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Classical limit of relativistic positive-energy theories with intrinsic spin*

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The classical (nonquantum) relativistic differential equations obtained from positive-energy relativistic wave equations in the correspondence limit are analyzed. The essential role of the Majorana representation of SO(3,2) in the known examples is expounded, and a translation of this role into constraints on the classical-limit motion is developed. It is found that the quantum interpretation of these theories in terms of composite models of elementary particles has a natural analog on the classical level. The structural constituents are found to be massless and to undergo a classical *Zitterbewegung* consisting of a planar corotation whose amplitude and frequency depend on the spin of the quantum state from which the limit was taken. This motion is exactly that which would be described, at the classical level, as the rigid rotation of a massless "string" whose ends achieve the velocity of light. The model serves to define a relativistic, nonquantum intrinsic spin.

I. INTRODUCTION

Until quite recently,[1,2] all of the known examples of (massive) relativistic wave equations have had the property that the spectrum of their solutions included either negative-energy states or spacelike states. Prototype theories with these properties are the well-known Dirac electron equation,[3] with negative-energy states, and the perhaps less well-known Majorana equation,[4] with spacelike solutions. The two theories are somewhat logically related in that both are based on Lorentz-invariant linear SO(3, 2) wave equations.

Of course, the negative-energy solutions of the Dirac theory have been quite properly interpreted in terms of antiparticles at the level of second quantization, etc. Nevertheless, the fact remains that any theory of the Dirac type is not a consistent one-particle theory, but rather a multiparticle theory, at any level. (The spacelike *free* states of the Majorana theory are unphysical in any interpretation.)

It is also well known[5] that essentially the same feature of the Dirac theory which permits the appearance of antiparticles also causes a rather involved process to be required in order to obtain a nonrelativistic quantum limit. The other possible limit of the Dirac theory, a relativistic nonquantum limit, has been obtained by various authors,[6] using primarily wave packet expectation techniques. To our knowledge, the limits of the Majorana theory have been only partially discussed.[7]

Now, however, since the advent, in 1971 and also in 1974 of Lorentz-covariant wave equations[1,2] whose solutions describe states of definite mass, definite spin, and strictly positive energies, the possibility of obtaining consistent classical-limit pictures of the structure of elementary particles has arisen. In this report we shall discuss the relativistic nonquantum theory obtained from these new theories in the classical limit.

Our interest in this subject arises principally from our desire to gain a better understanding of the content of the new positive-energy theories. It is most intriguing that Biedenharn, Han, and van Dam[8] have interpreted the positive-energy theories in terms of states containing partonlike constituents and suggested[9] as well a relation of these theories to the dual resonance model. Our results here suggest that, even at the classical level, a certain substructure is present.

Recently, Hanson and Regge[10] have constructed a mechanical, relativistic Lagrangian model of the spinning top. One of their solutions has a fixed angular momentum, which may therefore be interpreted as an intrinsic spin. The model which we shall obtain is quite different, being obtained via a limit from a relativistic wave equation, but it does exhibit the characteristics of a classical model of an intrinsically spinning particle, and suggests, moreover, that the constituents of Biedenharn, Han, and van Dam may be massless.

Further motivation may be found in the fact that while the spinless positive-energy equation recently proposed by Dirac[1] does not permit minimal coupling to electromagnetism, such a coupling is not forbidden to the newer[2] spin-$\frac{1}{2}$ example. Now, wave equations defining an energy spectrum which covers only the half-open real line describe states which cannot be exactly localized.[11] Given, then, the essentially local nature of electromagnetism, it follows that the coordinate variables which appear in these wave equations must have a very complicated nature. Our results will verify that the positive-energy theories exhibit a type of *Zitterbewegung*, even at the classical level. The motion, in this case, constitutes a planar rotation at the velocity of light, and it is this rotational mode which defines a spin.

We may finally mention that the classical model may be of some interest in its own right, divorced from its origins. The no-go theorems[12] of Currie *et al.* suggest that any consistent relativistic model is worthy of inspection. Also, the paucity of models of this type has been one of the factors contributing to our continued, and continuing, struggle to untangle the intricacies of the various relativistic coordinate operators.[13]

We shall begin our development, in Sec. II, with a discussion of an SO(3, 2) wave equation of the Dirac-Majorana type. Most of our remarks will concern a pair of equations which serve to define a single definite-mass definite-spin state of the Majorana equation.[14] We follow this line of development because the known examples of positive-energy wave equations define theories which, in the case of no electromagnetic interaction, are exactly equivalent[15] to projections upon the Majorana solution spectrum.

In Sec. III we develop the differential equations obtaining in the classical relativistic limit and present their solution. Section IV contains the crux of our analysis. We show that the positive-energy nature of the quantum theory, which depends crucially upon the nature of the Majorana representation of SO(3, 2), imposes severe constraints upon the classical motion which, in particular, result in a classical analog of intrinsic spin.

Section V contains a classical analysis of the motion of those variables which have been interpreted as describing, on the quantum level, partonlike constituents by Biedenharn, Han, and van Dam.[8] Also included in this section is a heuristic analysis of the quantum motion using the classical results.

We have added two appendixes which contain short proofs on the nature of certain representations of both the quantum and classical SO(3, 2) Lie algebras, some of which are known but are widely scattered in the literature. We incorporate these results in our discussions to indicate the uniqueness features of our results.

II. LINEAR SO(3,2) WAVE EQUATIONS

The general relativistic wave equation of the Dirac-Majorana type[3,4] may be written[16]

$$(\Gamma_\mu P^\mu - \kappa m)\psi = 0\,, \tag{2.1}$$

where κ and m are positive constants.

The wave function ψ is required to transform multiplicatively under the action of the Lorentz group generated by the operators

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}\,, \tag{2.2}$$

where $L_{\mu\nu}$ denotes the usual space-time generators, and $S_{\mu\nu}$ accounts for the spin degrees of freedom of the states.

The space-time independent operators Γ_μ transform as a vector under the action of the Lorentz group. In the Dirac and Majorana cases, they form, with $S_{\mu\nu}$, the Lie algebra of SO(3, 2):

$$\begin{aligned}
&[\Gamma_\mu, \Gamma_\nu] = iS_{\mu\nu}\,,\\
&[S_{\mu\nu}, \Gamma_\alpha] = i(g_{\mu\alpha}\Gamma_\nu - g_{\nu\alpha}\Gamma_\mu)\,,\\
&[S_{\mu\nu}, S_{\alpha\beta}] = i(g_{\mu\alpha}S_{\nu\beta} - g_{\mu\beta}S_{\nu\alpha} + g_{\nu\beta}S_{\mu\alpha} - g_{\nu\alpha}S_{\mu\beta})\,.
\end{aligned} \tag{2.3}$$

The first of Eqs. (2.3) represents an additional requirement[17] upon the representation of the operators Γ_μ and $S_{\mu\nu}$ over those strictly necessary for the Lorentz invariance of Eq. (2.1). This additional restriction, standing alone, is not sufficient to meaningfully restrict the spectrum of solutions of Eq. (2.1).

We shall not follow the historically divergent paths of Dirac and Majorana toward imposing restrictions upon the state content of their respective equations. Rather we remark, as is well known, that the end result of their efforts may be categorized, with historical hindsight, as the placing of severe restrictions upon the Lorentz representation content[18] of the SO(3, 2) operators Γ_μ and $S_{\mu\nu}$.

