac{3\phi_0(x)}{\lambda^2}G(0;xx), \tag{6.6}$$

where we used the fact that $(\psi_0(x),\,\phi_0(x)G(0;xx))=0$.

Next, we will compute the leading term of the one-meson matrix element $\langle p'|\Phi|p;\omega_n\rangle$, where p is the total momentum of the soliton and meson and $\omega_n=(k^2+2)^{1/2}$ is the meson energy. This matrix element is equal to

$$\langle p'|\phi_0(x-X(0))|p;\omega_n\rangle + \langle p'|\chi(0,x-X(0))|p,\omega_n\rangle \tag{6.7}$$

and here the first classical term has no contribution to the leading order. To evaluate the second term we expand the field $\chi(\rho,t)$ in terms of the eigenfunctions ψ_n:

$$\chi(t,\rho)=q(t)\psi_1(\rho) + {\sum_n}''\frac{1}{(2\omega_n)^{1/2}}\left[a_n(t)\psi_n(\rho)+a_n^\dagger(t)\psi_n^*(\rho)\right], \tag{6.8}$$

where the $n=0$ mode is omitted because of the δ-function conditions in Eq. (2.14). Now the $n=1$ mode describes the internal soliton degree of freedom and not a quantum particle. Therefore, the soliton can have energetically excited states. The continuum modes correspond to the meson degrees of freedom since, neglecting the interaction a_n satisfies the equation $\ddot{a}_n(t)=-\omega_n{}^2a_n$, with $\omega_n{}^2=(k^2+2)^{1/2}$. Thus, in the first approximation the one-meson matrix element is

$$\left\langle p'\left|{\sum_n}''\frac{1}{(2\omega_n)^{1/2}}\left[\psi_n(x-X(0))a_n(0)+\psi_n^*(x-X(0))a_n^\dagger(0)\right]\right|p;\omega_n\right\rangle = \frac{1}{(2\omega_n)^{1/2}}\langle p'|\psi_n(x-X(0))|p;\omega_n\rangle \tag{6.9}$$

so that

$$\langle p'|\phi(0,x)|p;\omega_n\rangle = \int dy\frac{e^{i(p-p')y}}{(2\omega_n)^{1/2}}\,\psi_n(x-y) \tag{6.10}$$

is in agreement with the ansatz of Ref. 7.

The n-point Green's functions in the one-soliton sector can also be computed perturbatively. For simplicity we will calculate the two-point function and it is then trivial to generalize the result to the arbitrary n-point function. Let us consider the Fourier-transformed form:

$$G(p',k';p,k)=\int dt'\,dx'\,e^{i[\omega(k')t'-k'x']}\int dt\,dx\,e^{-i[\omega(k)t-kx]}\langle p'|T[\phi(t',x')\phi(t,x)]|p\rangle . \tag{6.11}$$

Making the canonical transformation in the operator form, the time-ordered product is equal to

$$T[\phi_0(x'-X(t'))\phi_0(x-X(t))]+T[\phi_0(x'-X(t'))\chi(t,x-X(t))]$$
$$+T[\chi(t',x'-X(t'))\phi_0(x-X(t))]+T[\chi(t',x'-X(t'))\chi(t,x-X(t))], \tag{6.12}$$

where X is now the coordinate operator. We will first evaluate the contribution from the first term which is of the order $O(\lambda^2)$, while the others give smaller-order contributions. The time-ordered product can be split into two parts,

$$\theta(t'-t)\phi_0(x'-X(t'))\phi_0(x-X(t))+\theta(t-t')\phi_0(x-X(t))\phi_0(x'-X(t')) . \tag{6.13}$$

