Mean-field theory for self-interacting relativistic Luttinger fermions

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We investigate a class of quantum field theories with relativistic Luttinger fermions and local self-interaction in scalar channels. For an understanding of possible low-energy phases, we first classify the set of mass terms arising from scalar fermion bilinears. For large flavor numbers, we show that each of our models features a coupling branch in which the theory is asymptotically free. In order to address the long-range behavior, we use mean-field theory which is exact in the limit of large flavor numbers. We identify two models which undergo dimensional transmutation, interconnecting the asymptotically free high-energy regime with an ordered low-energy phase sustaining a vacuum condensate. We also study the analytic structure of the Luttinger-fermionic propagator in the various possible gapped phases.

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I. INTRODUCTION

Luttinger fermions are effective degrees of freedom of nonrelativistic solid state physics [1,2] used to describe, e.g., materials with quadratic band touching/crossing points involving general spin-orbit couplings [3–5]. These systems can feature a rich set of quantum critical phenomena [6–16]. Inspired by the diverse set of structures emerging from such long-range degrees of freedom, the generalization of Luttinger fermions to fundamental degrees of freedom of relativistic quantum field theories has recently been studied [17].

Since the resulting relativistic Luttinger operator is quadratic, the mass dimension of the field agrees with that of standard scalar fields which allows for the construction of a large number of perturbatively renormalizable quantum field theories in $3+1$ dimensional spacetime. Specifically, self-interacting theories of Luttinger fermions are renormalizable and can also be asymptotically free [17]. As a consequence, such quantum field theories can serve as a novel building block for high-energy complete theories for particle physics.

Another unorthodox feature of these theories becomes visible in the pole structure of the propagator where properties familiar from those of a higher-derivative theory [18–21] can be found [17]. For a standard mass term, not only the standard particle pole but also a tachyonic pole appears. The latter comes with a negative residue, characterizing a so-called ghost. Naively, this is often taken as an indication of nonunitarity interpreted as a consequence of Ostrogradsky's theorem [22], even though many different viewpoints on such ghost states exist in the literature, see e.g., [19,21,23–34].

In the present work, we concentrate on a set of simple example theories involving self-interacting relativistic Luttinger fermions. More specifically, we concentrate on massless classical actions with local scalar or pseudoscalar interactions. In addition to an investigation of the high-energy behavior characterized by the beta functions of the couplings, we explore the long-range behavior of these theories using the mean-field approximation as a simple tool, being exact in the limit of large flavor number N_{f}. We pay specific attention to the possible condensates and the long-range phase diagrams. Specifically, we identify two models that feature asymptotic freedom in the ultraviolet (UV), undergo dimensional transmutation in the sense of Coleman and Weinberg [35], and exhibit condensate formation in the long-range limit.

Since we expect the long-range phases to be characterized by a massive spectrum we start our exploration with a classification of possible mass terms for the relativistic Luttinger fermions. As the relativistic Abrikosov algebra needs to be spanned by a reducible representation of the underlying Clifford algebra, there is a larger set of possible mass terms. The latter is reminiscent to mass terms of relativistic $2+1$ dimensional Dirac

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materials where several mass terms can describe different patterns of gap formation [36–39].

Interestingly, the richer set of mass terms also goes along with a more intricate analytic structure of the corresponding propagators. We observe that the two asymptotically free models with low-energy condensate formation at the mean-field level do not feature tachyonic mass poles but a complex pair of poles or a branch cut.

Our paper is organized as follows: We begin in Sec. II with a short summary of relativistic Luttinger fermions following [17]. In Sec. III, we present a set of different mass terms for relativistic Luttinger fermions. Section IV introduces the set of models discussed in the present work. Here, we verify that each one features an asymptotically free coupling branch by computing the perturbative one-loop beta function. In Sec. V, we solve each model in a mean-field approximation exploring their potential for condensate formation. In Sec. VI, we study the analytic structure of the gapped Luttinger propagators in the complex momentum square plane. Conclusions are given in Sec. VII.

II. RELATIVISTIC LUTTINGER FERMIONS

We define field theories of relativistic Luttinger fermions in terms of their classical action. Focusing on four-dimensional spacetime, the action of the free theory reads [17]

$$S = \int d^4x [\bar{\psi} G_{\mu\nu}(i\partial^\mu)(i\partial^\nu)\psi], \tag{1}$$

where ψ denotes a spinor with d_γ components. Correspondingly, $G_{\mu\nu}$ represents a set of $d_\gamma \times d_\gamma$ matrices labeled by the Lorentz indices $\mu, \nu = 0, \ldots, 3$. These matrices satisfy the relativistic version of the Abrikosov algebra [2,7,17]

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = -\frac{2}{3} g_{\mu\nu} g_{\kappa\lambda} + \frac{4}{3}(g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}), \tag{2}$$

where the right-hand side involves the Minkowski metric $g = \mathrm{diag}(+,-,-,-)$ and is also implicitly understood to be proportional to the identity $\mathbb{1}_{d_\gamma}$ in spinor space. With respect to the Lorentz indices, the matrices $G_{\mu\nu}$ are symmetric, $G_{\mu\nu} = G_{\nu\mu}$, and traceless, $G^\mu{}_\mu = g^{\mu\nu} G_{\mu\nu} = 0$, implying that 9 linearly independent elements are needed to span the Abrikosov algebra (2). With respect to the spin indices, we can choose G_{0i} anti-hermitean whereas G_{00} and G_{ij} can be chosen hermitean for all i, $j = 1, 2, 3$.

Finally, the conjugate spinor in Eq. (1) is defined by $\bar{\psi} = \psi^\dagger h$ involving the spin metric h. Choosing h hermitean $h^\dagger = h$, the requirement that the classical action is real, $S \in \mathbb{R}$, imposes the conditions

$$\{h, G_{0i}\} = 0, \qquad [h, G_{ij}] = 0, \qquad [h, G_{\underline{\mu}\underline{\mu}}] = 0, \tag{3}$$

where underscored indices are exempted from Einstein's sum convention.

Both sets of algebraic conditions (2) and (3) can be spanned by a Euclidean Dirac algebra,

$$\{\gamma_A, \gamma_B\} = 2\delta_{AB}, \tag{4}$$

with d_e hermitean elements, $A, B = 1, 2, \ldots, d_e$. Whereas the irreducible representation of the Abrikosov algebra would, in principle, require only $d_e = 9$, the additional reality conditions (3) demands for $d_e = 11$. The latter implies that $d_\gamma = 2^{\lfloor d_e/2 \rfloor} = 32$ characterizes the irreducible representation of relativistic Luttinger fermions. An explicit representation of the $G_{\mu\nu}$ in terms of the Euclidean Dirac matrices γ_A is given in Appendix A. Setting, for instance, $G_{0i} = i\sqrt{\frac{2}{3}}\gamma_{A=i}$, all other $G_{\mu\nu}$ are real linear combinations of $\gamma_{A=4,\ldots,9}$, and the spin metric can be chosen as

$$h = \gamma_1 \gamma_2 \gamma_3 \gamma_{10}. \tag{5}$$

The free field equation for Luttinger fermions derived from Eq. (1) reads

$$G_{\mu\nu} \partial^\mu \partial^\nu \psi = 0. \tag{6}$$

Using the Abrikosov algebra, it follows straightforwardly that the Luttinger operator squares to (the square of) the Klein-Gordon operator, $(G_{\mu\nu}\partial^\mu\partial^\nu)^2 = (\partial^2)^2$, implying that each of the 32 components of ψ satisfies a relativistic wave equation.

III. MASS TERMS

In order to classify different possibilities of gap formation potentially occurring in self-interacting models studied below, let us first investigate the different mass terms that can be constructed for Luttinger fermions. As basic requirements, we are interested in Lorentz invariant bilinear and real terms that we can add to the action.

For this, let us first recall that the Abrikosov algebra is separately invariant under Lorentz transformations

$$G_{\mu\nu} \to G_{\kappa\lambda} \Lambda^\kappa{}_\mu \Lambda^\lambda{}_\nu, \tag{7}$$

where $\Lambda^\kappa{}_\mu$ is the transformation matrix of Lorentz tensors, as well as spin-base transformations [40–43]

$$G_{\mu\nu} \to \mathcal{S} G_{\mu\nu} \mathcal{S}^{-1}, \qquad \mathcal{S} \in \mathrm{SL}(32, \mathbb{C}). \tag{8}$$

Analogous to the conventional way of defining Lorentz transformations of, e.g., Dirac spinors (leaving the Dirac matrices constant), we can identify the Lorentz transformations S_{Lor} of Luttinger spinors as the subgroup of the spin-base group $\mathrm{SL}(32,\mathbb{C})$ which rotates the Lorentz transformed $G_{\mu\nu}$ matrices back to their original constant forms. This implies the identity

$$S_{\mathrm{Lor}}^{-1} G_{\mu\nu} S_{\mathrm{Lor}} = G_{\kappa\lambda} \Lambda^{\kappa}{}_{\mu} \Lambda^{\lambda}{}_{\nu}. \tag{9}$$

Correspondingly, $\psi \to S_{\mathrm{Lor}}\psi$ and $\bar{\psi} \to \bar{\psi} S_{\mathrm{Lor}}^{-1}$ denote the Lorentz transformation of Luttinger spinors.

Let us start now with the standard form of the mass term $\sim\bar{\psi}\psi$ first discussed in [17], leading to a free Lagrangian of the form

$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^{\mu}\partial^{\nu}\psi - m^2 \bar{\psi}\psi. \tag{10}$$

This mass term is invariant under spin-base and thus also under Lorentz transformations and real as a consequence of the spin metric being hermitean $h = h^{\dagger}$.

The corresponding equation of motion for the field ψ reads in momentum space

$$(G_{\mu\nu} p^{\mu} p^{\nu} - m^2)\psi = 0. \tag{11}$$

Multiplying by $(G_{\kappa\lambda} p^{\kappa} p^{\lambda} + m^2)$ from the left yields

$$\begin{aligned}(G_{\kappa\lambda} p^{\kappa} p^{\lambda} + m^2)&(G_{\mu\nu} p^{\mu} p^{\nu} - m^2)\psi \\ &= (p^4 - m^4)\psi = (p^2 - m^2)(p^2 + m^2)\psi = 0.\end{aligned} \tag{12}$$

In addition to the expected massive relativistic dispersion relation $p^2 = m^2$, this mass term also gives rise to tachyonic solutions with $p^2 = -m^2$. An explicit check confirms that both types of solutions occur with multiplicity 16 [44].