We define, then, the Lorentz scalar operators

$$\begin{aligned}
&F \equiv \tfrac{1}{4} S_{\mu\nu}S^{\mu\nu}\,,\\
&G \equiv \tfrac{1}{8}\epsilon^{\mu\nu\alpha\beta}S_{\mu\nu}S_{\alpha\beta}\,,\\
&D \equiv \Gamma_\mu \Gamma^\mu\,,
\end{aligned} \tag{2.4}$$

the first two of which are the Casimir operators of the Lorentz group.

Following, then, the logical path illuminated by hindsight, we make the unifying assertion that *the Dirac and Majorana equations are distinguished by the requirement that the Lorentz-group Casimir operator F of Eq. (2.4) be a c number in the representation of the SO(3,2) Lie algebra.*

In the interest of completeness, it is proved in Appendix A that only two nontrivial representations of SO(3, 2) with this property exist: that of Dirac and that of Majorana. The Dirac representation comprises a nonunitary, reducible representation of the Lorentz group which may be conveniently realized with $\Gamma_\mu = \frac{1}{2}\gamma_\mu$, where the γ_μ

re the usual Dirac matrices. If the constant κ in Eq. (2.1) is set to $\frac{1}{2}$, then the constant m is the mass and the usual Dirac equation results. In this case $F=\frac{3}{4}$, $G=-\frac{3}{4}i\gamma_5$, and $D=1$.

The only other nontrivial possibility, that of Majorana, comprises[18] a unitary, reducible representation of the Lorentz group with $F=-\frac{3}{8}$, $G=0$, and $D=-\frac{1}{2}$. This representation is sometimes realized in terms of infinite-dimensional Hermitian matrices. However, we choose to consider here a realization as differential operators[19] on the space of L_2 functions of two (dimensionless) variables, q_1 and q_2. We define the quantum conjugates $\eta_j=(1/i)\partial/\partial q_j$, so that $(j,k=1,2)$

$$[q_j,\eta_k]=i\delta_{jk} . \tag{2.5}$$

Then the Hermitian operators Γ_μ and $S_{\mu\nu}$ have the realization

$$\begin{aligned}
\Gamma_0&=\tfrac{1}{4}(q_1^2+q_2^2+\eta_1^2+\eta_2^2),\\
\Gamma_1&=\tfrac{1}{2}(-q_1\eta_1+q_2\eta_2),\\
\Gamma_2&=\tfrac{1}{2}(q_1\eta_2+q_2\eta_1),\\
\Gamma_3&=\tfrac{1}{4}(q_1^2+q_2^2-\eta_1^2-\eta_2^2),\\
S_{10}&=\tfrac{1}{4}(q_1^2-\eta_1^2-q_2^2+\eta_2^2),\\
S_{20}&=\tfrac{1}{2}(\eta_1\eta_2-q_1q_2),\\
S_{30}&=\tfrac{1}{2}(q_1\eta_1+\eta_2q_2),\\
S_{12}&=\tfrac{1}{2}(q_1\eta_2-q_2\eta_1),\\
S_{31}&=\tfrac{1}{4}(q_2^2+\eta_2^2-q_1^2-\eta_1^2),\\
S_{23}&=-\tfrac{1}{2}(q_1q_2+\eta_1\eta_2).
\end{aligned} \tag{2.6}$$

When the realization (2.6) is used in Eq. (2.1), the wave function is $\psi=\psi(x^\mu,q_1,q_2)$, a single function of the indicated arguments.

The Majorana equation, as is well known, has many interesting features,[18] most of which lead to unphysical results. The operators Γ_μ all being Hermitian, any of them may be diagonalized so that spacelike solutions exist. Timelike momentum eigenstates ($p^2>0$) may be transformed to the rest frame, so that Eq. (2.1) may be brought into the form

$$(\Gamma_0p_0-\kappa m)\psi(\text{rest})=0 . \tag{2.7}$$

The spectrum of the operator Γ_0 is $(s+\frac{1}{2})$, $(s=0,\frac{1}{2},1,\frac{3}{2},\ldots)$. A desirable physical result is that *timelike momentum-eigenstate solutions have strictly positive energies*. On the negative side, the spin of these momentum eigenstatesis given by the number s (see the discussion below), so that Eq. (2.7) defines an unphysical mass-spin spectrum: mass ~ (spin)$^{-1}$.

Recently, however, two Lorentz-covariant wave equations have been reported[1,2] whose noninteracting solutions are uniquely the timelike spin-zero and spin-$\frac{1}{2}$ states, respectively, of the Majorana equation. (For a complete discussion, see Ref. 2.) These new equations then serve to define projections upon the Majorana equation which pick out the only agreeable feature, that of restricting the energy spectrum of timelike states to positive values.

Rather than discuss these two very different and somewhat involved new wave equations, it is sufficient for our purpose here to note that their net effect, *in the noninteracting case*, is to define a theory in which the wave function ψ simultaneously satisfies the two wave equations[20]

$$\begin{aligned}
&(P_\mu P^\mu-m^2)\psi=0,\\
&(\Gamma_\mu P^\mu-\kappa m)\psi=0,
\end{aligned} \tag{2.8}$$

for particular, fixed values of the constant κ.

In subsequent sections we will consider the classical (relativistic) limit of the theory defined by Eqs. (2.8). In particular, it should be clear that the state described by ψ is a timelike positive-energy state, so that the passage to a classical limit requires no machinations to remove residual effects due to negative-energy solutions.

The momentum-eigenstate solution of Eqs. (2.8) has the property that in the rest system $p_0=+m$. It then follows from Eq. (2.7) and the attendant discussion that a solution to Eqs. (2.8) exists if and only if the value of the fixed constant κ is one of the numbers $(s+\frac{1}{2})$, $(s=0,\frac{1}{2},1,\frac{3}{2},\ldots)$.

The intrinsic spin of the state is found by considering the Pauli-Lubanski operator following from Eq. (2.2):

$$W^\mu=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}P_\beta . \tag{2.9}$$

Then

$$W^\mu W_\mu=-\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}P^2+S_{\mu\alpha}S^{\mu\beta}P^\alpha P_\beta . \tag{2.10}$$

The special properties [(A1) and (A6) with $F=-\frac{3}{8}$ and $D=-\frac{1}{2}$] of the Majorana representation may be used, with the result that

$$W^2=\tfrac{1}{4}P^2-(\Gamma_\mu P^\mu)^2 . \tag{2.11}$$

Equations (2.8) then imply

$$W^2\psi=-(\kappa^2-\tfrac{1}{4})m^2\psi, \tag{2.12}$$

so that the intrinsic spin of the state, s, is uniquely given by $\kappa=s+\frac{1}{2}$.

The two new relativistic wave equations exhibited recently and referred to above result in Eqs. (2.8) for the values $\kappa=\frac{1}{2}$ (zero spin) and $\kappa=1$ (spin $\frac{1}{2}$), respectively. We shall leave κ arbitrary except that it must be fixed to one of its allowed values, thereby fixing the spin of the state considered.

Before closing this section we should make two remarks. The first is that we lose no generality by considering Eqs. (2.8) rather than the new relativistic wave equations (in the noninteracting case), since their solutions[15] are *uniquely* specified by the pair (2.8). In contrast, a complete analysis of the usual Dirac equation obviously cannot be made on the basis of the first of Eqs. (2.8) coupled with Eq. (2.12) (for $\kappa = 1$).

