Letter

Kinematic Hopf algebra and Bern-Carrasco-Johansson numerators at finite α'

Gang Chen,[1,*] Laurentiu Rodina,[2,†] and Congkao Wen[3,‡]

[1]*Niels Bohr International Academy, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[2]*Beijing Institute of Mathematical Sciences and Applications (BIMSA), Beijing 101408, China*
[3]*Centre for Theoretical Physics, Department of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom*

(Received 10 April 2024; accepted 21 July 2024; published 15 August 2024)

We significantly extend recent work on the kinematic Hopf algebra, a structure that was shown to underlie the color-kinematics duality in Yang-Mills (YM) theory coupled to two heavy scalars, known as the heavy-mass effective theory (HEFT) limit. First, staying in the HEFT limit, we show the same kinematic Hopf algebra can be used to obtain Bern-Carrasco-Johansson (BCJ) numerators in $\mathrm{DF}^2 + \mathrm{YM}$ theory, a theory containing an infinite number of higher derivative corrections to YM, used to obtain string amplitudes via the double copy. Second, we exploit the intricate structure induced by the massive poles to obtain an efficient and direct expression for local BCJ numerators in pure YM based on the same kinematic Hopf algebra. This demonstrates that the kinematic Hopf algebra works even beyond the HEFT limit, strongly suggesting this structure universally underlies the color-kinematics duality.

DOI: 10.1103/PhysRevD.110.L041902

Introduction. The discovery of color-kinematics duality and the double-copy [1,2] has revolutionized the computation of amplitudes, particularly in gravity. The duality relies on finding kinematic numerators that satisfy the same algebraic relations as the corresponding color factors, such as the Jacobi identity. Then the double-copy procedure allows the computation of various amplitudes by mixing and matching different types of such numerators. However, the origin of this duality remains deeply mysterious, in particular because the nature of the kinematic algebra [3–24] at the heart of the duality is not yet understood.

Important progress could be accomplished by finding a more systematic, universal, and efficient approach to constructing Bern-Carrasco-Johansson (BCJ) numerators, and recent work suggests this may involve utilizing a combinatorial algebra perspective. A novel kinematic algebra, known as the quasishuffle Hopf algebra [25–29], has been discovered within the framework of heavy-mass effective field theory (HEFT) [30,31]. Besides providing compact formulas for BCJ numerators, this approach phrases the kinematic algebra in terms of the rich structure of Hopf algebras, opening a completely new avenue to understand this duality. For instance, a Hopf algebra is a structure that is both an algebra and a coalgebra, and in our case the coalgebra is responsible for the factorization property of numerators, a property that is normally expected only for full amplitudes. Other properties of this kinematic Hopf algebra are discussed in detail in [32].

This structure has been subsequently extended to amplitudes and form factors incorporating finite massive scalar and fermionic contributions [33–36], and also uncovered in a geometrical context [37]. Recently, the same algebraic structure has been identified in the $\alpha'\mathrm{F}^3 + \alpha'^2\mathrm{F}^3$ theory [38,39], which can be viewed as lower-order terms in the α' expansion of $\mathrm{DF}^2 + \mathrm{YM}$ theory, where the α' dependence enters via massive propagators [41]. This theory was used to construct various conformal (super)gravity theories via the double-copy procedure, as well as amplitudes for bosonic and heterotic string theory [40,42–51]).

In this Letter, to further explore the applicability and universality of this approach, we reveal the presence of the same kinematic Hopf algebra in two contexts: the HEFT limit of $\mathrm{DF}^2 + \mathrm{YM}$ theory, and in pure Yang-Mills (YM), this time directly away from the HEFT limit. The crux of constructing the BCJ numerator from the kinematic Hopf algebra lies in identifying an evaluation map [31,33]. This map connects the abstract combinatorial algebra generators to physically meaningful, gauge-invariant functions that exhibit only physical poles. We find the evaluation map is fully determined by imposing the relabeling symmetry and consistent factorization conditions on the massive and

*Contact author: gang.chen@nbi.ku.dk
†Contact author: laurentiu.rodina@gmail.com
‡Contact author: c.wen@qmul.ac.uk

massless gluon poles. Consequently, we first determine the BCJ numerator, applicable for an arbitrary number of external gluons (with two heavy particles, which can be removed through a factorization limit). By expanding in terms of α', we obtain local BCJ numerators for massless gluon amplitudes.

Finally, by exploiting the specific form of the mapping rule, we also derive a novel formula directly for BCJ numerators in pure YM theory, which is compact, manifestly local, and relabeling symmetric. This is a significant improvement from the original results in [30,31], as considering HEFT numerators and taking a factorization limit is no longer a required intermediate step, thus also avoiding the appearance of any spurious poles. Interestingly, this formula shares several similarities with the expression obtained through completely different methods in [52], such as in the numbers of terms (i.e., twice the Fubini numbers) and the distribution of gluon labels.