A second possible local fermionic bilinear is given by $\bar{\psi}\gamma_{10}\psi$. In order to add such a term in a way that the action stays real, it is instructive to verify the hermiticity properties of this bilinear. We observe that

$$\begin{aligned}(\bar{\psi}\gamma_{10}\psi)^{\dagger} &= \psi^{\dagger}\gamma_{10}^{\dagger}\bar{\psi}^{\dagger} = \psi^{\dagger}\gamma_{10} h\psi = -\psi^{\dagger} h\gamma_{10}\psi \\ &= -\bar{\psi}\gamma_{10}\psi,\end{aligned} \tag{13}$$

where we have used the unitarity of γ_{10} and h as well as the explicit form of our choice for h in Eq. (5). Therefore, reality of the action implies to choose a Lagrangian of the form

$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^{\mu}\partial^{\nu}\psi - i m_{10}^2 \bar{\psi}\gamma_{10}\psi. \tag{14}$$

The equation of motion in momentum space reads

$$(G_{\mu\nu} p^{\mu} p^{\nu} - i m_{10}^2 \gamma_{10})\psi = 0. \tag{15}$$

Since all $G_{\mu\nu}$ anticommute with γ_{10}, we multiply the equation of motion by $(G_{\kappa\lambda} p^{\kappa} p^{\lambda} - i m_{10}^2 \gamma_{10})$ and find

$$\begin{aligned}(G_{\kappa\lambda} p^{\kappa} p^{\lambda} - i m_{10}^2 \gamma_{10})&(G_{\mu\nu} p^{\mu} p^{\nu} - i m_{10}^2 \gamma_{10})\psi \\ &= (p^4 - m_{10}^4)\psi = (p^2 - m_{10}^2)(p^2 + m_{10}^2)\psi = 0,\end{aligned} \tag{16}$$

i.e., we again obtain solutions with both a regular massive as well as a tachyonic dispersion relation; also the corresponding multiplicities are 16 modes each, as for the standard mass term above. In fact, this is not astonishing, since both mass terms are connected by a discrete chiral/axial transformation. For this we first note, that the kinetic term (1) features a continuous $\mathrm{U}(1)_{10}$ symmetry,

$$\psi \to e^{i\vartheta\gamma_{10}}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{i\vartheta\gamma_{10}}, \tag{17}$$

which is broken by each of the mass terms discussed above. However, starting from the massive theory (10) and performing a $\mathrm{U}(1)_{10}$ transformation (17) with the choice $\vartheta = \frac{\pi}{4}$, we obtain the Lagrangian (14) upon the identification $m^2 \to m_{10}^2$. This also explains, why the solution spectra and multiplicities match upon this identification.

The situation is somewhat analogous to conventional Dirac theory, where mass terms of the form $-m\bar{\psi}\psi$ and $-m\bar{\psi}\gamma_5\psi$ are connected by an analogous discrete axial transformation.

Next, we can also use the eleventh Euclidean Dirac matrix γ_{11} in order to form a bilinear mass term. Using the fact that $[h, \gamma_{11}] = 0$, we can verify the reality property

$$(\bar{\psi}\gamma_{11}\psi)^{\dagger} = \psi^{\dagger}\gamma_{11} h\psi = \psi^{\dagger} h\gamma_{11}\psi = \bar{\psi}\gamma_{11}\psi. \tag{18}$$

The corresponding free Lagrangian now reads

$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^{\mu}\partial^{\nu}\psi - m_{11}^2 \bar{\psi}\gamma_{11}\psi, \tag{19}$$

giving rise to the equation of motion

$$(G_{\mu\nu} p^{\mu} p^{\nu} - m_{11}^2 \gamma_{11})\psi = 0. \tag{20}$$

Since $G_{\mu\nu}$ anticommutes with γ_{11}, we multiply by $(G_{\kappa\lambda} p^{\kappa} p^{\lambda} - m_{11}^2 \gamma_{11})$, yielding this time

$$\begin{aligned}(G_{\kappa\lambda} p^{\kappa} p^{\lambda} - m_{11}^2 \gamma_{11})&(G_{\mu\nu} p^{\mu} p^{\nu} - m_{11}^2 \gamma_{11})\psi \\ &= (p^4 + m_{11}^4)\psi = 0.\end{aligned} \tag{21}$$

In contrast to the previous cases, the dispersion relation is now solved by two complex zeros $p^2 = \pm i m_{11}^2$. Both types occur with multiplicity 16. Neither a standard massive nor a tachyonic mode are present. It is interesting to note that the Lagrangian (19) is invariant under $\mathrm{U}(1)_{10}$ transformations (17) as well.

Finally, we can use a product of the matrices γ_{10} and γ_{11} to construct another independent bilinear, for which we also check its reality properties based on the identities used above,

$$(\bar{\psi}\gamma_{10}\gamma_{11}\psi)^\dagger = \psi^\dagger\gamma_{11}\gamma_{10}h\psi = -\psi^\dagger\gamma_{10}\gamma_{11}h\psi = \psi^\dagger h\gamma_{10}\gamma_{11}\psi = \bar{\psi}\gamma_{10}\gamma_{11}\psi. \tag{22}$$

For convenience, let us introduce the hermitean product

$$\gamma_{01} := -i\gamma_{10}\gamma_{11}, \qquad \gamma_{01}^\dagger = \gamma_{01}, \tag{23}$$

which satisfies

$$[G_{\mu\nu},\gamma_{01}]=0, \quad \{h,\gamma_{01}\}=\{\gamma_{10},\gamma_{01}\}=\{\gamma_{11},\gamma_{01}\}=0. \tag{24}$$

Correspondingly, the free real Lagrangian containing the new bilinear can be written as

$$\mathcal{L} = -\bar{\psi}G_{\mu\nu}\partial^\mu\partial^\nu\psi - im_{01}^2\bar{\psi}\gamma_{01}\psi, \tag{25}$$

yielding the equation of motion

$$(G_{\mu\nu}p^\mu p^\nu - im_{01}^2\gamma_{01})\psi = 0. \tag{26}$$

As $G_{\mu\nu}$ commutes with γ_{01}, we multiply by $(G_{\kappa\lambda}p^\kappa p^\lambda + im_{01}^2\gamma_{01})$ and obtain

$$(G_{\kappa\lambda}p^\kappa p^\lambda + im_{01}^2\gamma_{01})(G_{\mu\nu}p^\mu p^\nu - im_{01}^2\gamma_{01})\psi = (p^4 + m_{01}^4\gamma_{01}^2)\psi = (p^4 + m_{01}^4)\psi = 0, \tag{27}$$

since γ_{01} squares to one. As in the preceding case, we observe complex conjugate zeros in the momentum plane p^2, implying solutions with a dispersion relation $p^2 = \pm im_{01}^2$. Each type of solution has again multiplicity 16. Also, the Lagrangian (25) is invariant under the $\mathrm{U}(1)_{10}$ symmetry.

It is tempting to expect that each of the dispersion relations found for the different free massive theories corresponds to a generic analytic pole structure in the complex p^2 plane. Whether or not this is the case is discussed in Sec. VI.

It is suggestive to introduce two further U(1) transformations, namely,

$$\mathrm{U}(1)_{11}: \psi \to e^{i\vartheta\gamma_{11}}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\vartheta\gamma_{11}}, \tag{28}$$

$$\mathrm{U}(1)_{01}: \psi \to e^{i\vartheta\gamma_{01}}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\vartheta\gamma_{01}}. \tag{29}$$

We observe that the four mass terms can be connected via discrete versions of these transformations: e.g., the m_{01} mass is connected to the m_{10} mass via a $\mathrm{U}(1)_{11}$ transformation with $\vartheta = \frac{\pi}{4}$.

However, it is important to emphasize that the transformations (28) and (29) do not represent symmetries of the kinetic term and thus are no symmetries of the Lagrangians if taken at face value. Some of these transformations may, nevertheless, be uplifted to a symmetry, if combined with a simultaneous transformation of the spin metric. E.g., we observe that a discrete $\mathrm{U}(1)_{01}$ transformation with $\vartheta = \frac{\pi}{4}$ transforms the kinetic term into an analogous kinetic term with the spin metric being replaced by $h \to \gamma_1\gamma_2\gamma_3\gamma_{11}$. The latter is also a valid choice for the spin metric satisfying all necessary conditions of Eq. (3).

The existence of a set of different masslike terms is similar to that for Dirac fermions in reducible representation, with the $d = 3$ case with $d_\gamma = 4$ being the most well-studied case [36–39]. In contrast to this, the present case of Luttinger fermions is not a reducible representation: though the Abrikosov algebra (2) could be represented by 16-dimensional matrices in $d = 4$, the spin metric cannot and thus requires a 32-dimensional representation. From a technical viewpoint the properties of the spin metric are also responsible for the fact that the transformations (28) and (29) do not correspond to symmetries of the action. Hence, there is also no extended flavor symmetry such as $\mathrm{U}(2N_\mathrm{f})$ as in the case of $d = 3$ reducible Dirac fermions.

Let us finally remark that the existence of further mass terms is conceivable; e.g, if the fermionic field satisfies additional reality constraints, mass terms analogous to Majorana masses in the Dirac case may be allowed.

IV. SELF-INTERACTING FERMIONIC MODELS

Let us introduce a set of massless theories of self-interacting relativistic Luttinger fermions with interactions defined in terms of the above-mentioned spinor bilinears. For a first glance at the quantum theory, we perform a one-loop analysis of their renormalization group (RG) flow concentrating on the large-N_f limit for simplicity.

In the present section, we work in the Euclidean domain in order to use Wilsonian RG techniques. Note that the definition of the following models in the Euclidean differs by a minus sign in the interaction terms from the formulation in Minkowskian spacetime, cf. Appendix B.

We start with the simplest interaction term which is reminiscent to that of the standard Gross-Neveu [45] model, cf. [17]

$$S = \int d^4x \left[-\bar{\psi}G_{\mu\nu}\partial^\mu\partial^\nu\psi + \frac{\bar{\lambda}}{2}(\bar{\psi}\psi)^2\right]. \tag{30}$$

Flavor indices are suppressed for simplicity here and in the following; all bilinears written in terms of parentheses are assumed to be flavor singlets, i.e., $(\bar{\psi}\psi) = \bar{\psi}^a\psi^a$, where $a = 1, \ldots, N_f$. This, as well as all subsequent models, therefore features a global $\mathrm{U}(N_\mathrm{f})$ symmetry which will remain trivially present in all of the subsequent discussion.