Our second remark is that once the classical limit is taken, then there is nothing in the classical theory *by itself* which restricts the possible values of the constant κ. The effect is that the classical spin, while fixed by the value of κ, has a continuous range of possible values.

III. TRANSITION TO THE CLASSICAL EQUATIONS

We begin with the commutation relations[16]

$$[x^\mu, P^\nu] \equiv -ig^{\mu\nu}, \tag{3.1}$$

along with those among Γ_μ and $S_{\mu\nu}$ in Eq. (2.3). We will make the transition from the quantum to the classical level via the well-known[21] substitution of classical Poisson bracket (PB) (denoted here by curly brackets) relations for commutation relations:

$$[O, O']/i \to \{O, O'\} . \tag{3.2}$$

Possible difficulties with the consistency of any theory following a transition of this type have been discussed at length by Dirac[22] and more recently by Hanson and Regge.[10] However, the difficulty invariably arises in ensuring a consistent quantum theory obtained via the transition from the classical level. We shall proceed from a well-defined quantum theory to the classical level.

We obtain, then, from Eq. (3.1) the classical, relativistic PB relations

$$\{x^\mu, p^\nu\} = -g^{\mu\nu} . \tag{3.3}$$

It should be emphasized that the x^μ of Eq. (3.1) are the elements of a real 4-field, while the P^μ of the same equation, and also of Eq. (2.8) are differential operators upon that field. In no sense is the operator P_0 considered to be a Hamiltonian operator. Similarly, the classical vector p^μ of Eq. (3.3) is an energy-momentum vector with p_0 being the energy, not the Hamiltonian function.

The classical PB of Eq. (3.3) is well defined in terms of a classical procedure[21]—that of taking partial derivatives with respect to the classical real variables x^μ and p^μ. What then is proposed for the operators Γ_μ and $S_{\mu\nu}$ and the Lie algebra (2.3)? Here we take advantage of the fact that the unitary (or infinite-dimensional) nature of the Majorana representation of these operators permitted their definition as differential operators. We consider the q_i, η_j of Eqs. (2.5) and (2.6) on the same footing as the x^μ, p^ν of Eq. (3.1) and define the classical limit of the Lie algebra via Eq. (3.2). The essential point is that whenever the operators $\Gamma_\mu, S_{\mu\nu}$ may be expressed as differential operators, then classical PB relations may be unambiguously defined in terms of partial derivatives with respect to a set of classical real variables q_1, η_1, q_2, η_2 which satisfy the defining PB relation $(i, j = 1, 2)$

$$\{q_i, \eta_j\} = \delta_{ij} . \tag{3.4}$$

We cannot emphasize this point enough. A classical limit of the Dirac realization of the same Lie algebra in terms of PB's defined as partial derivatives is clearly impossible. The classical limit of a noncommuting set of finite-dimensional matrix operators simply does not exist.

For the Majorana realization then, Eq. (2.6), we may take the *same definitions* of Γ_μ and $S_{\mu\nu}$ given in that equation directly to the classical level as functions of real conjugate variables, and consistently realize, *by direct partial differentiation,* the PB relations

$$\begin{aligned} \{\Gamma_\mu, \Gamma_\nu\} &= S_{\mu\nu} , \\ \{S_{\mu\nu}, \Gamma_\alpha\} &= g_{\mu\alpha}\Gamma_\nu - g_{\nu\alpha}\Gamma_\mu , \\ \{S_{\mu\nu}, S_{\alpha\beta}\} &= (g_{\mu\alpha}S_{\nu\beta} - g_{\mu\beta}S_{\nu\alpha} + g_{\nu\beta}S_{\mu\alpha} - g_{\nu\alpha}S_{\mu\beta}) , \end{aligned} \tag{3.5}$$

which of course follow also from the prescription (3.2).

We are now prepared to consider the dynamics defined by Eqs. (2.8). Following Dirac,[7] we define the generalized Hamiltonian $(\kappa > 0)$:

$$\Phi = \kappa m - \Gamma_\mu P^\mu . \tag{3.6}$$

The unitary transformation generated by Φ, with parameter τ, then defines a Heisenberg picture. In this picture the dynamical development of the operators is given by Heisenberg equations to which the classical-limit prescription (3.2) may be applied. We obtain the equations

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \{x^\mu, \Phi\} , \\ \frac{dp^\mu}{d\tau} &= \{p^\mu, \Phi\} , \\ \frac{d\Gamma^\mu}{d\tau} &= \{\Gamma^\mu, \Phi\} , \\ \frac{dS^{\mu\nu}}{d\tau} &= \{S^{\mu\nu}, \Phi\} . \end{aligned} \tag{3.7}$$

Further constraints on the dynamics are provided by the classical equivalents of (2.8):

$$p^\mu p_\mu \equiv m^2$$
$$\Phi = \kappa m - \Gamma_\mu p^\mu \equiv 0\,, \tag{3.8}$$

relations which may not be employed until after all PB's have been evaluated.[22]

The differential equations (3.7) read, with (3.3), (3.5), and (3.6),

$$\begin{aligned} \frac{dp^\mu}{d\tau} &= 0\,, \\ \frac{dx^\mu}{d\tau} &= \Gamma^\mu\,, \\ \frac{d\Gamma^\mu}{d\tau} &= -S^{\mu\nu}p_\nu\,, \\ \frac{dS^{\mu\nu}}{d\tau} &= \Gamma^\mu p^\nu - \Gamma^\nu p^\mu\,. \end{aligned} \tag{3.9}$$

Still following Dirac, we observe that Eqs. (3.9) may be iterated, and (3.8) used to obtain the second-order equation

$$\frac{d^2\Gamma^\mu}{d\tau^2} + m^2\Gamma^\mu = \kappa m p^\mu\,, \tag{3.10}$$

whose solution is obvious.

With this solution in hand, Eqs. (3.9) may be solved serially to obtain the set of functions

$$\begin{aligned} x^\mu &= \kappa(p^\mu/m)\tau + (A^\mu/m)\sin m\tau \\ &\quad - (B^\mu/m)\cos m\tau + C^\mu\,, \\ \Gamma^\mu &= \kappa(p^\mu/m) + A^\mu\cos m\tau + B^\mu \sin m\tau\,, \\ S^{\mu\nu} &= (A^\mu p^\nu - A^\nu p^\mu)m^{-1}\sin m\tau \\ &\quad - (B^\mu p^\nu - B^\nu p^\mu)m^{-1}\cos m\tau \\ &\quad + [M^{\mu\nu} + C^\mu p^\nu - C^\nu p^\mu]\,, \end{aligned} \tag{3.11}$$

where A^μ, B^μ, C^μ, and $M^{\mu\nu}$ are τ independent constants of integration. (The peculiar form of the bracketed constant of integration in $S_{\mu\nu}$ is convenient below.) The set of linear equations (3.9) imposes constraints upon the integration constants, which read

$$\begin{aligned} A^\mu p_\mu &= B^\mu p_\mu = 0\,, \\ M^{\mu\nu}p_\nu &= p^\mu(p^\nu C_\nu) - m^2 C^\mu\,. \end{aligned} \tag{3.12}$$

Using the usual definition $L^{\mu\nu} \equiv x^\nu p^\mu - x^\mu p^\nu$ to define, via the correspondence principle, the classical orbital angular momentum tensor, it may be verified that Eqs. (3.11) imply, as a functional identity, the result

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}\,. \tag{3.13}$$

The carefully chosen integration constant $M_{\mu\nu}$, then, represents the total angular momentum tensor constant of the classical motion.

