Letter

Five dimensional Weyl double copy

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The Weyl double copy (WDC) relation connects the Weyl tensor of the gravity theory and the field strength tensor of the Maxwell theory, which provides a concrete realization of the classical double copy. Although intensively investigated, the WDC is only limited in four-dimensional spacetime. In this Letter, we generalize the WDC relation to five-dimensional spacetime, which offers the first example of the WDC in higher dimensions. We show that a special class of five-dimensional type N vacuum solutions admits a special class of degenerate Maxwell field that squares to give the Weyl tensor. The five-dimensional WDC relation defines a scalar field that satisfies the source-free Klein-Gordon equation on the curved background. We further verify that for five-dimensional pp-wave solution and Kundt solutions, the Maxwell fields and the scalar fields also satisfy the Maxwell's equations and the wave equation on five-dimensional Minkowski spacetime.

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Introduction. Gauge and gravity theories are fundamentally important in our understanding of physical phenomena. It is intriguing to observe that they are intimately linked. One such example is the celebrated AdS/CFT correspondence [1], which relates the quantum gravity in the anti–de Sitter (AdS) space to the conformal field theory (CFT) on the AdS boundary. Another striking example is the double copy relation, which interprets the perturbative scattering amplitudes of gravity as a product of two scattering amplitudes of gauge theory [2–4]. The double copy perspective suggests a more efficient and elegant way for computing the gravity scattering amplitudes than the traditional calculations in general relativity, see, e.g., [5–8], which plays a crucial role in the recently established amplitudes-based methods to derive the state-of-the-art results of interest to the gravitational wave community [9–12].

The remarkable success of the double copy relation has motivated investigations at the classical level connecting classical solutions of gauge and gravity theories which pioneers a classical double copy relation from the Kerr-Schild construction [13]. The classical double copy reveals exact relations between solutions beyond perturbative approach, which has a broader range of interests outside of the amplitudes community and significantly extends the scope of the double copy relation, see, e.g., [14–37]. Shortly, a second version of classical double copy, the Weyl double copy [38], was proposed by connecting the Weyl tensor of the gravity theory and the field strength tensor of the Maxwell theory.

The advantage of the WDC is resided in its gauge invariant nature, which provides a coordinate independent, hence more general method for revealing the exact mathematical connections between two classical theories. The WDC has drawn intensive attentions from various perspectives [39–57], which underpins the intrinsic connection between the Einstein equations and the Maxwell's equations. Although, the WDC was expected to exist in higher dimensions since it was proposed [38] and the higher-dimensional spinorial formalism was adapted in view of this goal [58], the current investigations are still restricted in four-dimensional spacetime [59]. Since the independent components of the Weyl tensor increase much faster than the metric as the spacetime dimension increases, building exact relations for the Weyl tensor to realize a classical double copy in higher dimensions is much more challenging than the Kerr-Schild construction, see, e.g., the known examples of the later [19,28,64,65]. Whether the WDC relation exists in higher dimensions is a vital issue for its scope, origin [42,44], and compatibility with the Kerr-Schild double copy [52,53]. More importantly, it restricts how general the classical relation between gauge and gravity theories is.

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In this Letter, we initialize the study of a five-dimensional Weyl double copy. The construction of the four-dimensional WDC formula constricts that it is only valid for the algebraic type D and type N spacetime [66]. We start from the simpler case in five dimensions, the type N vacuum solution of Einstein gravity in the Coley-Milson-Pravda-Pravdova (CMPP) classification [67–70]. In the five-dimensional spinorial formalism [58], we propose an algebraic construction of the WDC formula for type N solutions. We find a self-contained reduction of the type N spacetime, where one can confirm that any solution of this special class admits a special class of degenerate Maxwell field that squares to give the Weyl tensor. Moreover, a complex scalar field is defined from this five-dimensional WDC relation and the scalar field satisfies the wave equation on the curved background. Hence, a concrete WDC relation is uncovered in five dimensions. We then present two examples of exact solutions in five dimensions, which are a special class of the pp-wave and Kundt solutions. Remarkably, the Maxwell fields and the scalar fields for those two cases also satisfy the Maxwell's equations and the wave equation on the flat Minkowski spacetime. Our results confirm the existence of the higher-dimensional WDC and consolidate the robustness of the classical double copy relation.