The Luttinger-Gross-Neveu (LGN) model additionally exhibits a discrete axial symmetry of the type of Eq. (17) with the choice $\vartheta = \frac{\pi}{2}$. Under such a transformation, the kinetic term is invariant. The scalar bilinear transforms as $(\bar{\psi}\psi) \to -(\bar{\psi}\psi)$ which leaves the interaction term in Eq. (30) invariant, but forbids the occurrence of a mass term. In analogy to the standard Gross-Neveu model, it is tempting to speculate that this discrete symmetry might be broken depending on the sign and the strength of the initial value for the coupling $\bar{\lambda}$.

Another rather similar model is given by a scalar interaction involving the γ_{10} matrix,

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - \frac{\bar{\lambda}}{2} (\bar{\psi}\gamma_{10}\psi)^2 \right]. \qquad (31)$$

Here and in the following, the sign in front of the coupling is chosen such that the one-loop beta functions computed below have the same form. Also, we use the same letter $\bar{\lambda}$ for the coupling for simplicity, even though the couplings in all the models considered here are unrelated. Also this γ_{10} model has the same discrete axial symmetry as the Luttinger-Gross-Neveu model: under the transformation (17) with the choice $\vartheta = \frac{\pi}{2}$, the Lagrangian in Eq. (31) remains invariant, whereas a mass term of the m_{10} type as in Eq. (14) would change sign and thus break the symmetry.

Next, we introduce the γ_{11} model in terms of the action

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - \frac{\bar{\lambda}}{2} (\bar{\psi}\gamma_{11}\psi)^2 \right]. \qquad (32)$$

This γ_{11} model is invariant under the full continuous $\mathrm{U}(1)_{10}$ symmetry (17). However, already the m_{11} mass term in Eq. (19) is invariant under this symmetry, hence the realization of this symmetry does not serve as an indicator for gap formation. Instead, this role is played by a combined discrete symmetry involving both a discrete $\mathrm{U}(1)_{11}$ transformation (28) with $\vartheta = \frac{\pi}{2}$ and the replacement $\psi \to -\psi$ and $\bar{\psi} \to \bar{\psi}$ (treating ψ and $\bar{\psi}$ as independent variables in the quantum theory). The γ_{11} model (32) is invariant under this discrete transformation while an m_{11} mass term is not. This discrete symmetry is somewhat similar to the discrete symmetry of the 3d Gross-Neveu model with irreducible Dirac fermions [46].

As a fourth action, we consider the γ_{01} model:

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{\bar{\lambda}}{2} (\bar{\psi}\gamma_{01}\psi)^2 \right]. \qquad (33)$$

Also the γ_{01} model is invariant under the full continuous $\mathrm{U}(1)_{10}$ symmetry (17), as we observed already for the m_{01} mass term in Eq. (25). Hence, the status of this symmetry is not indicative for mass generation. In fact, none of the transformations discussed in the previous sections is a suitable ingredient for constructing an indicator symmetry for mass gap formation as each of them acts similarly on the kinetic and the mass term. Still, we have checked explicitly that the interaction does not generate an m_{01} mass term at one-loop order. This implies that either a mass-protecting symmetry exists or an m_{01} mass term may be generated at higher-loop order.

Finally, we note that the models can, of course, also be combined such that continuous symmetries emerge. An example is given by a Luttinger-fermionic analog of the Nambu-Jona-Lasinio (NJL) model [47], which features a full continuous $\mathrm{U}(1)_{10}$ symmetry (17), as first discussed in [17],

$$S = \int d^4x \left\{ -\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{\bar{\lambda}}{2} [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_{10}\psi)^2] \right\}. \qquad (34)$$

In this model, the $\mathrm{U}(1)_{10}$ symmetry forbids corresponding mass terms such that the status of the symmetry can be expected to be indicative of gap formation.

For each of these theories, we compute the one-loop beta function. While this can straightforwardly be done with any conventional quantum field theory method, we use the functional renormalization group here, as it can be generalized straightforwardly to future nonperturbative studies. Specifically for fermionic theories, the computational techniques based on the Wetterich equation [48] are well developed [49–51] and have found manifold nonperturbative applications [46,52–61]. Starting from the Wetterich equation for the effective average action Γ_k,

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr}[\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1}], \qquad (35)$$

the regulator function R_k implements the regularization in the Euclidean momentum domain at a regularization scale k; here $\partial_t = k\frac{d}{dk}$. Provided Γ_k is fixed in terms of the bare microscopic action S for $k \to \Lambda$ as a UV boundary condition, the full quantum effective action Γ is approached for $k \to 0$ in the IR. Importantly, it can be chosen in such a way that the symmetries of the kinetic term are respected by the regularization procedure. Using this as well as standard methods as detailed in [17], we project the Wetterich equation for each of the models onto a theory space defined by the ansatz

$$\Gamma_k = \int_x [-Z_\psi \bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \mathcal{L}_{\mathrm{int}}], \qquad (36)$$

where $\mathcal{L}_{\mathrm{int}}$ denotes the interaction term of the corresponding model including the scale-dependent coupling $\bar{\lambda}$ and a wave function renormalization Z_ψ. Introducing the renormalized coupling

$$\lambda = \frac{\bar{\lambda}}{Z_\psi^2}, \qquad (37)$$

we find for each of the five models the beta function

$$\partial_t \lambda = -\frac{4N_f}{\pi^2} \lambda^2 \qquad (38)$$

in the large-N_f limit. [To one-loop order, the anomalous dimension $\eta_\psi = -\partial_t \ln Z_\psi$ vanishes $\eta_\psi = 0$, which completes the flow in the theory space spanned by the ansatz (36).] Equation (38) demonstrates that each of these models is asymptotically free for positive $\lambda > 0$, approaching the Gaussian fixed point towards the ultraviolet as a high-energy fixed point. Asymptotic freedom guarantees that the models can be extended to arbitrarily high energy scales. Towards low energies, the coupling λ grows larger and the true behavior of the models has to be analyzed by nonperturbative means.

By contrast, the Gaussian fixed point is infrared attractive for negative couplings, $\lambda < 0$. Correspondingly, the couplings diverge to negative infinity towards high energies (Landau poles); thus, a nonperturbative analysis is necessary to search for a possible UV completion or to prove triviality of the models in this coupling branch.

Of course, the present analysis can straightforwardly be generalized to finite N_f values. However, a consistent treatment in this regime requires one to include a Fierz-complete set of interaction channels. This has, e.g., been done for the Luttinger-Gross-Neveu model in [17,62] which required the inclusion of a tensor channel $\sim(\bar\psi G_{\mu\nu}\psi)^2$. The property of coupling branches where the theory is asymptotically safe then generalizes to higher dimensional regions in the space of all couplings. We expect similar properties to hold for each of the models studied here.

V. MEAN-FIELD THEORY

In order to investigate the possible occurrence of gap formation in the models defined in the previous section, we use mean-field theory which becomes exact in the large-N_f limit. For this, we bilinearize the fermionic actions given above using auxiliary scalar fields with a Gaussian action and a Yukawa coupling to the fermionic fields. In the large-N_f limit, the auxiliary scalar field integral is dominated by the classical configurations, i.e., the extrema of the action which in turn is governed by the fermion determinant. Since the true expansion parameter of the large-N_f limit also involves the dimensionality of the Clifford algebra [56], the expansion is in powers of $\frac{1}{d_\gamma N_\mathrm{f}} = \frac{1}{32 N_\mathrm{f}}$ which is already a small parameter for $N_\mathrm{f} = 1$.

In view of various forms of possible mass terms discussed in Sec. III, we expect radiatively generated gaps to occur in the complex momentum plane. Therefore, we perform the mean-field analysis in Minkowski space, using propertime methods for the regularization. Of course, for all models, the sign change of the interaction term when comparing the Euclidean description used in Sec. IV with the present Minkowskian analysis has to be accounted for, cf. Appendix B.

A. Luttinger-Gross-Neveu model

Let us start with the Luttinger-Gross-Neveu model, the action of which in Minkowski spacetime reads

$$S = \int d^4x \left[-\bar\psi G_{\mu\nu}\partial^\mu\partial^\nu\psi - \frac{\bar\lambda}{2}(\bar\psi\psi)^2 \right]. \tag{39}$$

We bilinearize the interaction term using a Hubbard-Stratonovich (HS) transformation introducing an auxiliary real scalar field, such that the action reads

$$S_\mathrm{FB} = \int d^4x \left[-Z_\psi\bar\psi G_{\mu\nu}\partial^\mu\partial^\nu\psi + \bar h\phi\bar\psi\psi - \frac{1}{2}\bar m^2\phi^2 \right]. \tag{40}$$

First, we note that the action (40) is manifestly real if the coupling $\bar h$ is real, since both ϕ and $\bar\psi\psi$ are real. Also, the sign of the scalar mass term is such that it corresponds to a positive mass term and thus a stable potential in Minkowski space. The discrete axial symmetry of the fermionic description is also preserved by the action (40) if the scalar field transforms as $\phi \to -\phi$. The theories defined by Eq. (39) and Eq. (40) are identical both on the classical as well as on the quantum level, provided the coupling constants satisfy a matching condition. The matching condition can, e.g., be derived from the classical equation of motion for the scalar field (corresponding to a Gaussian integration on the quantum level) which reads

$$\frac{\delta S}{\delta\phi} = \bar h\,\bar\psi\,\psi - \bar m^2\phi = 0. \tag{41}$$

Inserting the solution for ϕ into Eq. (40) leads us back to Eq. (39) provided the matching condition

$$\bar\lambda = -\frac{\bar h^2}{\bar m^2} \tag{42}$$

is satisfied. Incidentally, the Yukawa coupling could be set to a unit scale by rescaling the scalar field; we keep it for reasons of generality. The minus sign in (42) implies that the HS transformation can be meaningfully performed in the standard fashion only for negative values of the coupling $\bar\lambda$. This confines the following analysis to the nonasymptotically free branch $\bar\lambda < 0$ of the Luttinger-Gross-Neveu model. While this corresponds to the branch where the Gaussian fixed point is IR attractive, we are still free to assume that the initial value of the coupling $\bar\lambda$ is sufficiently large to potentially introduce a nontrivial long-range behavior.