We may also define a classical Pauli-Lubanski tensor

$$W^\mu \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}M_{\nu\alpha}p_\beta = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}p_\beta\,, \tag{3.14}$$

where the last equality represents again a functional identity. Of course, W_μ is a τ-independent constant of the motion.

Finally, we remark that the integration constant C^μ obviously represents only a choice of origin in the parameter τ. Accordingly, to simplify what follows, we set it equal to zero. The relations (3.12) then become

$$C^\mu = A^\mu p_\mu = B^\mu p_\mu = M^{\mu\nu}p_\nu = 0\,. \tag{3.15}$$

IV. THE NATURE OF THE CLASSICAL MOTION

In this section we shall undertake the evaluation of the integration constants and an analysis of the motion.

In deriving (3.11) we made use of the PB algebra of the functions Γ_μ and $S_{\mu\nu}$, but we have not made any essential use of the fact that we are investigating the classical limit of a particular representation of these operators.

In Sec. II we remarked that the uniquely distinguishing feature of the Majorana and Dirac representations of SO(3, 2) is that the Lorentz-scalar operator, F, of Eq. (2.4) is a c number. In the Majorana case, all of the operators of Eq. (2.4) are c numbers, with the particular values $F = -\frac{3}{8}$, $G = 0$, and $D = -\frac{1}{2}$.

Now, in order to make an important point, let us suppose that we are attempting to construct a consistent classical relativistic theory using the relations (3.3) and (3.5) through (3.8), but not, in particular, the special functional forms given by Eq. (2.6). We assert that the correct correspondence-principle procedure is to carry over to the classical level the distinguishing feature that the function F, defined by Eq. (2.4) in the same way, is a pure number.

We prove in Appendix B that the classical bracket relations (3.5), coupled with the requirement that F be a number (rather than a function), imply the unique values $F = G = D = 0$ for the functions defined by Eq. (2.4). An explicit classical evaluation of these quantities using the particular (Majorana) functions defined in Eq. (2.6) yields, of course, these same values. Our point is twofold. The first is the technical remark that any attempt to carry the quantum values of these quantities over to the classical level is incorrect[7] and results eventually in inconsistencies. More importantly, we have shown that *any consistent classical theory of this type is equivalent to one defined using a Majorana functional representation, no matter what types*[23] *of classical brackets are defined.*

The remaining task is now quite straightforward. Using the functions (3.11) and the conditions (3.15), we may directly evaluate the quantity D, for instance, to obtain

$$\begin{aligned} D &= \Gamma^\mu \Gamma_\mu \\ &= \kappa^2 + A^2 \cos^2 m\tau + B^2 \sin^2 m\tau \\ &\quad + 2A^\mu B_\mu \sin m\tau \cos m\tau \,. \end{aligned} \tag{4.1}$$

Then the representation requirement $D=0$ yields the information

$$\begin{aligned} &A^2 = B^2 = -\kappa^2 \,, \\ &A^\mu B_\mu = 0 \,. \end{aligned} \tag{4.2}$$

Similarly, the requirement $F=0$, along with (4.2) implies

$$M_{\mu\nu} M^{\mu\nu} = 2\kappa^2 \,. \tag{4.3}$$

Finally, the requirement $G=0$, in the form of (B19), implies

$$\kappa M^{\mu\nu} = A^\mu B^\nu - A^\nu B^\mu \,. \tag{4.4}$$

It follows then from (3.14) that ($\kappa > 0$)

$$W^\mu = (1/\kappa)\epsilon^{\mu\nu\alpha\beta} A_\nu B_\alpha p_\beta \,, \tag{4.5}$$

so that

$$W^2 = -\kappa^2 m^2 \,. \tag{4.6}$$

The set of vectors p^μ, A^μ, B^μ, and W^μ then forms an orthogonal tetrad of space-time 4-vectors.

In order to interpret the meaning of these results, it is convenient to consider the motion viewed in the "proper" Lorentz frame, defined to be that frame in which the 3-momentum $\vec{p}$ vanishes, and $p_0 = +m$ (we do not employ the term "rest frame," since our results indicate that the term is inappropriate). The orthogonality of the tetrad defined above then implies that in this frame $A_0 = B_0 = W_0 = 0$. Equations (4.2) then yield the values

$$\begin{aligned} &(\vec{A})^2 = (\vec{B})^2 = \kappa^2 \,, \\ &\vec{A} \cdot \vec{B} = 0 \,. \end{aligned} \tag{4.7}$$

The functions x^μ of Eq. (3.11) reduce to

$$\begin{aligned} &x^0 = \kappa\tau \equiv t_p \,, \\ &\vec{x} = m^{-1}(\vec{A} \sin m\tau - \vec{B} \cos m\tau) \,, \end{aligned} \tag{4.8}$$

where we have defined the time variable in the proper system, t_p.

It is clear from Eqs. (4.7) and (4.8) that the 3-vector $\vec{x}$ describes a rotation in the plane defined by the orthogonal 2-basis $\vec{A}$ and $\vec{B}$. The Pauli-Lubanski 3-vector in this frame is

$$\vec{W} = (m/\kappa)(\vec{A} \times \vec{B}) \,, \tag{4.9}$$

with magnitude κm and differing from the 3-vector quantity which may be obtained from the space-space parts of $M_{\mu\nu}$ only by the mass factor.

The other useful quantities, Γ_μ, are in this frame

$$\begin{aligned} &\Gamma_0 = \kappa \,, \\ &\vec{\Gamma} = \vec{A} \cos m\tau + \vec{B} \sin m\tau \,. \end{aligned} \tag{4.10}$$

The 3-vector $\vec{\Gamma}$ has a magnitude κ.

Let us now inquire whether the rotational motion described by $\vec{x}$ can be interpreted as the motion of some constituents, i.e., as a classical model of intrinsic spin. The crucial quantity here[24] is the velocity of the constituents undergoing this relative motion:

$$\vec{v} \equiv \frac{d\vec{x}}{dt_p} = \frac{1}{\kappa} \frac{d\vec{x}}{d\tau} = (1/\kappa)\vec{\Gamma} \,, \tag{4.11}$$

so that

$$(\vec{v})^2 = (1/\kappa^2)(\vec{\Gamma})^2 = 1 \,. \tag{4.12}$$

In these units, any constituent matter is therefore orbiting at the velocity of light.

This is a very interesting result, since a classical model of *intrinsic* spin is almost a contradiction in terms. It implies, at the classical level, a rotational motion which cannot be brought to rest. But physically, the only objects which cannot be stopped in their motion *without being simultaneously destroyed* are those which are exactly massless and so have a speed c. For this reason, if one speculates that a classical model of the quantum phenomenon of intrinsic spin might exist, then a model obtained as a well-defined classical limit of a quantum theory and which gives just the result above is the only possibility. Therefore, we are inclined to take this interpretation quite seriously.

We view the rest mass m in this interpretation as being completely dynamical, reflecting the energy bound in the rotational mode of the massless constituents. In our opinion, this is a no-less-attractive feature of the model.

It is pertinent here to comment on a possible objection to our identification of p^μ with the 4-momentum at the classical level, particularly since its conjugate variable has the behavior (4.8) in the "proper" frame. We take the position that the quantum theory, being the fundamental theory, has already fixed this identification. In fact, whenever the results of the correspondence limit have proved surprising, we have attempted to expand our classical intuitions, particularly since a relativistic mechanics of bound massless particles is not available for comparison.