4D WDC relation for type N solutions. The WDC relation is better appreciated in the spinorial formalism [38,41]. In four-dimensional spacetime, the homomorphism between the Lorentz group and $\mathrm{SL}(2,\mathbb{C})$ allows one to convert the spacetime indices $\mu, \nu \ldots$ into spinor indices $A, B, \ldots$ and their conjugate $A', B', \ldots$, where the Van der Waerden matrices $\sigma^{\mu}_{AA'}$ are applied to transform between them. The spinorial version of the Weyl tensor is fully determined by the totally symmetric Weyl spinor ψ_{ABCD} and its complex conjugate. The WDC relation is interpreted by the decomposition of the Weyl spinor [38]

$$\psi_{ABCD} = \frac{3c}{S}\phi_{(AB}\phi_{CD)}, \tag{1}$$

where ϕ_{AB} is a totally symmetric two-spinor defined from the Maxwell tensor $F_{\mu\nu}$ and S is a (complex) scalar field. The WDC formula in (1) can be written in a null frame system, such as the Newman-Penrose (NP) formalism [71]. The null bases $(l, n, m, \bar{m})$ of the NP formalism is constructed from the spinor bases $\{o_A, \iota_A\}$ as

$$l^{\mu} \sim \sigma^{\mu}_{AA'} o^A \bar{o}^{A'}, \qquad n^{\mu} \sim \sigma^{\mu}_{AA'} \iota^A \bar{\iota}^{A'},$$
$$m^{\mu} \sim \sigma^{\mu}_{AA'} o^A \bar{\iota}^{A'}, \qquad \bar{m}^{\mu} \sim \sigma^{\mu}_{AA'} \iota^A \bar{o}^{A'}. \tag{2}$$

The spinor indices are raised and lowered by the two-dimensional Levi-Civita tensor which can be decomposed by the spinor bases as $\epsilon_{AB} = o_A \iota_B - o_B \iota_A$.

For a type N spacetime, the only nonzero Weyl scalar is $\Psi_4 = \psi_{ABCD}\iota^A\iota^B\iota^C\iota^D = C_{\mu\nu\rho\sigma} n^{\mu}\bar{m}^{\nu} n^{\rho}\bar{m}^{\sigma}$ when l is the principal null direction. And $\Phi_2 = \phi_{AB}\iota^A\iota^B = F_{\mu\nu}\bar{m}^{\mu}n^{\nu}$ is the only nonzero Maxwell scalar of a degenerate Maxwell field. The type N WDC relation in the NP formalism is simply given by [41]

$$\Psi_4 = \frac{3c}{S}\Phi_2\Phi_2. \tag{3}$$

This relation seems revealing a trivial connection, because for any given Maxwell field, its square is naturally associated to the Weyl tensor from the algebraic derivation of the scalar field $S = 3c\Phi_2\Phi_2/\Psi_4$. The key point of the WDC is that if one is given a type N spacetime, how can one construct the Maxwell field which gives the Weyl tensor from its square, and what is the property of the scalar field. Those questions are well addressed in the seminal work [41]. Inserting the WDC relation into the Bianchi identity of the Weyl tensor and assuming that the Maxwell scalar solves the Maxwell's equation, one can obtain the equations for the scalar field S as

$$D\log S - \rho = 0, \qquad \delta \log S - \tau = 0. \tag{4}$$

If a scalar field is a solution of (4), it must solve the wave equation $\Box S = 0$ on the type N background [41]. For any scalar field solves (4), a degenerate Maxwell field can be derived from the WDC relation (3) as the square-root $\sqrt{S\Psi_4/(3c)}$. The integrability of equations in (4) guarantees that all four-dimensional type N vacuum solutions admit a degenerate Maxwell field that squares to give the Weyl tensor and the zeroth copy (the scalar field S) solves the wave equation. The decomposition of the Weyl tensor is not unique in this case.

The aim of the present work is to extend the above analysis to five dimensions. However, the four-dimensional WDC relation cannot imply any relation in five dimensions. The Weyl spinor in four dimensions can always be decomposed in terms of four rank-one spinors, which provides an alternative viewpoint of the Petrov classification from the alignment of the rank-one spinors. Hence, the four-dimensional WDC can only be constructed for algebraic type D and type N solutions for two equal Maxwell fields, which implies the intrinsic relationship between the four-dimensional WDC and the algebraic classification, and also simplifies the verification of the WDC, namely one just needs to consider two types of algebraically special vacuum solutions. In five dimensions, the algebraic classification is only relevant to the little group four-spinors of the Weyl tensor rather than the Weyl spinor [58]. If one insists on the relation (1) in five dimensions, the algebraic classification would not be of benefit to the verification of the WDC at all. A relevant issue is that there are more components of the Weyl tensor in five dimensions which makes the proposal of the WDC much more challenging. The very first issue about a five-dimensional WDC relation is that it should involve the Weyl spinor or the little group spinors. Actually, the existence of the WDC in five dimensions is also questionable. The four-dimensional WDC is argued to be originated from the twistor space [42]. Though it has higher-dimensional generalization, the twistor space has less direct connection to spacetime in higher dimensions. So, it is very doubtable that a higher-dimensional twistor space can yield a higher-dimensional WDC that indicates a decomposition of the spacetime Weyl tensor.