For this, we investigate the effective action of the scalar field upon integrating out the fermions. This functional integral yields the one-loop contribution $\Gamma_{1\ell}$ to the effective action in terms of the fermion determinant. We evaluate the latter for a constant scalar field, $\phi = \phi_0 = \text{constant}$:

$$\Gamma_{1\ell} = -i \ln \det[-G_{\mu\nu}\partial^\mu\partial^\nu + \bar{h}\phi_0] = -\frac{i}{2}\ln \det[-(\partial^2)^2 + (\bar{h}\phi_0)^2], \tag{43}$$

where in the last step we have used the γ_{10} hermiticity of the kinetic term, $\gamma_{10}G_{\mu\nu}\partial^\mu\partial^\nu\gamma_{10} = -G_{\mu\nu}\partial^\mu\partial^\nu$. Now, we employ $\ln\det = \mathrm{Tr}\ln$ and perform a trivial vacuum subtraction such that $\Gamma_{1\ell}[\phi_0 = 0] = 0$. Going to Fourier space and evaluating the functional trace we arrive at

$$\Gamma_{1\ell} = -\frac{i}{2}N_f d_\gamma \Omega \int \frac{d^4p}{(2\pi)^4}\ln\left(\frac{p^4 - (\bar{h}\phi_0)^2}{p^4}\right) = \frac{N_f d_\gamma \Omega}{2^6\pi^2}\int \frac{dt}{t^2}(1 - e^{(\bar{h}\phi_0)^2 t}). \tag{44}$$

For the last step, we have rotated the momentum integral to the Euclidean and used the Schwinger propertime representation of the logarithm. This representation is both infrared and ultraviolet divergent. We can cure the UV divergence with the introduction of a UV cutoff scale Λ, i.e., introduce a lower bound of the integral at $1/\Lambda^4$. Of course, the UV divergence is indicative for the renormalization of the couplings.

The IR divergence signals the existence of tachyonic modes. This is already obvious from the first line of Eq. (44) where the argument of the logarithm becomes negative for momenta with $p^4 < (\bar{h}\phi_0)^2$. We deal with the IR divergence by studying the integral in the complex $(\bar{h}\phi_0)^2$ plane where it exists for all $\mathrm{Re}(\bar{h}\phi_0)^2 < 0$; we then continue the result analytically back to real values of ϕ_0. As a result, the effective action picks up an imaginary part indicating that the assumption of a finite scalar mean field $|\phi_0| > 0$ would correspond to an unstable vacuum state. Expanding the resulting expression in inverse powers of the UV cutoff Λ, we obtain

$$\Gamma_{1\ell} = -\frac{N_f d_\gamma \Omega}{2^6\pi^2}(\bar{h}\phi_0)^2\left[1 - \gamma - \ln\left(\frac{(\bar{h}\phi_0)^2}{\Lambda^4}\right) - i\pi\right] + O((\bar{h}\phi_0)^2/\Lambda^4), \tag{45}$$

where Ω denotes the spacetime volume. Following Schwinger [63], the imaginary part of the effective action is a measure for the decay rate of a state with finite ϕ_0 with $\exp(-2\,\mathrm{Im}\,\Gamma)$ quantifying the probability for the state to persist. This tells us already that within our assumptions only the $\phi_0 = 0$ state can be an equilibrium state.

In order to further study the stability of this state, we consider the (real part of the) effective mean-field potential including the classical scalar mass term but ignoring all terms that vanish in the limit $\Lambda \to \infty$,

$$V_{\mathrm{eff}}(\phi_0) = \frac{1}{2}\phi_0^2\left\{\bar{m}^2 + \frac{N_f d_\gamma}{2^5\pi^2}\bar{h}^2\left[1 - \gamma - \ln\left(\frac{(\bar{h}\phi_0)^2}{\Lambda^4}\right)\right]\right\}, \tag{46}$$

where $\gamma \simeq 0.5772\ldots$ denotes the Euler-Mascheroni constant. With Λ being the largest scale (to be sent to infinity), we observe already in this unrenormalized expression that the term dominating the effective potential at large fields is positive, $V_{\mathrm{eff}}(\phi_0) \sim -\phi_0^2 \ln[(\bar{h}\phi_0)^2/\Lambda^4] > 0$. Also, all other terms are positive for finite ϕ_0 and vanish only for $\phi_0 = 0$ in the validity regime of Eq. (46) with $\Lambda^2 \gg (\bar{h}\phi_0)$. Therefore, the zero-field mean-field state $\phi_0 = 0$ is the minimum of the (real part of the) effective potential.

Of course, we can also introduce renormalized quantities by defining a renormalized mass at some renormalization scale μ,

$$m^2(\mu) \coloneqq \bar{m}^2 - \frac{N_f d_\gamma}{2^5\pi^2}\bar{h}^2 \ln\frac{\mu^4}{\Lambda^4}. \tag{47}$$

We emphasize that both terms on the right-hand side are strictly positive, since $\mu \ll \Lambda$ for a meaningful renormalization scale well below the UV cutoff. The correspondingly renormalized effective potential reads

$$V_{\mathrm{eff}}(\phi_0) = \frac{\phi_0^2}{2}\left\{m^2(\mu) + \frac{N_f d_\gamma}{2^5\pi^2}\bar{h}^2\left[1 - \gamma - \ln\frac{(\bar{h}\phi_0)^2}{\mu^4}\right]\right\}. \tag{48}$$

For small fields $(\bar{h}\phi_0)^2 < \mu^4$, we again observe that the term in curly brackets remains positive, hence $\phi_0 = 0$ is a local minimum of the effective potential. On the other hand, for large fields $(\bar{h}\phi_0)^2 \gg \mu^4$, it naively seems that the effective potential is not bounded from below, since the logarithm can become arbitrarily large. However, this is an artifact of this representation. As discussed above, the mean-field potential in the representation (46) stays positive for all $\bar{m}^2 > 0$, since Λ is the largest scale in the game and eventually goes to infinity. Indeed, for $\bar{m}^2 > 0$, $m^2(\mu)$ grows logarithmically for decreasing μ, which compensates the logarithmic increase of $\ln(\frac{(\bar{h}\phi_0)^2}{\mu^4})$, keeping (48) positive. A plot of the effective potential V_{eff} is depicted in Fig. 1, confirming that the trivial vacuum $\phi_0 = 0$ is the global minimum in the validity range of the computation (solid line).

The present discussion is very similar to that of mean-field or Coleman-Weinberg-type effective potentials in Dirac-Yukawa theories [64–66], where a naive look at the renormalized form can be misleading if the ultimate existence of a UV cutoff is ignored.

In summary, we conclude that $\phi_0 = 0$ is not only a local, but the global minimum of V_{eff} in the mean-field approximation for the $\bar{\lambda} < 0$ branch of the Luttinger-Gross-Neveu

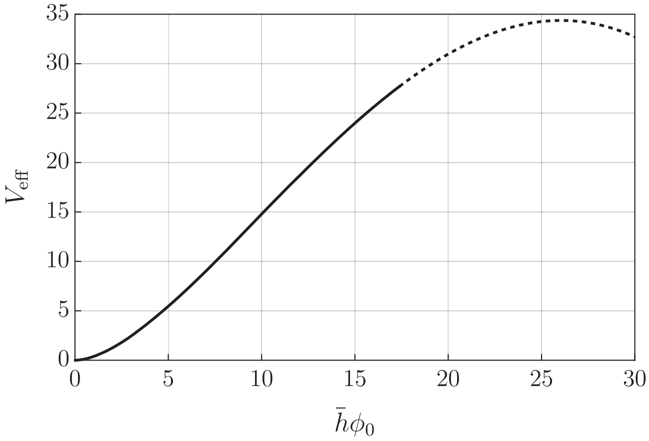

FIG. 1. Renormalized effective potential V_{eff} of the Luttinger-Gross-Neveu model for $N_{\text{f}} = 1$ and $d_\gamma = 32$. The plot is obtained by setting the renormalization scale to a small value, namely $\mu = 0.5$ and the ratio $m^2(\mu)/\bar{h}^2$ is set to 1. Moreover, since the mean-field analysis is done for (possibly large) negative values of the bare coupling $\bar{\lambda}$, we set the bare mass parameter to its lower bound, i.e. $\bar{m} = 0$ (larger values would correspond to couplings closer to zero). Choosing $\bar{h}\phi_0 < 0.5\Lambda^2$ as an *ad hoc* criterion for the validity regime of our analysis requiring, in principle, $|\bar{h}\phi_0| \ll \Lambda^2$, the effective potential is depicted with a solid line in the validity region, and with a dashed line where the assumptions are violated.

model. At mean field level, the fermions develop neither a mass nor a tachyonic mode in this branch of the model even if the initial value of the bare coupling $\bar{\lambda} < 0$ has a large absolute value at the UV scale Λ. Since the standard HS transformation cannot be applied to the positive coupling branch, we obtain no information about the status of the model in the asymptotically free branch where the couplings grows towards the IR.

As a consistency check, we can compute the β function at mean-field level by introducing the scale-dependent coupling $\lambda(\mu) = -\bar{h}^2/m^2(\mu)$, which for $d_\gamma = 32$ results in

$$\mu \frac{\partial}{\partial\mu} \lambda(\mu) = -\frac{4N_f}{\pi^2} \lambda^2(\mu). \tag{49}$$

Strictly speaking, we have derived this result for negative values of λ only. But it agrees with the result (49) of the preceding section for all values of λ, thereby reproducing the one-loop RG flow in the large-N_{f} limit including asymptotic freedom in the positive coupling branch.

B. γ_{10} model

Let us now study the second model discussed above, with Minkowskian action

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{\bar{\lambda}}{2} (\bar{\psi}\gamma_{10}\psi)^2 \right]. \tag{50}$$

As before, we aim at the mean-field potential in order to explore the possibility of gap formation. This time, the HS transformation leads us to

$$S_{\text{FB}} = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + i\bar{h}\phi\bar{\psi}\gamma_{10}\psi - \frac{1}{2}\bar{m}^2\phi^2 \right], \tag{51}$$

for the partially bosonized version of the model. The factor of i in front of the Yukawa interaction guarantees that the action is real, cf. Eq. (14). The discrete axial symmetry of Eq. (50) again induces a $\mathbb{Z}_2$ symmetry for the scalar $\phi \to -\phi$.

It turns out that the equivalence of the two actions requires the same matching condition (42). Again, only the negative coupling branch $\bar{\lambda} < 0$ can be studied in mean-field theory.

It is also straightforward to verify that the mean-field analysis leads to the same one-loop effective action including the imaginary part for finite ϕ_0 as well as the same effective potential as in the Luttinger-Gross-Neveu model (46), with a global minimum at $\phi_0 = 0$. Also in this case, the Luttinger fermions remain ungapped and do not exhibit a tachyonic mode in the negative coupling branch. While we again cannot address the long-range physics in the positive coupling branch, the mean-field analysis yields the correct β function (49) for all values of the coupling.

With hindsight, the fact that the two models behave identically is not too surprising, since the discrete $U(1)_{10}$ transformation with $\vartheta = \frac{\pi}{4}$ discussed below Eq. (17) transforms the Luttinger-Gross-Neveu model into the γ_{10} model at each stage of the analysis.