Inserting the identity $\int dy|y,t\rangle\langle y,t|=1$ into the matrix element of the first part,

$$\langle p'|\theta(t'-t)\phi_0(x'-X(t'))\phi_0(x-X(t))|p\rangle , \tag{6.14}$$

for both t' and t we see that it is equal to

$$\int dy'\,dy\,e^{-i[p'y'-E(p')t']}\phi_0(x'-y')\langle y',t'|y,t\rangle\,e^{i[py-E(p)t]}\phi_0(x-y), \tag{6.15}$$

and after translation of the time and space variables, we get the following contribution to the Green's function:

$$(2\pi)^2\delta(E(p')+\omega(k')-E(p)-\omega(k))\delta(p'+k'-p-k)$$
$$\times\int dx_1\,e^{-k_1x_1}\,\phi_0(x_1)\int dx\,e^{-i(p'+k')x}\int dt\,e^{i[E(p')+\omega(k')]t}\langle x,t|0,0\rangle\int dx_2\,e^{-ikx_2}\,\phi_0(-x_2). \tag{6.16}$$

The presence of the δ functions shows that transitional invariance is indeed respected by the formalism. Recognizing the nonrelativistic propagator we get for the factor multiplying the δ functions

$$\tilde\phi_0(k')\,\frac{i}{E(p')+\omega(k')-M_0-(p'+k')^2/2M_0}\,\tilde\phi_0(k) . \tag{6.17}$$

There is also a similar term coming from the second part of the ordered product (6.13):

$$\tilde\phi_0(k)\,\frac{i}{E(p')-\omega(k')-M_0-(p'-k)^2/2M_0}\,\tilde\phi_0(k'). \tag{6.18}$$

This is the classical part of Green's function and the first quantum correction can also be computed in a similar way. It is of the order $O(\lambda^0)$ and comes from the last three terms in Eq. (6.12). We will just demonstrate for example how one computes the corrections coming from the last term of Eq. (6.12) which is the Fourier transform of

$$\langle p'|T[\chi(t',x'-X(t'))\chi(t,x-X(t))]|p\rangle . \tag{6.19}$$

This rather unconventional form can be understood if we write down the corresponding path-integral expression, since the canonical transformation was originally carried out in the path-integral formalism. Then we see that one has first to find the "vacuum" expectation value of $T[\chi(t',\rho')\chi(t,\rho)]$ and next after substitution $\rho=x-X(t)$ and $\rho'=x'-X(t')$ to evaluate this operator between the one-soliton states. Since the operators $X(t)$ and $X(t')$ do not commute, this operator between the soliton states has to be time-ordered. So, in the first approximation, we have that the $O(\lambda^0)$ contribution to the matrix element (6.19) is given by

$$\langle p'|TG(t'-t;x'-X(t'),x-X(t))|p\rangle , \tag{6.20}$$

where $G(t;\rho'\rho)$ is the propagator. This gives the following contribution to the Green's function:

$$(2\pi)^2\delta(E(p')+\omega(k')-E(p)-\omega(k))\delta(p'+k'-p-k)\int dx\,e^{i[\omega(k')t-k'x]}\langle p'|TG(t;x-X(t),-X(0))|p\rangle . \tag{6.21}$$

Now we can continue in the same way as in the preceding calculation, obtaining for the $\theta(t)$ part of the time-ordered product

$$\sum_n{}' \bar{\psi}_n(-k') \int \frac{d\omega}{2\pi} \frac{i}{\omega^2-\omega_n{}^2} \frac{i}{\omega+E(p')+\omega(k')-M_0-(p'+k')^2/2M_0} \bar{\psi}_n(-k) \tag{6.22}$$

and a similar formula for the $\theta(-t)$ part of the time-ordered product.

ACKNOWLEDGMENT

We would like to thank P. Senjanovic for his valuable help and interest in this work.

APPENDIX

In this appendix we prove

$$\sum_{n=0}^{N} \omega_n \sum_{m=N+1}^{\infty} (\psi_n, \rho\psi_m)(\psi_m, \psi_n{}') = -\frac{3\sqrt{2}}{4\pi} \tag{A1}$$

and

$$\sum_{n,m}{}'' \frac{S_{nm}(\psi_m, \bar{\psi}_n)}{\omega_n+\omega_m} = O\left(\frac{1}{L^2}\right). \tag{A2}$$

In (A1) N is a sufficiently large integer such that $N/L \gg 1$.