5D WDC relation for type N solutions. We follow closely [58] for the spinorial formalism in five dimensions. Five γ matrices are chosen as the bases to connect the spacetime indices and spinor indices. They are given by

$$\gamma^{\hat{\mu}}_{AB} = \begin{pmatrix} 0 & \sigma^{\hat{\mu}\alpha}{}_{\beta'} \\ \tilde{\sigma}^{\hat{\mu}}{}_{\alpha'}{}^{\beta} & 0 \end{pmatrix}, \qquad \gamma^{4}_{AB} = -i\begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\alpha'\beta'} \end{pmatrix}, \tag{5}$$

where $\hat{\mu} = 0, 1, 2, 3$ denotes the first four components of a five-dimensional vector, $A, B = 1, 2, 3, 4$ are the spacetime spinor indices and α, β are now the spinor indices in the van der Waerden matrices. Correspondingly, the spinor bases in five dimensions are chosen as

$$k^A_1 = \begin{pmatrix} 0 \\ \bar{o}^{\alpha'} \end{pmatrix}, \qquad k^A_2 = \begin{pmatrix} o_\alpha \\ 0 \end{pmatrix},$$
$$n^A_1 = \begin{pmatrix} \iota_\alpha \\ 0 \end{pmatrix}, \qquad n^A_2 = -\begin{pmatrix} 0 \\ \bar{\iota}^{\alpha'} \end{pmatrix}. \tag{6}$$

One can simply package them as $k^A_a = (k^A_1, k^A_2)$ and $n^A_a = (n^A_1, n^A_2)$, where indices a and b are packaged from two basis spinors and are referred to as little group spinor indices. A rank-two tensor Ω_{AB} is introduced to raise or lower the spinor index [58], which we decompose as

$$\Omega_{AB} = (k_{Aa}n_{Bb} - k_{Ba}n_{Bb})\epsilon^{ab}. \tag{7}$$

One can verify that the right hand side of (7) fulfills all the properties of Ω_{AB}.

The algebraic classification in the spinorial formalism has been constructed in [58], which reproduces the full structure of the CMPP classification. For the five-dimensional type N spacetime, the nonzero little group four-spinors for the Weyl tensor are $\psi^{(4)}_{abcd} = C_{\mu\nu\rho\sigma}\sigma^{\mu\nu}{}_{AB}\sigma^{\rho\sigma}{}_{CD}n^A{}_a n^B{}_b n^C{}_c n^D{}_d$ [58], where

$$\sigma^{\mu\nu}{}_{AB} = \frac{1}{2}(\gamma^\mu{}_{AC}\gamma^{\nu C}{}_B - \gamma^\nu{}_{AC}\gamma^{\mu C}{}_B). \tag{8}$$

A natural generalization of the WDC formula to five dimensions for the type N spacetime can be written as

$$\psi^{(4)}_{abcd} \propto \phi^{(2)}_{(ab}\phi^{(2)}_{cd)}, \tag{9}$$

where $\phi^{(2)}_{ab} = F_{\mu\nu}\sigma^{\mu\nu}{}_{AB}n^A{}_a n^B{}_b$ are the nonzero little group two-spinors of a degenerate Maxwell tensor [58]. Here we have not specified the scalar field as the case in four dimensions since there are more nonzero Weyl scalars in five dimensions. In principle, there could be more scalar fields involved. Note that the four-dimensional case of the WDC relation for two equal electromagnetic fields is only valid for type D and type N spacetime where only one (complex) Weyl scalar is nonzero. More explicitly, we propose

$$\psi^{(4)}_{1111} = \frac{3c}{S_1}\phi^{(2)}_{11}\phi^{(2)}_{11}, \qquad \psi^{(4)}_{2222} = \frac{3c}{S'_1}\phi^{(2)}_{22}\phi^{(2)}_{22},$$
$$\psi^{(4)}_{1112} = \frac{3c}{S_2}\phi^{(2)}_{11}\phi^{(2)}_{12}, \qquad \psi^{(4)}_{2221} = \frac{3c}{S'_2}\phi^{(2)}_{22}\phi^{(2)}_{21},$$
$$\psi^{(4)}_{1212} = \frac{c}{S_3}[\phi^{(2)}_{11}\phi^{(2)}_{22} + 2(\phi^{(2)}_{12})^2]. \tag{10}$$

We have introduced two independent complex scalar fields S_1, S_2 and one real scalar field S_3. Then we will transform to a null frame to test the relations in (10) by the Bianchi identity and the Maxwell's equation.