C. γ_{11} model

Now, the γ_{11} model turns out to behave rather differently. We start with the corresponding action in Minkowski spacetime

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{\bar{\lambda}}{2} (\bar{\psi}\gamma_{11}\psi)^2 \right]. \tag{52}$$

This time, the HS transformation leads us to the partially bosonized action

$$S_{\text{FB}} = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \bar{h}\phi\bar{\psi}\gamma_{11}\psi - \frac{1}{2}\bar{m}^2\phi^2 \right] \tag{53}$$

which is equivalent to (52), provided the matching condition

$$\bar{\lambda} = \frac{\bar{h}^2}{\bar{m}^2} \tag{54}$$

is satisfied. We observe that the HS transformation is now tied to the positive asymptotically free $\bar{\lambda}$ branch. Again, the discrete symmetry of the fermionic formulation inhibiting a bare mass term induces a $\mathbb{Z}_2$ symmetry for the scalar field such that S_{FB} is invariant under the combined transformation. Choosing the scalar field to be constant $\phi = \phi_0 = \mathrm{const}$, the fermion determinant yielding the one-loop contribution to the effective action reads

$$\begin{aligned}\Gamma_{1\ell} &= -i \ln \det[-G_{\mu\nu}\partial^\mu \partial^\nu + \bar{h}\phi_0 \gamma_{11}] \\ &= -\frac{i}{2} \ln \det[-(\partial^2)^2 - (\bar{h}\phi_0)^2]. \end{aligned} \tag{55}$$

In the last step, we have used the γ_{10}-hermiticity of the kinetic term, as well as the anticommutator properties $\{G_{\mu\nu}, \gamma_{11}\} = \{\gamma_{10}, \gamma_{11}\} = 0$. Evaluating the resulting trace in Fourier space, performing the vacuum subtraction, and using the propertime representation, we arrive at

$$\begin{aligned}\Gamma_{1\ell} &= -\frac{i}{2} N_f d_\gamma \Omega \int \frac{d^4 p}{(2\pi)^4} \ln\left(\frac{p^4 + (\bar{h}\phi_0)^2}{p^4}\right) \\ &= \frac{N_f d_\gamma \Omega}{2^6 \pi^2} \int \frac{dt}{t^2} (1 - e^{-(\bar{h}\phi_0)^2 t}). \end{aligned} \tag{56}$$

In the first line, it is already obvious that a finite value of ϕ_0 does neither induce tachyonic modes nor an imaginary part of the action. Consequently, the propertime representation (second line) requires only a UV cutoff, implemented by a $1/\Lambda^4$ lower bound at the t integral, whereas the action is IR finite. Correspondingly, the unrenormalized effective potential including the classical scalar mass term can straightforwardly be computed. Ignoring the terms that vanish in the large-Λ limit, we find

$$V_{\mathrm{eff}}(\phi_0) = \phi_0^2 \left\{ \frac{\bar{m}^2}{2} - \frac{N_f d_\gamma}{2^6 \pi^2} \bar{h}^2 \left[1 - \gamma - \ln\left(\frac{(\bar{h}\phi_0)^2}{\Lambda^4}\right) \right] \right\}. \tag{57}$$

In order to renormalize the effective potential, we first introduce a renormalized mass parameter at some renormalization scale μ,

$$m^2(\mu) \coloneqq \bar{m}^2 + \frac{N_f d_\gamma}{2^5 \pi^2} \bar{h}^2 \ln \frac{\mu^4}{\Lambda^4}. \tag{58}$$

Note that $m^2(\mu)$ can take values with either sign in contrast to the renormalized mass in the previously discussed models, cf. Eq. (47). The effective potential can then be written as

$$V_{\mathrm{eff}} = \frac{1}{2} \phi_0^2 \left\{ m^2(\mu) - \frac{N_f d_\gamma}{2^5 \pi^2} \bar{h}^2 \left[1 - \gamma - \ln\left(\frac{(\bar{h}\phi_0)^2}{\mu^4}\right) \right] \right\}. \tag{59}$$

The latter displays a nontrivial minimum satisfying $V'_{\mathrm{eff}}(\phi_0 = v) = 0$ at

$$(\bar{h}v)^2 = \mu^4 e^{-\frac{2^5 \pi^2}{N_f d_\gamma} \frac{m^2(\mu)}{\bar{h}^2} - \gamma}. \tag{60}$$

Using Eq. (58), it is straightforward to verify that this minimum is RG invariant,

$$\mu \frac{d}{d\mu} v^2 = 0. \tag{61}$$

In terms of the minimum, the renormalized potential can also be brought into an RG invariant form

$$V_{\mathrm{eff}} = \frac{N_{\mathrm{f}} d_\gamma}{2^6 \pi^2} (\bar{h}\phi_0)^2 \ln \frac{(\bar{h}\phi_0)^2}{e(\bar{h}v)^2}, \tag{62}$$

where e denotes the Euler number.

The effective potential is plotted in Fig. 2; it is bounded from below and exhibits the nontrivial minimum at $\phi_0 = v$. In this quantum-induced ground state, the discrete $\mathbb{Z}_2$ symmetry is spontaneously broken, giving rise to a fermionic m_{11} mass term. We can read off from Eq. (53) that

$$m_{11}^2 = \bar{h}v. \tag{63}$$

In fact, the product $\bar{h}v$ sets the scale for all dimensionful quantities occurring in Eqs. (62), (63). Since the original theory, the γ_{11} model, has no intrinsic scale on the classical level, this is a textbook example for dimensional transmutation. As discussed in Sec. III, the quantity m_{11}^2 gaps the fermionic spectrum by a complex conjugate pair of offsets from zero $p^2 = \pm i m_{11}^2$.

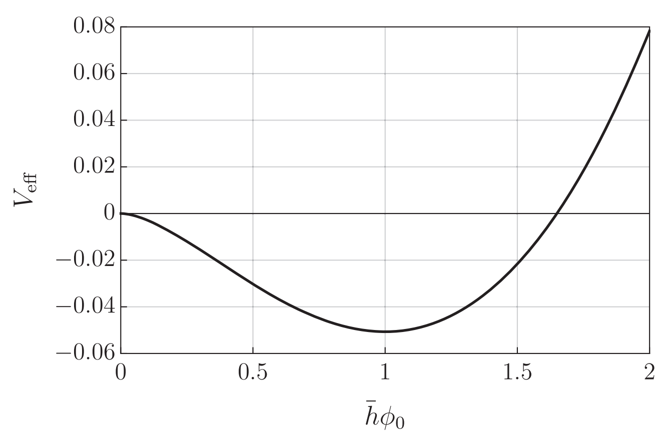

FIG. 2. Renormalized effective potential of the γ_{11} model for $N_{\mathrm{f}} = 1$ and $d_\gamma = 32$. The position of the minimum has been set to 1, namely $\bar{h}v = 1$, in units of the square of an arbitrary dimensionful scale. The presence of a minimum at $\phi_0 = v$ indicates the formation of a mass gap in the fermionic spectrum.

In principle, the mean-field analysis also gives access to the curvature of the effective potential at the minimum. From Eq. (62), we obtain

$$V''(\phi_0 = v) = \frac{2N_\mathrm{f}}{\pi^2}\bar{h}^2, \tag{64}$$

where the scale is set purely in terms of the (dimensionful) Yukawa coupling. In the present setting, this result does not acquire an independent meaning. In order to interpret Eq. (64) as a mass of a scalar σ-type excitation on top of the condensate v, we would also need the correspondingly induced kinetic term for this excitation. For instance, if the fluctuation induced kinetic term read $S[\sigma] = \int \frac{1}{2} Z_\sigma \partial_\mu \sigma \partial^\mu \sigma$ with a wave function renormalization Z_σ, the result for the scalar excitation would be $m_\sigma^2 = \frac{2N_\mathrm{f}}{\pi^2}\frac{\bar{h}^2}{Z_\sigma}$.

Finally, as a self-consistency check, we can derive the β function for the scale-dependent coupling $\lambda(\mu) = \bar{h}^2/m^2(\mu)$ within mean-field theory using Eq. (58), yielding

$$\mu\frac{\partial}{\partial\mu}\lambda(\mu) = -\frac{4N_f}{\pi^2}\lambda^2(\mu), \tag{65}$$

in agreement with previous results. For the γ_{11} model, the mean-field computation proceeds fully in the asymptotically free $\lambda > 0$ branch of the model.

D. γ_{01} model

Let us now study the fourth model with a scalar self-interaction channel on the mean-field level. The computation is interesting, since it requires slightly different techniques. The action reads in the Minkowskian domain

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu}\partial^\mu\partial^\nu\psi - \frac{\bar{\lambda}}{2}(\bar{\psi}\gamma_{01}\psi)^2\right]. \tag{66}$$

Bilinearizing this action by an HS transformation, we arrive at

$$S_\mathrm{FB} = \int d^4x \left[-\bar{\psi} G_{\mu\nu}\partial^\mu\partial^\nu\psi + i\bar{h}\phi\bar{\psi}\gamma_{01}\psi - \frac{1}{2}\bar{m}^2\phi^2\right], \tag{67}$$

where the i in front of the Yukawa term renders the action real as in Eq. (51), cf. also Eq. (25). The two actions are equivalent if the matching condition (54) is satisfied. The mean-field approximation thus gives us information about the asymptotically free $\lambda > 0$ branch.