Finally, we remark that the "spin," here being interpreted as due to massless constituents, is given by Eq. (4.6), with κ a fixed number, and depends in the final analysis only upon Eqs. (3.13) and (3.14) and the correspondence principle. In particular, the fact that $\vec{x}$ does not appear to represent the "center-of-mass" motion is completely immaterial. In fact, the new spin-$\frac{1}{2}$ positive-energy relativistic wave equation[2] apparently permits minimal coupling to electromagnetism, at the point x^μ. It would then appear that the classical motion of the vector $\vec{x}$ in following a presumably charged constituent[25] supports the requirement of *local* electromagnetic interactions in a theory which is necessarily nonlocal owing to its positive-energy character.

V. NULL-PLANE CONSTITUENT MOTION

In 1971 Dirac presented[1] a new relativistic wave equation which admitted only positive-energy spinless solutions, and defined a projection upon the Majorana spectrum. Subsequently, Biedenharn, Han, and van Dam[8] have shown that the solution to Dirac's new equation is also the ground-state solution of a (Galilean) Hamiltonian problem in 2 + 1 dimensions which arises naturally when one describes relativistic quantum-mechanical initial conditions on the 3-surface defined by a plane-wave front advancing with velocity c along (say) the z direction, rather than on the usual equal-time hyperplane.

The existence of such an alternative formulation of classical relativistic mechanics had been pointed out some years ago by Dirac,[26] who termed it the "front form" of relativistic mechanics. The quantum-mechanical formulation of the relativistic problem along these same lines has been termed "quantum front subdynamics" by the authors of Ref. 8. Their essential point is that the 3-surface of the plane-wave front (sometimes termed the null plane) contains a timelike dimension, so that there exists on the quantum-mechanical level the possibility of a dynamics being defined entirely within this subspace.

The subset of seven generators of the Poincaré group which generate transformations mapping the front 3-space into itself have the Lie algebra of the (2 + 1) Galilean group and include the combinations $P_+ = P_0 + P_3$ and $P_- = P_0 - P_3$. The spacetime variable $x_- = \frac{1}{2}(x_0 - x_3)$, conjugate to P_+, defines by its fixed values the particular light front considered. The variable $x_+ = \frac{1}{2}(x_0 + x_3)$, which is conjugate to P_-, may then be used to define a dynamics which takes place entirely within the manifold of the (2 + 1)-dimensional null plane. The "Hamiltonian" $H_G \equiv \frac{1}{2}P_-$ which describes this quantum front subdynamics includes a part which can be interpreted as a "center-of-mass" term, containing $(P_1^2 + P_2^2)$, at the Galilean level. The remainder of H_G resembles interaction terms between two constituent objects, and contains the dimensionless variables q_1 and q_2, which may be interpreted, with suitable dimensional factors, as the two components of a 2-dimensional relative-coordinate vector. The conjugate relative 2-momentum has components proportional to η_1 and η_2. The specific form of the interaction described in Ref. 8 is that of a harmonic oscillator in two dimensions.

The particular Poincaré generator $P_- = 2H_G$ of these authors has been integrated into an (interacting) Lie algebra of the complete Poincaré group.[27] The full set of eigenfunctions of H_G define, at the Poincaré level, a set of states having a Chew-Frautschi spectrum ($m^2 \sim s$), with each different mass-spin state being channeled into a separate[27] Poincaré Hilbert space. Now within each of these Hilbert spaces, the theory is completely equivalent to that defined by the two relations (2.8), for appropriate values of the constant κ. An infinite set of projections of the Majorana theory is thus defined, so that the full set of eigenfunctions of H_G is a unified entity only when analyzed in the submanifold of the null-plane light front, and not on the full Poincaré 4-space level.

In order to make contact with this interpretation, we will consider the classical motion of the quantities q_1, q_2, η_1, and η_2 obtained from the classical function definitions (2.6), the classical conjugate definition (3.4), and the dynamics defined by the generalized classical Hamiltonian Φ, of Eq. (3.8). Then the equations of motion of these quantities may be computed from relations of the type

$$\frac{dq_1}{d\tau} = \{q_1, \Phi\}, \text{ etc.} \tag{5.1}$$

The results are

$$\begin{aligned}
\frac{dq_1}{d\tau} &= -\tfrac{1}{2}(p_+\eta_1 + p_1 q_1 - p_2 q_2),\\
\frac{dq_2}{d\tau} &= -\tfrac{1}{2}(p_+\eta_2 - p_2 q_1 - p_1 q_2),\\
\frac{d\eta_1}{d\tau} &= \tfrac{1}{2}(p_- q_1 + p_1\eta_1 - p_2\eta_2),\\
\frac{d\eta_2}{d\tau} &= \tfrac{1}{2}(p_- q_2 - p_2\eta_1 - p_1\eta_2),
\end{aligned} \tag{5.2}$$

where the null-plane variables $p_+ = p_0 + p_3$ and $p_- = p_0 - p_3$ arise naturally because of the functional form chosen[19] in Eq. (2.6). The quantities p_+, p_-, and also $p^\mu p_\mu = p_+ p_- - p_1^2 - p_2^2 = m^2$ are all

τ independent.

The first-order equations (5.2) may be iterated to obtain a set of second-order equations, each of which has the same form:

$$\frac{d^2q_1}{d\tau^2}+\frac{m^2}{4}q_1=0,\ \text{etc.} \tag{5.3}$$

It follows that $(j=1,2)$

$$\begin{aligned} q_j&=\alpha_j\cos(m\tau/2)+\beta_j\sin(m\tau/2),\\ \eta_j&=\lambda_j\cos(m\tau/2)+\sigma_j\sin(m\tau/2), \end{aligned} \tag{5.4}$$

where the α_j, β_j, λ_j, and σ_j are integration constants.

The four linear differential equations (5.2) then imply four independent relations among the eight constants of integration. Taking, say, the λ_j and σ_j to be independent, we may express these relations as

$$\begin{aligned} p_-\alpha_1&=m\sigma_1-p_1\lambda_1+p_2\lambda_2,\\ p_-\alpha_2&=m\sigma_2+p_2\lambda_1+p_1\lambda_2,\\ p_-\beta_1&=-m\lambda_1-p_1\sigma_1+p_2\sigma_2,\\ p_-\beta_2&=-m\lambda_2+p_2\sigma_1+p_1\sigma_2. \end{aligned} \tag{5.5}$$

To simplify the discussion, we specialize these results to the "proper" frame, where $p_+=p_-=m$, $p_1=p_2=0$. Then Eqs. (5.4) and (5.5) yield $(j=1,2)$

$$\begin{aligned} q_j&=\sigma_j\cos(m\tau/2)-\lambda_j\sin(m\tau/2),\\ \eta_j&=\lambda_j\cos(m\tau/2)+\sigma_j\sin(m\tau/2). \end{aligned} \tag{5.6}$$