In higher-dimensional spacetime, it is convenient to construct a null frame system for describing the classification of the Weyl tensor [67–70]. The basis vectors are typically chosen as

$$l = e_0 = e^1, \qquad n = e_1 = e^0,$$
$$m_i = e_i = m^i = e^i, \quad i = 2, \ldots, n-1, \tag{11}$$

with two null vectors l and n, and $n-2$ spacelike vectors m_i. They satisfy the orthogonal conditions $l \cdot l = n \cdot n = l \cdot m^i = n \cdot m^i = 0$, and the normalization conditions $l \cdot n = 1, m^i \cdot m^j = \delta^{ij}$. The quantities $L_{\mu\nu} = \nabla_\nu l_\mu, N_{\mu\nu} = \nabla_\nu n_\mu, M^k_{\mu\nu} = \nabla_\nu m^k_\mu$ are defined to specify the spin coefficients.

For a type N solution, the nonzero Weyl scalars are $\psi_{ij} = C_{\mu\nu\rho\sigma}n^\mu m^{i\nu} n^\rho m^{j\sigma}$ in the null frame [67–70], see also [72]. And $\phi_i = F_{\mu\nu}n^\mu m^{i\nu}$ are the nonzero Maxwell scalars of a degenerate Maxwell field. Converting the WDC formulas (10) to the null frame yields [72]

$$\psi_{22} - \psi_{33} - 2i\psi_{23} = \frac{3c}{S_1}[(\phi_2)^2 - (\phi_3)^2 - 2i\phi_2\phi_3],$$
$$\psi_{22} - \psi_{33} + 2i\psi_{23} = \frac{3c}{S'_1}[(\phi_2)^2 - (\phi_3)^2 + 2i\phi_2\phi_3],$$
$$\psi_{34} + i\psi_{24} = \frac{3c}{S_2}(\phi_3\phi_4 + i\phi_2\phi_4),$$
$$\psi_{34} - i\psi_{24} = \frac{3c}{S'_2}(\phi_3\phi_4 - i\phi_2\phi_4),$$
$$\psi_{44} = \frac{c}{S_3}[(\phi_2)^2 + (\phi_3)^2 + 4(\phi_4)^2]. \tag{12}$$

The above equations present a natural extension of the WDC formulas to five dimensions for the type N spacetime. However, we are not able to extend the verification of [41] to a generic five-dimensional type N spacetime. Because the equations for the scalar fields are highly entangled with the Maxwell scalars. We cannot separate the equations for each scalar field. In the next section, we will show a self-contained reduction of the type N solution where one can realize a concrete WDC.

Reduced type N solution and the WDC relation. We now consider a 4D-like reduction [73] of the five-dimensional type N spacetime where we impose the restrictions $\psi_{4i} = 0$ for the Weyl scalars. The traceless property of the Weyl tensor then yields $\psi_{22} = -\psi_{33}$. Correspondingly, the Maxwell scalars and zeroth copies should be $\phi_4 = 0$, $S_3 \to \infty$, where the divergent scalar field S_3 is chosen to turn off the extra component from the doubling in five dimensions. We postpone commenting on this curious point to the end of this section. A higher-dimensional Goldberg-Sachs-like theorem [74,75] guarantees that for the CMPP type N spacetime the Weyl aligned null direction is geodesic (but no longer shear-free in general). Thus we can affine parametrize the geodesic and then take an appropriate spin transformation and a null rotation to make other basis vectors parallely propagated along l. We introduce the following definitions

$$\begin{aligned}
&\Psi_4 = -(\psi_{22} + i\psi_{23}), \qquad \bar{\Psi}_4 = -(\psi_{22} - i\psi_{23}),\\
&\varphi_2 = -\frac{1}{\sqrt{2}}(\phi_2 + i\phi_3), \qquad \bar{\varphi}_2 = -\frac{1}{\sqrt{2}}(\phi_2 - i\phi_3),\\
&\delta = \frac{1}{\sqrt{2}}(\delta_2 - i\delta_3), \qquad \bar{\delta} = \frac{1}{\sqrt{2}}(\delta_2 + i\delta_3),\\
&\hat{\delta} = i\delta_4, \qquad S = \frac{S_1}{c}.
\end{aligned} \tag{13}$$