As before, we write the mean-field quantum contribution to the effective action in terms of the fermion determinant,

$$\Gamma_{1\ell} = -i \ln\det[-G_{\mu\nu}\partial^\mu\partial^\nu + i\bar{h}\phi_0\gamma_{01}]. \tag{68}$$

We have not found a way to rewrite the determinant in terms of scalar squares of the involved operators as no obvious γ_A-hermiticity property for a suitable value of A appears to be available. Hence, we keep the nontrivial spin structure for the propertime representation of the ln det. Assuming $\phi_0 = \text{const}$, and using that $(-G_{\mu\nu}\partial^\mu\partial^\nu)^{-1} = \frac{-G_{\mu\nu}\partial^\mu\partial^\nu}{(-\partial^2)^2}$, we can write the vacuum-subtracted expression as

$$\Gamma_{1\ell} = -i\mathrm{Tr}\ln\left(\mathbb{1} + i\frac{\bar{h}\phi_0}{(-\partial^2)}\frac{(-G_{\mu\nu}\partial^\mu\partial^\nu)}{(-\partial^2)}\gamma_{01}\right), \tag{69}$$

where ln det = Tr ln has been used. Having in mind that the coordinate/momentum trace will ultimately be performed in Euclidean momentum space, we note that the involved operators possess simple hermiticity properties. The latter imply that the eigenvalues of the total operator in parentheses in the Euclidean must be of the form $1 + ix$ with $x \in \mathbb{R}$. Hence, we can use the standard propertime representation of the logarithm such that we obtain in momentum space

$$\Gamma_{1\ell} = -i\mathrm{Tr}\int\frac{dt}{t}e^{-t}\left(\mathbb{1} - e^{-i\frac{\bar{h}\phi_0}{p^2}\frac{G_{\mu\nu}p^\mu p^\nu}{p^2}\gamma_{01}t}\right) \tag{70}$$

Since $(\frac{G_{\mu\nu}p^\mu p^\nu}{p^2}\gamma_{01})^2 = \mathbb{1}$, the last exponential in Eq. (70) can be decomposed as

$$e^{-i\frac{\bar{h}\phi_0}{p^2}\frac{G_{\mu\nu}p^\mu p^\nu}{p^2}\gamma_{01}t} = \mathbb{1}\cos\frac{\bar{h}\phi_0}{p^2}t - i\frac{G_{\mu\nu}p^\mu p^\nu}{p^2}\gamma_{01}\sin\frac{\bar{h}\phi_0}{p^2}t. \tag{71}$$

The contribution proportional to $\frac{G_{\mu\nu}p^\mu p^\nu}{p^2}$ vanishes, since the functional trace, i.e., the momentum integral requires the Lorentz tensor structure to be proportional to the metric $\int_p p^\mu p^\nu f(p^2) \sim g^{\mu\nu}$; however, the Lorentz trace of the $G_{\mu\nu}$ vanishes, $G^\mu{}_\mu = 0$. The trace in spinor and flavor space thus becomes trivial. Next, we can Wick rotate the momentum-space variables to the Euclidean domain, rescale the propertime $t \to p^2 t$, perform the momentum integral, and arrive at

$$\Gamma_{1\ell} = \frac{N_\mathrm{f} d_\gamma}{16\pi^2}\Omega\int_{1/\Lambda^2}^\infty \frac{dt}{t^3}(1 - \cos\bar{h}\phi_0 t), \tag{72}$$

where we have introduced a UV cutoff at the lower bound of the propertime integral. The integral can be evaluated analytically in terms of cosine integral functions. Expanding the result for large UV cutoff Λ and dropping the terms that vanish in the limit of $\Lambda \to \infty$, we obtain for the effective potential

$$\begin{aligned} V_\mathrm{eff}(\phi_0) &= \phi_0^2\left\{\frac{\bar{m}^2}{2} - \frac{N_f d_\gamma}{2^6\pi^2}\bar{h}^2\left[3 - 2\gamma - \ln\left(\frac{(\bar{h}\phi_0)^2}{\Lambda^4}\right)\right]\right\} \\ &= \frac{\phi_0^2}{2}\left\{m^2(\mu) - \frac{N_f d_\gamma}{2^5\pi^2}\bar{h}^2\left[3 - 2\gamma - \ln\left(\frac{(\bar{h}\phi_0)^2}{\mu^4}\right)\right]\right\}, \end{aligned} \tag{73}$$

where we have introduced the renormalized mass $m(\mu)$ using the same definition as in Eq. (58) for the γ_{11} model. As in this previous model, the effective potential features a nontrivial minimum at

$$(\bar{h}v)^2 = \mu^4 e^{-\frac{2^5\pi^2}{N_f d_\gamma}\frac{m^2(\mu)}{\bar{h}^2}+2-2\gamma}, \tag{74}$$

which is RG invariant, since $\mu\frac{d}{d\mu}v = 0$. In terms of this minimum, the effective potential for the present γ_{01} model can be written in the identical form of Eq. (62) as for the γ_{11} model. Accordingly, its graph is identical to that shown in Fig. 2. If the absence of the fermionic mass term is protected by a suitable symmetry, it is broken by the ground state $\phi_0 = v$ at the mean-field quantum level, inducing a fermion mass

$$m_{01}^2 = \bar{h}v. \tag{75}$$

Similar to the γ_{11} model, the model exhibits dimensional transmutation and induces a gap in the fermion spectrum in terms of a complex conjugate pair of offsets from zero $p^2 = \pm i m_{01}^2$, as below Eq. (27). Analogously, the model allows for massive σ-type scalar excitations on top of the ground state as is indicated by the curvature of the effective potential, cf. Eq. (64). Of course, the present mean-field analysis also passes the self-consistency check in terms of the β function for the scale-dependent coupling $\lambda(\mu) = \bar{h}^2/m^2(\mu)$ also yielding Eq. (65) as in all other cases.

E. LNJL model

Let us finally take a look at the mean-field result for the Luttinger-fermionic version of the NJL model, featuring a continuous axial $\mathrm{U}(1)_{10}$ symmetry. The action in Minkoski space reads

$$S = \int d^4x \left[-Z_\psi \bar{\psi} G_{\mu\nu}\partial^\mu\partial^\nu\psi - \frac{\bar{\lambda}}{2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_{10}\psi)^2]\right]. \tag{76}$$

Both interaction channels can be bilinearized by an HS transformation involving this time a complex massive scalar field, yielding the partially bosonized version of action

$$S_{\mathrm{FB}} = \int d^4x \left[-Z_\psi \bar{\psi} G_{\mu\nu}\partial^\mu\partial^\nu\psi + \bar{h}\phi\bar{\psi}\left(\frac{1-\gamma_{10}}{2}\right)\psi + \bar{h}^*\phi^*\bar{\psi}\left(\frac{1+\gamma_{10}}{2}\right)\psi - \bar{m}^2\phi^*\phi\right]. \tag{77}$$

Here we allow for a complex Yukawa-like coupling $\bar{h}$ for generality. Reality of the action (77) is again manifest, since the two fermion-boson interaction terms are complex conjugate to one another. The matching condition for the two models to be identical now reads

$$\bar{\lambda} = -\frac{|\bar{h}|^2}{2\bar{m}^2}, \tag{78}$$

where the additional factor of two in the denominator, e.g., in comparison to Eq. (42), is a consequence of the complex-field normalization. As for the LGN or the γ_{10} model, the HS transformation can be meaningfully performed only for negative values of the coupling $\bar{\lambda}$. Incidentally, we observe from Eq. (78) that $\bar{h}$ could have been chosen real from the outset. Also, any complex phase of $\bar{h}$ can be compensated by a global phase rotation of the field ϕ.

For the choice of the ground state in a mean-field computation, we also have a free phase parameter to choose. Therefore, we assume $\bar{h}$ as well as $\phi = \phi_0 \in \mathbb{R}$ as real without loss of generality, and compute the one-loop contribution $\Gamma_{1\ell}$ to the effective action for $\phi_0 = \mathrm{const.}$ In fact, we again arrive at the same result as for the LGN model in Eq. (43).

Also, all other conclusions such as the occurrence of an imaginary part of the effective action for finite ϕ_0 as a result of the tachyonic quantum modes, and $\phi_0 = 0$ being the only equilibrium state of the effective action are essentially the same as for the LGN or the γ_{10} model. For completeness, we state the resulting renormalized effective potential accounting for the factor of two difference of the field normalization

$$V_{\mathrm{eff}} = |\phi_0|^2\left\{m^2(\mu) + \frac{N_f d_\gamma}{2^6\pi^2}|\bar{h}|^2\left[1-\gamma-\ln\left(\frac{|\bar{h}\phi_0|^2}{\mu^4}\right)\right]\right\}, \tag{79}$$

where the renormalized mass $m^2(\mu)$ has been defined as in Eq. (47). We also have written the effective potential such that it is valid for any constant complex field value $\phi_0 \in \mathbb{C}$. We conclude that the LNJL model does not exhibit a gap formation on the negative $\bar{\lambda}$ branch that is accessible by the standard HS transformation.

We conclude this section by mentioning that the mean-field computation also gives access to the running of the renormalized LNJL coupling defined by $\lambda(\mu) = -|\bar{h}|^2/(2m^2(\mu))$, yielding the large-$N_{\mathrm{f}}$ beta function Eq. (38) as expected.

F. Summary of mean-field results

Let us summarize our findings for all the scalar self-interacting models at mean-field level in Table I. All models that we considered exhibit asymptotic freedom for positive values of the coupling λ; in fact, the sign conventions of the interaction terms have deliberately been chosen such that the one-loop RG flows exhibit the

TABLE I. Summary of mean-field level results for all fermionic models studied in this work. While all models feature an asymptotically free branch for $\lambda > 0$, the matching condition required by the standard Hubbard-Stratonovich transformation gives access to the mean-field analysis only for a specific branch. For the two models (γ_{11} and γ_{01}) for which we can analyze the long-range behavior of the asymptotically free branch, we find condensate and gap formation at mean-field level and a gap in the complexified fermion spectrum.

Model	Asymptotic freedom	Mean-field analysis	Gap formation in mean field	Fermionic spectrum
LGN	$\lambda > 0$	$\lambda < 0$	no	massless
γ_{10}	$\lambda > 0$	$\lambda < 0$	no	massless
γ_{11}	$\lambda > 0$	$\lambda > 0$	yes	complex gap
γ_{01}	$\lambda > 0$	$\lambda > 0$	yes	complex gap
LNJL	$\lambda > 0$	$\lambda < 0$	no	massless

same sign. The mean-field analysis has been performed with the aid of the Hubbard-Stratonovich transformation in the positive (asymptotically free) branch of the coupling λ for the γ_{11} and γ_{01} models, and in the negative (nonasymptotically free) branch for the LGN, γ_{10} and LNJL models.

In the mean-field approximation, only the γ_{11} and γ_{01} models feature the formation of a nonzero condensate in the effective potential and thereby gap formation in the fermion spectrum on the coupling branch accessible by the HS transformation. The corresponding gaps, however, do not correspond to a conventional real mass term, but to a complex pair of offsets in imaginary direction in the complex p^2 plane.

The other models do not undergo gap formation in the mean-field approximation, i.e. the fermions remain massless and the effective potential exhibits a global minimum at $\phi_0 = 0$. While the HS transformation gives access only to the branch which is not asymptotically free, condensation or gap formation is not observed at all even for arbitrarily large (negative) bare coupling values. In a sense, this result can be interpreted as a self-consistent behavior of these models: if gap formation had occurred, the fermionic spectrum would have featured tachyonic modes. At the same time, the effective action would have acquired imaginary parts indicating the instability of such a ground state.