Now the classical motion in the proper frame is such that the relations (4.10) must hold while at the same time the Γ_μ are defined by Eq. (2.6) in terms of the quantities in Eq. (5.6). The first of Eqs. (4.10) implies, for instance, that

$$\Gamma_0\equiv\tfrac{1}{4}(q_1^2+q_2^2+\eta_1^2+\eta_2^2)=\kappa,$$

which in turn yields the relation

$$\lambda_1^2+\lambda_2^2+\sigma_1^2+\sigma_2^2=4\kappa. \tag{5.7}$$

The remainder of Eqs. (4.10), in the same fashion, yields expressions for the independent vectors $\vec{A}$ and $\vec{B}$ $[\vec{A}\equiv(A^1,A^2,A^3)]$:

$$\begin{aligned} A_1&=\tfrac{1}{2}(\lambda_2\sigma_2-\lambda_1\sigma_1),\\ A_2&=\tfrac{1}{2}(\lambda_1\sigma_2+\lambda_2\sigma_1),\\ A_3&=\tfrac{1}{4}(\sigma_1^2+\sigma_2^2-\lambda_1^2-\lambda_2^2),\\ B_1&=\tfrac{1}{4}(\lambda_1^2-\lambda_2^2-\sigma_1^2+\sigma_2^2),\\ B_2&=\tfrac{1}{2}(\sigma_1\sigma_2-\lambda_1\lambda_2),\\ B_3&=-\tfrac{1}{2}(\lambda_1\sigma_1+\lambda_2\sigma_2). \end{aligned} \tag{5.8}$$

We may make a convenient choice of phase for the null-plane constituent motion by the choices $\lambda_1=-\sqrt{2\kappa}$, $\sigma_1=0$, $\lambda_2=0$, $\sigma_2=\sqrt{2\kappa}$. Then

$$\begin{aligned} q_1&=\sqrt{2\kappa}\sin(m\tau/2),\\ q_2&=\sqrt{2\kappa}\cos(m\tau/2),\\ \eta_1&=-\sqrt{2\kappa}\cos(m\tau/2),\\ \eta_2&=\sqrt{2\kappa}\sin(m\tau/2), \end{aligned} \tag{5.9}$$

so that, in the proper frame ($p_0=m$, $\vec{p}=0$, $A_0=B_0=W_0=0$, $t_p\equiv x^0=\kappa\tau$),

$$\begin{aligned} \vec{A}&=(0,\kappa,0),\\ \vec{B}&=(-\kappa,0,0),\\ \vec{W}&=(0,0,m\kappa),\\ \vec{x}&=((\kappa/m)\cos m\tau,(\kappa/m)\sin m\tau,0). \end{aligned} \tag{5.10}$$

Note that the classically observable quantities such as those of Eq. (5.10) ultimately involve quadratic expressions in the q_i,η_j, so that the particular relations (5.9) are not subject to direct verification. This is in keeping with the remarks above concerning the nonobservability[8,27] of the "quantum front subdynamics" at the full Poincaré level.

Nevertheless, we will attempt to gain some interpretive insight from these relations through an inspection of the quantum front Galilean Hamiltonian.[8] We consider the more convenient form given below and explicitly insert the dimensional factors $\hbar$ and c:

$$H_G\equiv\tfrac{1}{2}P_-=\frac{(p_1^2+p_2^2)}{2M}+\hbar\omega\left[\frac{\beta}{M}(\Gamma_0+\Gamma_3)+\frac{M}{\beta}(\Gamma_0-\Gamma_3)-\frac{2p_1}{mc}\Gamma_1-\frac{2p_2}{mc}\Gamma_2+\frac{\beta}{M}\frac{p_1^2+p_2^2}{m^2c^2}(\Gamma_0+\Gamma_3)\right], \tag{5.11}$$

where $M\equiv P_+/c$ plays the role of the Galilean "mass" and β represents a freedom of choice of scale (see Ref. 27).

If we consider the proper frame, then $p_1=p_2=0$, $M=m$. Now strictly speaking, (5.11) makes sense only as a quantum operator. However, we will make these substitutions anyway, in a totally heuristic spirit, following our quest for interpretive insight. We shall also fix the scale parameter at $\beta=m$ (a valid operation), which the results of Ref. 27 show to be a required choice to complement the definition adopted in Eq. (2.2).

We then "obtain"

$$H_G = \hbar\omega(2\Gamma_0) = \tfrac{1}{2}\hbar\omega[(\eta_1^2+\eta_2^2)+(q_1^2+q_2^2)]\ . \qquad (5.12)$$

Appropriate dimensional variables may be defined by the substitutions ($j=1, 2$)

$$\xi_j = (\hbar/m\omega)^{1/2} q_j\ ,\quad \pi_j = (\hbar m\omega)^{1/2}\eta_j\ , \qquad (5.13)$$

so that (5.12) may be reexpressed in the form

$$H_G = \frac{1}{2m}(\pi_1^2+\pi_2^2)+\tfrac{1}{2}m\omega^2(\xi_1^2+\xi_2^2)\ . \qquad (5.14)$$

The interpretation of Biedenharn, Han, and van Dam[8] in terms of two constituents bound by a 2-dimensional harmonic-oscillator interaction is obvious.

The mass-spin spectrum which arises naturally from the process of completing the Poincaré Lie algebra partially defined by Eq. (5.11) may be expressed in our notation as

$$m^2c^2 = 4M\hbar\omega\kappa\ , \qquad (5.15)$$

where we have used the earlier result that the constant κ is related to the spin s by $\kappa = s+\frac{1}{2}$, at the quantum level. The combination $M\hbar\omega$ is a Poincaré-invariant constant,[27] so that the spectrum is $m^2 \sim s$. This requires that the front "oscillation frequency" varies according to the 3-momentum of the state considered.

For the case of the proper frame, then,

$$4\hbar\omega\kappa = mc^2\ ,\quad \xi_j = 2(\hbar/mc)\sqrt{\kappa}\ q_j\ , \qquad (5.16)$$

or, with Eq. (5.9),

$$\xi_1 = 2\sqrt{2}\ (\kappa\hbar/mc)\sin(mc^2\tau/2\hbar)\ ,\quad \xi_2 = 2\sqrt{2}\ (\kappa\hbar/mc)\cos(mc^2\tau/2\hbar)\ , \qquad (5.17)$$

i.e., the front constituents "viewed" classically undergo harmonic motion with half the frequency of the vector $\vec{x}$, and with an amplitude and frequency dependent upon the spin of the state.

We might inquire as to the velocity of the 2-dimensional motion described by Eq. (5.17). We must then keep in mind that it is in terms of a null-plane time variable, where the motion is defined, that we must seek our answer. The variable conjugate to $H_G \equiv \frac{1}{2}P_-$ is $t_F \equiv x_0 + x_3$. Now $x_0 = \kappa\tau$ and, with our convenient choice of phases, Eq. (5.10) contains the result $x_3 = 0$ for this motion. Then $t_F = \kappa\tau$ and

$$\dot{\xi}_j \equiv \frac{d\xi_j}{dt_F} = (1/\kappa)\ \frac{d\xi_j}{d\tau}\ , \qquad (5.18)$$

or

$$\dot{\xi}_1 = \sqrt{2}\ c\cos(mc^2\tau/2\hbar)\ ,\quad \dot{\xi}_2 = -\sqrt{2}\ c\sin(mc^2\tau/2\hbar)\ , \qquad (5.19)$$

which implies an effective velocity

$$[(\dot{\xi}_1)^2+(\dot{\xi}_2)^2]^{1/2} = 2c\ . \qquad (5.20)$$

The magnitude of the result poses no difficulty since it represents the *relative* (Galilean) velocity between two objects and tends rather to suggest again that they may be massless. However, the significant feature of the result is not the particular magnitude obtained,[28] but rather the absence of a time dependence. This indicates that harmonic-oscillator front subdynamics is not[29] incompatible with an interpretation in terms of a rotational constituent motion at the Minkowski level.