Using the explicit expression (3.7) for ψ_n, one obtains

$$(\psi_m, \psi_n{}') = ik_n\delta_{nm} + \frac{3i\pi}{N_nN_m} \frac{(k_m{}^2-k_n{}^2)}{\sinh[(\pi/\sqrt{2})(k_m-k_n)]} \times(2+k_m{}^2+k_n{}^2). \tag{A3}$$

To the sum in (A1), only the second term contributes. The denominator of this term gets exponentially large for $m-n$ so that $m\sim n\sim N$ contribute dominantly in the sum of (A1). So we approximate $\psi_n \sim (1/\sqrt{L})e^{ik_n\rho}$ for the calculation of $(\psi_n, \rho\psi_m)$:

$$(\psi_n, \rho\psi_m) \sim \frac{(-1)^{m-n}}{i(k_m-k_n)}. \tag{A4}$$

Inserting (A3) and (A4) into (A1) and using the fact that the main contribution in (A1) comes from the region $m\sim N$ and $m-n=l\ll L$, one obtains the following expression for (A1):

$$\frac{3}{2}\sum_{l=1}^{\infty}(-1)^l \frac{k_l}{\sinh[(\pi/\sqrt{2})k_l]}. \tag{A5}$$

Next, we use the following variation of Euler-Maclaurin formula:

$$\sum_{n=1}^{N}(-1)^n g(k_n) = \frac{1}{2}[g(\Lambda)-g(0)] + \frac{\pi}{2L}g'(\Lambda) + O\left(\frac{1}{L^2}\right), \tag{A6}$$

where $\Lambda = 2\pi N/L$ and N is an even number. This expression can be proven easily be using the standard Euler-Maclaurin formula[13]

$$\sum_{m=1}^{M} f(m) = \int_0^M dm\, f(m) + \tfrac{1}{2}[f(M)-f(0)] + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}[f^{(2n-1)}(M)-f^{(2n-1)}(0)], \tag{A7}$$

where B_{2n}'s are Bernouilli numbers.

Application of (A6) to (A5) immediately leads to (A1).

The proof of (A2) can be done in a similar fashion by using (A6).

*Work supported in part by the National Science Foundation under Grant No. GP-3809 7X, and in part by the Research Foundation of the City University of New York under Grant No. 10633.

[1]J.-L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975).

[2]D. Bohm and D. Pines, Phys. Rev. 92, 609 (1953).

[3]G. Branco, B. Sakita, and P. Senjanovic, Phys. Rev. D 10, 2573 (1974).

[4]A similar method to ours was developed by C. Callan and D. Gross [Princeton report (unpublished)]. They also discussed the perturbation theory and one-loop calculation. We became aware of this some time after most of the present work was completed.

[5]In the Higgs model, the corresponding extended object is a closed string. J.-L. Gervais and B. Sakita, Nucl. Phys. B91, 301 (1975).

[6]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974).

[7]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[8]Reference 1 consists of two parts; the so-called nonrelativistic and relativistic formalism. In this paper we discuss only the nonrelativistic formalism.

[9]The same canonical transformation was investigated in the operator formalism by E. Tomboulis [Phys. Rev. D (to be published)].

[10]These "vacuum" diagrams are diagrams with no external legs in the theory with the Hamiltonian given by Eq. (4.3) where P is treated as a parameter. We use the word "vacuum" state for the state of the lowest eigenvalue of this Hamiltonian for a given P. This,

of course, does not mean the true vacuum. On the contrary, it represents the one-soliton state.

[11] *Cargèse Lectures in Physics*, 1970, edited by B. Bessis (Gordon and Breach, New York, 1972).

[12] The numerical evaluation of these expressions is carried out by C. Callan and D. Gross (Ref. 4).

[13] G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1966), p. 362.