VI. ANALYTIC STRUCTURE OF GAPPED PROPAGATORS

As discussed in Sec. III, the classical equations of motion of the free theory admits tachyonic solutions for the case of the standard mass term and the m_{10} mass. In the interacting cases of the LGN, the γ_{10}, and the LNJL model, these modes have the potential to trigger an instability associated with an imaginary part of the effective action, if the ground state developed a fermionic condensate. At mean-field level and for the negative coupling branch, this, however, did not happen, since the ground state remains trivial.

Nevertheless, the potential presence of an instability may be viewed as a manifestation of Ostrogradsky's theorem stating that Hamiltonians of higher-derivative theories are unbounded from below [22]. On the quantum level, higher-derivative theories generically go along with *ghosts*, i.e., states manifesting as poles in the propagator with negative spectral weight [18–21,67]. In the sense of the Lehmann-Källen spectral representation, such states do not allow for a probability interpretation potentially invalidating the existence of an S matrix and thus the validity of such theories as quantum field theories. Despite these serious issues at least for a perturbative construction, many concrete proposals have been made to deal with ghosts in a quantized fundamental theory [19,21,23–33]; moreover, theories with ghosts can be meaningfully discussed within effective field theory, e.g., in cases where the timescale of the instability is large compared to other timescales of interest; see [68–71] for applications in cosmology.

In order to study the possible occurrence of ghosts, let us determine the analytic structure of the propagators of the various free theories in the complex p^2 momentum plane. We start with the free theory with a standard mass term Eq. (10). The Minkowski space propagator $S(p)$ is related to the inverse Hessian of the action by

$$-iS(p) = \frac{1}{G_{\mu\nu}p^\mu p^\nu - m^2} = \frac{1}{p^4 - m^4}(G_{\mu\nu}p^\mu p^\nu + m^2), \quad (80)$$

which is, of course, matrix valued in spinor space. Using, for instance, the explicit representation of the Abrikosov

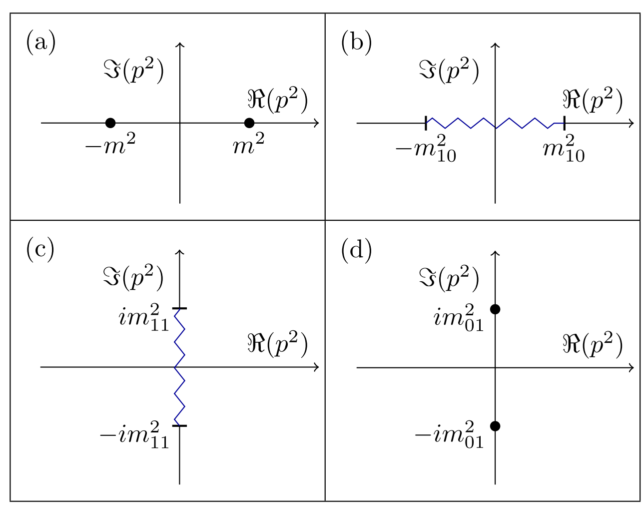

FIG. 3. Analytic structure of the propagators for the free theory with (a) a standard mass m, (b) an m_{10} mass, (c) an m_{11} mass, (d) an m_{01} mass depicted in the complex p^2 momentum plane. Black dots represent the position of simple poles in the eigenvalues of the propagator, and zigzag lines denote a branch cut.

algebra given in Appendix A, we can determine the eigenvalues of the propagator:

$$\mathrm{eig}(-iS(p)) = \left\{\frac{1}{p^2 - m^2}[\deg.16], \frac{-1}{p^2 + m^2}[\deg.16]\right\}, \tag{81}$$

where we find that the two different eigenvalues occur with a degree of degeneracy of 16 each. From Eq. (81), we can read off that the propagator features 16 poles in the complex p^2 plane that correspond to a standard massive dispersion relation $p^2 = m^2$, and 16 poles corresponding to tachyonic states with $p^2 = -m^2$, cf. Fig. 3(a). Moreover, the latter are, in fact, ghost poles as their residue is negative. Of course, causality requirements for the propagator may be implemented by suitable $i\epsilon$ prescriptions; however, the details are not relevant for the present discussion.

We conclude that the free theory with a standard mass term indeed exhibits the properties that are generically expected from a higher-derivative theory: it features tachyons and ghosts. For the present case, the tachyonic and the ghost states are identical; these properties are not necessarily linked, a counterexample is, e.g., given by certain versions of quadratic gravity [20,72].

As a second example, let us consider the free theory with a γ_{10} mass term of Eq. (14). Here, the propagator is given by

$$\begin{aligned} -iS(p) &= \frac{1}{G_{\mu\nu}p^\mu p^\nu - im_{10}^2\gamma_{10}} \\ &= \frac{1}{p^4 - m_{10}^4}(G_{\mu\nu}p^\mu p^\nu - im_{10}^2\gamma_{10}), \end{aligned} \tag{82}$$

with eigenvalue spectrum

$$\begin{aligned} \mathrm{eig}(-iS(p)) = \Bigg\{ &\frac{1}{\sqrt{(p^2 - m_{10}^2)(p^2 + m_{10}^2)}}[\deg.16], \\ &-\frac{1}{\sqrt{(p^2 - m_{10}^2)(p^2 + m_{10}^2)}}[\deg.16]\Bigg\}. \end{aligned} \tag{83}$$

We observe that the propagator has square root singularities at $p^2 = \pm m_{10}^2$ instead of simple poles. This implies that there is a branch cut in the complex p^2 plane. Choosing the cut to lie at negative values of the radicand, the branch cut extends from $p^2 = -m_{10}^2$ to $p^2 = m_{10}^2$ along the real axis, see Fig. 3(b). Half of the modes come with a minus sign such that we rediscover the ghost modes in the massless limit $m_{10}^2 \to 0$ as expected. However, there is no straightforward Lehmann-Källen spectral representation of the propagator for finite m_{10}^2 and thus no immediate probability interpretation in terms of asymptotic states.

Let us also study the propagator for the free theory including a γ_{11} mass term, cf. Eq. (19),

$$\begin{aligned} -iS(p) &= \frac{1}{G_{\mu\nu}p^\mu p^\nu - m_{11}^2\gamma_{11}} \\ &= \frac{1}{p^4 + m_{11}^4}(G_{\mu\nu}p^\mu p^\nu - m_{11}^2\gamma_{11}). \end{aligned} \tag{84}$$

Now, the eigenvalue spectrum reads

$$\begin{aligned} \mathrm{eig}(-iS(p)) = \Bigg\{ &\frac{1}{\sqrt{(p^2 - im_{11}^2)(p^2 + im_{11}^2)}}[\deg.16], \\ &-\frac{1}{\sqrt{(p^2 - im_{11}^2)(p^2 + im_{10}^2)}}[\deg.16]\Bigg\}. \end{aligned} \tag{85}$$

The propagator again supports a square-root type branch cut in the complex p^2 plane, this time ranging from $p^2 = -im_{11}^2$ to $p^2 = im_{11}^2$ along the imaginary p^2 axis, cf. Fig. 3(c). Also in this case, we observe the absence of a conventional spectral representation thus losing the interpretation of some of the modes as ghosts.

A fourth interesting example is given by the free theory with a γ_{01} mass, cf. Eq. (25) with the propagator

$$-iS(p) = \frac{1}{p^4 + m_{01}^4}(G_{\mu\nu}p^\mu p^\nu + im_{01}^2\gamma_{01}), \tag{86}$$

yielding a slightly more intricate eigenvalue spectrum

$$\begin{aligned} \mathrm{eig}(-iS(p)) = \Bigg\{ &\frac{1}{p^2 - im_{01}^2}[\deg.8], \frac{1}{p^2 + im_{01}^2}[\deg.8], \\ &\frac{-1}{p^2 - im_{01}^2}[\deg.8], \frac{-1}{p^2 + im_{01}^2}[\deg.8]\Bigg\}. \end{aligned} \tag{87}$$

For this mass term, we observe the naively expected simple poles on the imaginary axis at $p^2 = \pm im_{01}^2$. Half of the modes seems to have a ghost-type residue. However, since the poles are off the real axis, there is no conventional spectral representation and thus no straightforward probability interpretation.

At this point, we conclude that a naively expected link between a higher-derivative theory, the occurrence of ghosts, and an inferred breakdown of a consistent quantum field theory does not hold in general in theories with relativistic Luttinger fermions. While this link appears to be present in the case of theories with a standard mass term, where we do find tachyonic ghosts, all other mass terms do not give rise to a spectral representation. Therefore, we have no reason to infer that these theories feature ghost states.

Based on these observations read together with the findings of the previous sections, our interpretation at present is as follows:

(a) The tachyonic ghost states of relativistic Luttinger fermions with a standard mass term inhibit a straightforward perturbative construction of observables such as those derived from an S matrix. This is also reflected in our mean-field approach for the LGN or LNJL model by the potential occurrence of an imaginary part of the effective action as a consequence of tachyonic instabilites. Of course, this does not exclude the possibility that a successful quantization may be possible along the lines suggested for other higher-derivative theories.

(b) While we cannot make a statement about the possible (in-)existence of perturbative S-matrix based observables for Luttinger fermions with m_{10}^2 masses due to the lack of a spectral representation, the occurrence of tachyonic modes in the mean-field approach to the γ_{10} model suggests that such degrees of freedom lead to similar problems as those with the standard mass terms and tachyonic ghosts.

(c) For the models with m_{11}^2 or m_{01}^2 mass terms, we have observed a perfectly stable and consistent mean-field description for the corresponding γ_{11} and γ_{01} models going along with the absence of tachyonic mass poles. Also, we do not find ghost states in the sense of conventional mass poles with negative residue. From this perspective, we do not see any reason based on our analysis why such theories should not be consistent.

On the other hand, our results suggest that relativistic Luttinger fermions with m_{11}^2 or m_{01}^2 mass terms do not have a conventional Lehmann-Källen spectral representation and thus no conventional Lehmann-Symanzik-Zimmermann (LSZ) construction of the S matrix. Our preliminary interpretation of this finding is that such relativistic Luttinger fermions do not exist in the sense of asymptotic states. In fact, propagators with complex poles have been intensely discussed in the literature of the strong interactions where the fundamental variables of the QCD action, quarks and gluons, are not expected to exist as asymptotic states [73–89]. Nevertheless, the long-range physics can be described by asymptotic (bound) states such as hadrons in strong-interaction physics or a composite σ-type excitation in the present case of the γ_{11} or γ_{01} model.

VII. CONCLUSIONS

We have introduced and investigated a number of self-interacting quantum field theories with relativistic Luttinger fermions as fundamental degrees of freedom. Concentrating on models with scalar interaction channels, we find that each one features a coupling branch which is asymptotically free in four-dimensional spacetime. While we have worked at a large number of flavors where the restriction to the single scalar interaction channels is justified and quantitatively controlled, we expect our results on asymptotic freedom to generalize to full Fierz-complete local interaction bases, as has been shown in [17] for the LGN model.