We must reemphasize that this result, as well as all of the analysis from Eq. (5.11) on, constitutes a totally heuristic argument and that none of the results of the previous sections depend in any way upon these particular results. However, our interpretations are strongly supported, in a suggestive way, by this analysis.

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APPENDIX A

In this appendix we shall prove[30] that the only nontrivial representations of the Lie algebra of SO(3, 2), Eq. (2.3), with the property that the Lorentz Casimir operator F is a c number[31] are those of Majorana and of Dirac.

We begin with the definition

$$F \equiv \tfrac{1}{4}S^{\mu\nu}S_{\mu\nu}\ . \qquad (A1)$$

Since F is a Lorentz Casimir operator, our additional assumption means that

$$[\Gamma_\mu, F] = 0\ . \qquad (A2)$$

Combining (A1) and (A2), and using the Lie algebra, we immediately obtain the results

$$\Gamma^\mu S_{\mu\nu} = S_{\nu\mu}\Gamma^\mu = -\tfrac{3}{2}i\Gamma_\nu\ . \qquad (A3)$$

We define also the Lorentz-scalar operator

$$D \equiv \Gamma^\mu\Gamma_\mu\ . \qquad (A4)$$

Consideration of the quantity $\Gamma^\mu\Gamma^\nu S_{\mu\nu}$, along with the use of (A3) and the antisymmetry of $S_{\mu\nu}$, then results in the identity

$$F=\tfrac{3}{4}D, \tag{A5}$$

so that D is also a c number in any such representation.

An exactly similar calculation using the quantity $[\Gamma_\mu, \Gamma_\nu]S^{\mu\alpha}$ yields the useful result

$$S_{\mu\nu}S^{\mu}{}_{\alpha}=\tfrac{3}{2}iS_{\nu\alpha}-\Gamma_\nu\Gamma_\alpha+g_{\nu\alpha}D. \tag{A6}$$

The other Lorentz Casimir operator, G, has the definition

$$G\equiv\tfrac{1}{8}\epsilon^{\mu\nu\alpha\beta}S_{\mu\nu}S_{\alpha\beta}. \tag{A7}$$

Then

$$\Omega_\mu\equiv i[\Gamma_\mu, G] \tag{A8}$$

may be evaluated directly, with the result

$$\Omega^\mu=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}\Gamma_\beta. \tag{A9}$$

It follows immediately that

$$\Gamma_\mu\Omega^\mu=-\Omega^\mu\Gamma_\mu=2iG. \tag{A10}$$

Further, substitution of the identity (A3) for Γ_μ in the definition (A8) and evaluation of those commutators yields

$$S_{\mu\nu}\Omega^\nu=\Omega^\nu S_{\nu\mu}=-\tfrac{3}{2}i\Omega_\mu. \tag{A11}$$

A direct evaluation using (A9) also yields the result

$$\Omega^\mu\Omega_\mu=-2F(D+\tfrac{1}{2}). \tag{A12}$$

A logical chain may now be followed to establish the result that if the operator G is also a c number, then the representation is uniquely that of Majorana, or else it is completely trivial. Equation (A8) shows that if G is a c number, then Ω_μ vanishes. Then (A10) implies that G also vanishes, while (A12) implies that either $F=0$ or $D=-\tfrac{1}{2}$. Then (A5) selects either the trivial case $F=D=G=0$ or the Majorana case $F=-\tfrac{3}{8}$, $G=0$, $D=-\tfrac{1}{2}$.

The representation of Majorana is unitary and also reducible, containing both the principal series representation with lowest spin $\tfrac{1}{2}$ and the complementary series representation with lowest spin 0.[32]

In what follows, we assume that the operator G is nonzero, so that it is not a c number.

We contract the free index μ of (A11) with the operator $S^{\alpha\mu}$ and use (A6) and also (A11) again to obtain the results

$$\Omega^\mu D=D\Omega^\mu=2i\Gamma^\mu G. \tag{A13}$$

A direct evaluation of the quantity $\epsilon^{\mu\nu\alpha\beta}\Gamma_\alpha\Omega_\beta$ using (A9) and also (A3) yields

$$\epsilon^{\mu\nu\alpha\beta}\Gamma_\alpha\Omega_\beta=(D+\tfrac{1}{2})S^{\mu\nu}. \tag{A14}$$

Multiplication of (A14) with the c number D, and the use of (A13), results in the identity

$$G\epsilon^{\mu\nu\alpha\beta}S_{\alpha\beta}=-D(D+\tfrac{1}{2})S^{\mu\nu}. \tag{A15}$$

The assumption that G does not vanish then implies that the c number $D(D+\tfrac{1}{2})$ also cannot vanish. Contraction of (A15) with $S_{\mu\nu}$ and the use of (A5) yields the further information that

$$G^2=-\tfrac{3}{8}D^2(D+\tfrac{1}{2}). \tag{A16}$$

Therefore, the operator G^2 is a (nonzero) c number.

Now, the commutator of Γ_μ with Ω_ν may be directly calculated, with the result that

$$[\Gamma^\mu, \Omega^\nu]=-\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}S_{\alpha\beta}+\tfrac{1}{2}i\epsilon^{\nu\lambda\alpha\beta}S_{\alpha\beta}S^{\mu}{}_{\lambda}. \tag{A17}$$

Then (A15) and (A6) may be applied to (A17) to yield

$$G[\Gamma^\mu, \Omega^\nu]=[\Gamma^\mu, \Omega^\nu]G$$
$$=iD(D+\tfrac{1}{2})(\tfrac{1}{4}iS^{\mu\nu}+\tfrac{1}{2}\Gamma^\nu\Gamma^\mu-\tfrac{1}{2}g^{\mu\nu}D). \tag{A18}$$

Now G being a Lorentz Casimir operator, the commutation of G with (A6) yields the result

$$[\Gamma^\mu\Gamma^\nu, G]=0,$$

so that (A8) implies

$$\Gamma^\mu\Omega^\nu+\Omega^\mu\Gamma^\nu=0. \tag{A19}$$

Therefore,

$$[\Gamma^\mu, \Omega^\nu]=\Gamma^\mu\Omega^\nu+\Gamma^\nu\Omega^\mu, \tag{A20}$$

or, using (A13),

$$[\Gamma^\mu, \Omega^\nu]D=2i(\Gamma^\mu\Gamma^\nu+\Gamma^\nu\Gamma^\mu)G. \tag{A21}$$

Finally, equating the product of the c number D with (A18) to the product of G with (A21) yields the result

$$D^2(D+\tfrac{1}{2})\Gamma^\mu\Gamma^\nu=D^2(D+\tfrac{1}{2})(\tfrac{1}{2}iS^{\mu\nu}+\tfrac{1}{4}g^{\mu\nu}D). \tag{A22}$$

Now whenever G is nonzero, the quantity $D^2(D+\tfrac{1}{2})$ is also nonzero, so that

$$\Gamma^\mu\Gamma^\nu=\tfrac{1}{2}iS^{\mu\nu}+\tfrac{1}{4}g^{\mu\nu}D. \tag{A23}$$

It follows that the anticommutator of Γ_μ with Γ_ν is a c number,

$$\Gamma_\mu\Gamma_\nu+\Gamma_\nu\Gamma_\mu=\tfrac{1}{2}Dg_{\mu\nu}, \tag{A24}$$

and, with (A3), that $D=1$. Thus, either $G=0$ and the (nontrivial) representation is that of Majorana or else G is not a c number and, according to (A24), the representation is that of Dirac.