where D, Δ, and δ_i are the directional derivatives associated to the basis vectors l, n, and m_i, respectively. Then, the Bianchi identity and the Maxwell's equation could be rewritten as [72]

$$\begin{aligned}
D\Psi_4 &= -(L_{22} + iL_{23})\Psi_4,\\
\delta\Psi_4 &= -\frac{1}{\sqrt{2}}[(2L_{12} + 2M^2_{33} - L_{21})\\
&\quad + i(L_{31} - 2L_{13} + 2M^2_{32})]\Psi_4,\\
\hat{\delta}\Psi_4 &= [(M^3_{42} - 2M^3_{24}) + i(L_{41} - 2L_{14} + M^2_{42})]\Psi_4,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
D\varphi_2 &= -(L_{22} + iL_{23})\varphi_2,\\
\delta\varphi_2 &= -\frac{1}{\sqrt{2}}[(L_{12} + M^2_{33} - L_{21})\\
&\quad + i(L_{31} - L_{13} + M^2_{32})]\varphi_2,\\
\hat{\delta}\varphi_2 &= [(M^3_{42} - M^3_{24}) + i(L_{41} - L_{14} + M^2_{42})]\varphi_2.
\end{aligned} \tag{15}$$

The WDC formula in the reduced case is simply given by

$$\Psi_4 = \frac{1}{S}(\varphi_2)^2. \tag{16}$$

Now it is clear that the previously chosen divergent scalar field S_3 is to prevent the mixing term constructed from $\varphi_2\bar{\varphi}_2$ in the Weyl doubling. Such a term is not involved at all in the four-dimensional WDC relation. Hence, it is reasonable to have a special treatment in five dimensions by simply imposing that there is no Weyl scalar associated to the $\varphi_2\bar{\varphi}_2$ term.

Substituting the WDC relation (16) into the Bianchi identity and simplifying them with the Maxwell's equation on the type N background, we obtain the equations for the scalar field

$$\begin{aligned}
D\log S &= -(L_{22} + iL_{23}),\\
\delta\log S &= \frac{1}{\sqrt{2}}(L_{21} - iL_{31}),\\
\hat{\delta}\log S &= M^3_{42} + i(M^2_{42} + L_{41}).
\end{aligned} \tag{17}$$

Following closely the treatment in [41], we have proven that the integrability conditions for the above differential equations are satisfied [72], where the commutators of the operators D, δ_i in [76] and the Ricci identities in [77] are applied. Finally, it is straightforward to verify that any solution of (17) solves the Klein-Gordon equation $\Box S = 0$ on the type N background. This completes the zeroth copy of the five-dimensional WDC relation for the reduced type N solutions. In the next sections, we will present two examples of the five-dimensional WDC relation.

5D pp-wave solution. The line-element of the five-dimensional pp-wave solution with respect to the reduction introduced previously is given by [78]

$$\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{d}u[\mathrm{d}v + H(u, x_2, x_3, x_4)\mathrm{d}u]\\
&\quad + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2,\\
H(u, x_2, x_3, x_4) &= f(u)x_4 + g(u, z) + \bar{g}(u, \bar{z}),
\end{aligned} \tag{18}$$

where $z = x_2 - ix_3$. We choose the same null frame system as [65]. A generic solution to (17) in the spacetime (18) is $S = P(u, \bar{z})$. The Weyl scalar is given by $\Psi_4 = \partial^2_{\bar{z}}\bar{g}(u, \bar{z})$, which allows us to determine the Maxwell scalar as $\varphi_2 = \sqrt{P(u, \bar{z})\partial^2_{\bar{z}}\bar{g}(u, \bar{z})}$. We have checked directly that the Maxwell scalar satisfies the Maxwell's equation. Hence, a concrete WDC relation is constructed for the special pp-wave solution. It is easy to prove that the Maxwell field and the scalar field defined from the WDC for the pp-wave solution also satisfy the Maxwell's equation and the wave equation on five-dimensional Minkowski spacetime.