We provide large-N_f exact results for the mean-field effective potential for each model, identifying two models that are high-energy complete and undergo dimensional transmutation with a corresponding condensate and gap formation at low energies. Both models, the γ_{11} and γ_{01} model, do not exhibit any sign of instability at the present level of investigation, as might naively be expected for a higher-derivative theory. The reason for this lies in the fact that the generated mass term does not induce a tachyonic mass pole. This is corroborated by our study of the analytic structure of the propagators for the various versions of the massive theories.

It is interesting to observe that mass generation and gap formation does not happen at mean-field level for those theories where these masses would lead to tachyonic instabilities. However, since the mean-field analysis is confined to a specific coupling branch, other methods are needed to analyze the more interesting asymptotically free branch for those models. This should be possible with suitable techniques that allow for the inclusion of a more general bare potential of the scalar field such as the functional RG, see [50,50,59–61,90], or methods based on the gap equation.

These methods could also provide access to the spectrum of excitations above the nontrivial ground state in the stable and high-energy complete models. In the present cases of the γ_{11} and γ_{01} models, we expect the existence of a light σ mode. This would constitute an example of a UV-complete model with only marginal couplings that entails naturally light scalar long-range degrees of freedom in four-dimensional spacetime. Corresponding investigations are underway.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: RELATIVISTIC ABRIKOSOV ALGEBRA

In order to make the paper self-contained, let us summarize a few aspects of the relativistic version of the Abrikosov algebra [2] in Eq. (2) as derived in [17].

Whenever needed, we work with the explicit representation of the $G_{\mu\nu}$ matrices in terms of elements of a Euclidean Dirac algebra $\{\gamma_A, \gamma_B\} = 2\delta_{AB}$,

$$\begin{aligned}
&G_{0i} = i\sqrt{\frac{2}{3}}\gamma_{A=i}, \qquad i = 1, 2, 3,\\
&G_{12} = \sqrt{\frac{2}{3}}\gamma_4, \qquad G_{23} = \sqrt{\frac{2}{3}}\gamma_5, \qquad G_{31} = \sqrt{\frac{2}{3}}\gamma_6,\\
&G_{00} = \gamma_7, \qquad G_{11} = \frac{1}{3}\gamma_7 + \frac{2\sqrt{2}}{3}\gamma_8,\\
&G_{22} = \frac{1}{3}\gamma_7 - \frac{\sqrt{2}}{3}\gamma_8 + \sqrt{\frac{2}{3}}\gamma_9,\\
&G_{33} = \frac{1}{3}\gamma_7 - \frac{\sqrt{2}}{3}\gamma_8 - \sqrt{\frac{2}{3}}\gamma_9.
\end{aligned} \tag{A1}$$

This representation can be viewed as an appropriate Wick rotation of the one constructed for $d = 4$ Euclidean dimensions in [7]. It is straightforward to check that this representation satisfies Eq. (2). Whereas 9 elements $\gamma_{1,\ldots,9}$ were sufficient to satisfy the relativistic Abrikosov algebra, the reality conditions of the action demand for another anticommuting element for the construction of a spin metric h. This requires a $d_\gamma = 32$ dimensional representation for the Euclidean Dirac algebra (and correspondingly of the Abrikosov algebra) and thus in total with $d_e = 11$ anticommuting elements γ_A, with $A = 1, \ldots, 11$.

While we use γ_{10} for the construction of the spin metric, cf. Eq. (5), both additional elements γ_{10} and γ_{11} can serve for the construction of additional scalar bilinears and interactions. Alternatively, we could choose a different spin metric, e.g.,

$$\tilde{h} = \gamma_1\gamma_2\gamma_3\gamma_{11}, \tag{A2}$$

or, more generally, a linear combination $h' = \alpha h + \beta\tilde{h}$ with $\alpha^2 + \beta^2 = 1$ and $\alpha, \beta \in \mathbb{R}$ as the spin metric. This would induce a corresponding rotation of the interaction channels discussed in the main text, but the overall structure of different mass terms, interaction channels, and the existence of an axial symmetry of the kinetic term would remain the same.

APPENDIX B: EUCLIDEAN CONVENTIONS

For the Wilsonian renormalization group analysis in the main text, it is useful to have a manifestly Euclidean formulation of the models studied in the present work. For this, we need a Euclidean version of the Abrikosov algebra,

$$\{G_{\mu\nu}, G_{\kappa\lambda}\}\big|_E = -\frac{2}{d-1}\delta_{\mu\nu}\delta_{\kappa\lambda} + \frac{d}{d-1}(\delta_{\mu\kappa}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\kappa}), \tag{B1}$$

where Minkowski metric factors on the right-hand side are replaced by Kronecker symbols. For generality, we work in d-dimensional spacetime here. We also introduce τ as Euclidean time direction, related to the Minkowskian time t by a Wick rotation $\tau = it$. The reality condition of the Minkowskian action can then be rephrased for the Euclidean Lagrangian in terms of Osterwalder-Schrader (OS) reflection positivity [91],

$$\mathcal{L}_E^* = \hat{\mathcal{L}}_E, \tag{B2}$$

where $\hat{\mathcal{L}}_E$ arises from $\mathcal{L}_E$ by replacing the coordinates $x = (\tau, \vec{x})$ with $\hat{x} = (-\tau, \vec{x})$. In other words, in addition to complex conjugating the operator building blocks, we also need to flip the sign of the Euclidean time. Let us first check OS reflection positivity for the simple mass term. We extend the fields to the Euclidean domain $\psi = \psi(\tau, \vec{x})$ and look at the complex conjugate of the spinor bilinear

$$[(\bar{\psi}\psi)(x)]^* = (\psi^\dagger h^\dagger \psi)(\hat{x}). \tag{B3}$$

A simple choice to implement reflection positivity here is to define the Euclidean spin metric to be equivalent to the hermitean Minkowskian one, $h_E = h \equiv h^\dagger$, also preserving the definition $\bar{\psi} = \psi^\dagger h$.

Prior to looking at the kinetic term, let us establish the connection between the Euclidean $G_{\mu\nu}$ matrices, satisfying Eq. (B1), and their Minkowskian counterpart. From the algebra (B1), all the $G_{\mu\nu,E}$ matrices can be chosen hermitean in the Euclidean domain. Furthermore, by comparing (2) and (B1), we can find the explicit relation of the matrices in the Euclidean and Minkowskian domain. For example, by looking at the anticommutator

$$\{G_{00}, G_{00}\}|_M = 2 = \{G_{00}, G_{00}\}|_E, \tag{B4}$$

we infer that the choice $G_{00,E} \equiv G_{00,M}$ is a valid option. Next, since

$$\{G_{00}, G_{\underline{i}\,\underline{i}}\}\big|_M = 2/(d-1) = -\{G_{00}, G_{\underline{i}\,\underline{i}}\}\big|_E, \tag{B5}$$

we can take $G_{ii,E} \equiv -G_{ii,M}$. Another nontrivial anticommutator is

$$\{G_{0\underline{i}}, G_{0\underline{i}}\}\big|_M = -d/(d-1) = -\{G_{0\underline{i}}, G_{0\underline{i}}\}\big|_E, \tag{B6}$$

from which we read off the two possibilities $\pm iG_{0i,E} \equiv G_{0i,M}$. Lastly, noticing that

$$\{G_{\underline{i}\,\underline{j}}, G_{\underline{i}\,\underline{j}}\}\big|_M = d/(d-1) = \{G_{\underline{i}\,\underline{j}}, G_{\underline{i}\,\underline{j}}\}\big|_E, \tag{B7}$$

we get $\pm G_{ij,E} \equiv G_{ij,M}$. The last two ambiguities can be resolved by looking at the Luttinger kinetic operator $G_{\mu\nu}\partial^\mu\partial^\nu$. For the 00 and ii components, we have

$$G_{00}\partial^0\partial^0|_M = G_{00}|_E \frac{1}{(-i)^2}\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau} = -G_{00}\partial^0\partial^0|_E,$$
$$G_{\underline{i}\,\underline{i}}\partial^{\underline{i}}\partial^{\underline{i}}|_M = -G_{\underline{i}\,\underline{i}}|_E(-1)^2\frac{\partial}{\partial x^{\underline{i}}}\frac{\partial}{\partial x^{\underline{i}}}\bigg|_M = -G_{\underline{i}\,\underline{i}}\partial^{\underline{i}}\partial^{\underline{i}}|_E. \tag{B8}$$

Thus, for consistency with equation (B8), the choice of the minus sign also for the ij and $0i$ components fixes the signs of the remaining $G_{\mu\nu}$ matrices by $G_{0i,E} \equiv iG_{0i,M}$ and $G_{ij,E} \equiv -G_{ij,M}$. Ultimately, we obtain for the Luttinger kinetic term

$$\bar\psi G_{\mu\nu} i\partial^\mu i\partial^\nu\psi|_M = -\bar\psi G_{\mu\nu} i\partial^\mu i\partial^\nu\psi|_E. \tag{B9}$$

It is straightforward to check that the kinetic term (B9) is OS reflection positive in the Euclidean domain. In order for the weight functions of the functional integrals to undergo the transition e^{iS_M} to e^{-S_E}, we observe that

$$\begin{aligned} iS_M &= i\int d^4x(\bar\psi G_{\mu\nu} i\partial^\mu i\partial^\nu\psi + \mathcal{L}_{\rm int}) \\ &= \int d^4x|_E(-\bar\psi G_{\mu\nu} i\partial^\mu i\partial^\nu\psi|_E + \mathcal{L}_{\rm int}) \coloneqq -S_E \end{aligned} \tag{B10}$$

holds for any local nonderivative interaction (or mass) term $\mathcal{L}_{\rm int}$. We conclude that the Euclidean and Minkowskian actions differ by a minus sign with respect to the local nonderivative terms $\mathcal{L}_{\rm int}$,

$$S_E = \int d^4x(\bar\psi G_{\mu\nu} i\partial^\mu i\partial^\nu\psi - \mathcal{L}_{\rm int}). \tag{B11}$$

This sign change applies to all interacting models investigated in the present work, e.g., involving interaction terms such as $(\bar\psi\psi)^2$, $(\bar\psi\gamma_{10}\psi)^2$, $(\bar\psi\gamma_{11}\psi)^2$, $(\bar\psi\gamma_{01}\psi)^2$, $(\bar\psi G_{\mu\nu}\psi)^2$, ...

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