APPENDIX B

We shall prove in this appendix that the Poisson bracket[23] relations (3.5) when coupled with the assumption that the classical function

$$F \equiv \tfrac{1}{4} S^{\mu\nu} S_{\mu\nu} \tag{B1}$$

is a classical constant, implies that the three classical functions F, D, and G of Eq. (2.4) all must vanish.

A direct evaluation using Eq. (3.5) yields the result

$$\{\Gamma^\mu, F\} = \tfrac{1}{2}(S^{\mu\nu}\Gamma_\nu - S^{\mu\nu}\Gamma_\nu) = \Gamma_\nu S^{\nu\mu}, \tag{B2}$$

where we have used the antisymmetry of the $S_{\mu\nu}$ and recalled that there is no distinction in the ordering of products of classical functions. Then if F is identically a constant,

$$\Gamma_\mu S^{\mu\nu} = 0. \tag{B3}$$

We also define

$$\Omega^\mu \equiv \{G, \Gamma^\mu\}, \tag{B4}$$

and obtain directly the result

$$\Omega^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} S_{\nu\alpha}\Gamma_\beta. \tag{B5}$$

It follows, in the classical case, that

$$\Gamma_\mu \Omega^\mu = 0, \tag{B6}$$

as well as

$$\Omega^\mu \Omega_\mu = -2FD, \tag{B7}$$

with the use of (B3).

We may also evaluate the Poisson bracket of (B3) with Γ_α to obtain the identity

$$S^{\mu\nu} S_{\nu\alpha} = \Gamma^\mu \Gamma_\alpha - g^\mu_\alpha D. \tag{B8}$$

The subsequent use of (B3) implies

$$4F = 3D. \tag{B9}$$

Similarly, the Poisson bracket of (B3) with G yields

$$\Omega_\mu S^{\mu\nu} = 0, \tag{B10}$$

to which the sequential application of (B8) and (B6) results in the statement

$$D\Omega_\mu = 0. \tag{B11}$$

Two logical possibilities exist. Either $D = 0$, in which case (B9) implies that F also vanishes, or $\Omega_\mu = 0$, in which case (B7) and (B9) imply that both F and D vanish. In either event,

$$F = D = 0, \tag{B12}$$

so that (B8) now reads

$$\Gamma_\mu \Gamma_\nu = S_{\mu\alpha} S^\alpha{}_\nu. \tag{B13}$$

The Poisson bracket of G with (B13) yields, with (B4),

$$\Gamma_\mu \Omega_\nu + \Omega_\mu \Gamma_\nu = 0, \tag{B14}$$

so that the Poisson bracket of (B14) with Γ_μ yields, with (B10), the result that

$$\Gamma_\mu \{\Gamma^\mu, \Omega_\nu\} = -\{\Gamma^\mu, \Omega_\mu\}\Gamma_\nu. \tag{B15}$$

Now a direct calculation yields

$$\{\Gamma_\mu, \Omega^\nu\} = \tfrac{1}{2}\epsilon^{\nu\alpha\beta\sigma} S_{\alpha\beta} S_{\mu\sigma}. \tag{B16}$$

Then

$$\{\Gamma_\mu, \Omega^\mu\} = 4G \tag{B17a}$$

and

$$\Gamma^\mu \{\Gamma_\mu, \Omega^\nu\} = 0, \tag{B17b}$$

using (B3).

The identity (B15) then reads

$$G\Gamma_\mu = 0. \tag{B18}$$

The logical possibilities attendant upon (B18) both have the result that G vanishes. Equation (B4) then yields also the information that Ω_μ, given by (B5) also vanishes, so that

$$S_{\mu\alpha}\Gamma_\nu - S_{\mu\nu}\Gamma_\alpha + S_{\alpha\nu}\Gamma_\mu = 0. \tag{B19}$$

Finally, the existence of the Majorana functional realization (2.6) of the Poisson bracket algebra precludes the possibility that the only result is the trivial case that all quantities vanish.[33]

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[14]Somewhat different generalizations of Dirac's positive-energy equation to arbitrary spins have been reported by S. Browne, Nucl. Phys. B79, 70 (1974). In these examples, the wave-function manifold is larger than Minkowski space ⊗ Majorana representation variables. However, the positive-energy property is obtained from the properties of the Majorana representation, so that a straightforward modification of our analysis applies as well to these examples.

[15]The crux of this equivalence is the no-interaction specification. Sets of equations which restrict a theory to positive energies only could have been written down long ago, but the consistent introduction of interactions into a set of equations which are not directly logically related is very difficult. A complete discussion is given in Ref. 2.

[16]We use $\hbar = c = 1$, $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $P^\mu \equiv i\partial^\mu$, and $\epsilon^{0123} = +1$.

[17]For an analysis of the completely general case, see for instance, R. C. Hwa, Nuovo Cimento 56A, 107 (1968); 56A, 127 (1968).

[18]An excellent discussion may be found in the lecture of A. Böhm, reported in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. X-B.

[19]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).

[20]A generalization of Majorana's equation has been suggested by A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. 167, 1527 (1968). If attention is focused on a particular timelike state, then Eqs. (2.8) apply also to this theory.

[21]A modern, excellent discussion may be found in R. J. Finkelstein, *Nonrelativistic Mechanics* (Benjamin, New York, 1973).

[22]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva Univ. Press, New York, 1964).

[23]Many different types of classical bracket operations may be defined. (See Refs. 10 and 22.) Our analysis in Appendix B, however, makes use only of the algebra of the brackets.

[24]Recall that the Dirac electron equation implies a quantum velocity operator, each of whose components has eigenvalues of magnitude c. An *improper* classical-limit would then imply a velocity of $3c$. See also our remarks in Ref. 7.

[25]A discussion of the possible nature of a charged substructure which may be inferred from the properties of particular timelike states of another unitary wave equation has been given by A. O. Barut and H. Duru, Phys. Rev. D 10, 3448 (1974).

[26]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[27]L. P. Staunton, Phys. Rev. D 8, 2446 (1973).

[28]This being a heuristic argument, one could take another point of view and identify the argument of the harmonic functions of Eq. (5.19) as ωt_F. Using then (5.16), one would conclude that $t_F = 2\kappa\tau$ and obtain instead a constant velocity of magnitude c.

[29]The authors of Ref. 25 have argued that minimal electromagnetic interactions seem to require a rotational constituent motion, and have attempted to conclude that any subdynamics should therefore be of the Coulomb-force type.

[30]Many of the particular equations which we shall obtain represent known properties of the Dirac and Majorana representations. See, for instance, A. Böhm, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. X-B. Our purpose here is to establish the converse of the usual theorems and to justify our use of a weaker-than-usual hypothesis. See also, L. Jaffe, J. Math. Phys. 12, 882 (1971), and Ref. 31.

[31]Interlocking representations of SO(3,2) for which each irreducible part satisfies this criterion have been termed strongly interlocking by M. Lorente, P. L. Huddleston, and P. Roman, J. Math. Phys. 14, 1495 (1973). Our analysis shows that the only such irreducible representations are those of Dirac and of Majorana.

[32]The fact that the complementary series is not complete in the group manifold may well represent the ultimate reason that minimal coupling to electromagnetism is forbidden to the positive-energy wave equation of Dirac, Ref. 1, but not to the spin-$\frac{1}{2}$ equation of Ref. 2.

[33]Of course, it then follows that the quantum-mechanical Dirac representation is mapped into the trivial classical realization.