5D Kundt solution. The type N Kundt solution in five dimensions with reduction is given by [79]

$$\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{d}u[\mathrm{d}v + H(u,v,x^k)\mathrm{d}u + W_i(u,v,x^k)\mathrm{d}x^i] \\
&\quad + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad i,j,k = 2,3,4, \\
H(u,v,x^k) &= \frac{v^2}{2y^2} + f(u,x^2,x^3) + g(u,x^2,x^4), \\
g(u,x^2,x^4) &= \frac{1}{2}B_{43}(u)^2(x^4)^2 + g_1(u)x^2x^4 \\
&\quad + B_{43}(u)C_3(u)x^4 + g_2(u,x^2), \\
y = x^2, &\qquad W_2(u,v,x^k) = -2\frac{v}{y}, \\
W_m(u,v,x^k) &= x^n B_{nm}(u) + C_m(u), \qquad m,n = 3,4. \qquad (19)
\end{aligned}$$

Projecting the solution into the null frame system defined in [65], one can obtain a generic solution to (17) for the Kundt solution (19) as $S = P(u, x_2 - ix_3)/x_2$. The Weyl scalar of the Kundt solution is given by

$$\begin{aligned}
\Psi_4 = \frac{1}{(x_2)^3}&[(x_2 + ix_3)B_{43}(u)^2 - iB_{43}(u)C_4(u) \\
&- i\partial_{x_3}f(u,x_2,x_3) - x_2\partial_{x_3}^2 f(u,x_2,x_3) \\
&+ ix_2\partial_{x_2}\partial_{x_3}f(u,x_2,x_3)]. \qquad (20)
\end{aligned}$$

Then, one can recover the Maxwell scalar from the square-root $\varphi_2 = \sqrt{S\Psi_4}$. It can be verified directly that the Maxwell scalar satisfies the Maxwell's equation. We further verify that the Maxwell and scalar fields defined from the Kundt solution also satisfy the Maxwell's equation and the wave equation on five-dimensional Minkowski spacetime [80].

Discussions. In this Letter, we offer the first realization of a higher-dimensional Weyl double copy relation, which significantly enlarges the scope of the WDC relation. The formulation of the WDC in four-dimensional spacetime is closely related to the algebraic classification. However, the algebraic classification of solutions in five dimensions is only relevant to the little group four-spinors of the Weyl tensor [58]. For type N spacetime, the little group four-spinor of the Weyl tensor is totally symmetric and is irreducible representations of $SU(2)$. Generically, the little group four-spinors of the Weyl tensor are decomposed into irreducible representations of $SU(2)$, which could consist of totally symmetry four-spinors, symmetric bi-spinors, and a scalar [58]. The WDC formula proposed in (10) is not enough for algebraically special spacetime of types other than N in five dimensions, nor even type N solutions in dimensions higher than five. One should also deal with the extra symmetric bi-spinors and scalar fields. Actually, this is the main obstacle for constructing the WDC in higher dimensions where the structure of the classification of the Weyl tensor is much richer than four dimensions. Nevertheless, the lesson from our construction is that one can always expect a 4D-like reduction where the WDC can be realized in any dimensions, at least, for algebraic type D and type N solutions, and the reduction would impose constraints on the components associated to all the extra dimensions. Such idea has been recently applied to recover the WDC for five-dimensional algebraic type D solutions [81]. At the end of the day, the reduction reveals the remarkable fact that interesting features in four-dimensional spacetime are compatible with extra dimensions. The reduction proposed for the WDC could provide a refined algebraic classification of higher-dimensional spacetime, which should be useful for finding new exact solutions in higher dimensions [82].

As a future direction, it is interesting to investigate the compatibility of the five-dimensional WDC with the Kerr-Schild double copy, which is vital to an important aspect of the classical double copy that the single and zeroth copies are solutions on the flat background spacetime or the full curved spacetime. The verification of the WDC relation in the present work (also the generic construction in [41]) is purely at the equations of motion level in the vielbein formalism. We have never specified any exact solutions in a coordinates system. Then, the directional derivatives D, Δ, δ_i are not specified and their connections to the directional derivatives associated to the flat background spacetime are not known. Hence, we cannot use the WDC formulas to test any relations on the flat background spacetime. Nevertheless, we have verified for two exact solutions that the Maxwell fields and the scalar fields defined from the WDC formula also satisfy the Maxwell's equation and the wave equation on the flat background spacetime. Moreover, it is easy to verify that if one considers linearized gravity theory, our construction will lead to a WDC relation for the linearized Weyl tensor and the Maxwell tensor on the flat background spacetime, which is consistent with the investigations from the twistor perspective in [42,44].

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