Universal kinematic Hopf algebra. It has been shown that BCJ numerators in a manifest gauge-invariant form originate from the quasishuffle Hopf algebra in YM theory coupled with two heavy particles [31–33] as well as theories with higher-derivative corrections [38]. The BCJ numerators

$$\mathcal{N}([12\ldots n-2],v) \tag{1}$$

can be naturally written in the form of a quasishuffle product of algebraic generator $T_{(i)}$,

$$\langle T_{(1)}\star T_{(2)}\star\cdots\star T_{(n-2)}\rangle, \tag{2}$$

where $i=1,2,\ldots,n-2$ represents the external gluons, v is the velocity of the heavy particles, and $\langle\bullet\rangle$ is the evaluation map from the algebra generators to a manifestly gauge invariant function. More details are presented in the Supplemental Material [53]. Already from the study of lower orders in α' expansion [38] we find the BCJ numerators are constrained by the heavy-mass factorization behavior and relabeling symmetry. These properties impose stringent constraints on the evaluation map, which then lead to a recursive form for the BCJ numerator [38],

$$\mathcal{N}([1\alpha],v)=(-1)^{n-3}\frac{G_{1\alpha}(v)}{v\cdot p_1}+\sum_{\tau_L\subset\alpha}(-1)^{|\tau_R|}\frac{\mathcal{N}([1\tau_L],v)G_{\tau_R}(p_{\Theta(\tau_R)})}{v\cdot p_{1\tau_L}}, \tag{3}$$

where $\{1\tau_L\}\cup\{\tau_R\}=\{1\alpha\}\equiv\{12\ldots n-2\}$, $|\tau_R|$ denotes the number of gluons in τ_R, and $[\bullet]$ denotes the left nested commutator, e.g., $[123]\equiv 123–213-312+321$; and $\Theta(\tau_i)=(1\tau_L)\cap\{1,\ldots,\tau_{i[1]}\}$ that consists of all indices to the left of τ_i and smaller than the first index in τ_i, denoted as $\tau_{i[1]}$. Finally, the G function takes the following form [38]:

$$G_\tau(x)=x\cdot F_\tau\cdot v+\sum_{\sigma_1 i\sigma_2 j\sigma_3=\tau}x\cdot F_{\sigma_1}\cdot p_i W(i\sigma_2 j)p_j\cdot F_{\sigma_3}\cdot v, \tag{4}$$

with

$$W(\sigma)=\sum_{r=1}^{\lfloor|\sigma|/2\rfloor}\sum_{i_1\rho_1 j_1\sigma_2 i_2\rho_2 j_2\ldots i_r\sigma_r j_r\rho_r=\sigma}\times\left(\prod_{k=1}^{r}W'(i_k\rho_k j_k)\right)\left(\prod_{k=2}^{r}p_{j_{k-1}}\cdot F_{\sigma_k}\cdot p_{i_k}\right), \tag{5}$$

where we have further introduced the W' function, which we assume contains all the α' dependence in form of massive propagator factors inherited directly from the DF^2 theory. We will fix the exact form of these functions in the next sections. Importantly, we will also later show this function can be directly used to obtain BCJ numerators in pure YM, circumventing the need to start from HEFT amplitudes.

Below are a few examples of the G functions:

$$\begin{aligned}G_{12}(x)&=x\cdot F_{12}\cdot v+x\cdot p_1W'(12)p_2\cdot v,\\ G_{123}(x)&=x\cdot F_{123}\cdot v+x\cdot F_1\cdot p_2W'(23)p_3\cdot v\\ &\quad+x\cdot p_1W'(12)p_2\cdot F_3\cdot v+x\cdot p_1W'(123)p_3\cdot v.\end{aligned} \tag{6}$$

The amplitudes obtained via color-kinematics duality then can be expressed as [30–32]

$$A(1\alpha,v)=\sum_{\beta\in S_{n-3}}\mathrm{m}(1\alpha,1\beta)\mathcal{N}([1\beta],v), \tag{7}$$

where $\mathrm{m}(1\alpha,1\beta)$ is the Kawai-Lewellen-Tye propagator matrix [54].

As required by the relabeling symmetry of this BCJ numerator, the W' function satisfies the following relations [38]:

$$\begin{aligned}W'(\rho_1 i_1\rho_2 i_{n-2}\rho_3)&=(-1)^{|\rho_3|+\delta_{0,|\rho_1|}}W'(i_1[\rho_1]\rho_2[\rho_3^{\mathrm{rev}}]i_{n-2}),\\ W'(\rho_1 i_{n-2}\rho_2 i_1\rho_3)&=(-1)^{n-2}W'(\rho_3^{\mathrm{rev}}i_1\rho_2^{\mathrm{rev}}i_{n-2}\rho_1^{\mathrm{rev}}),\end{aligned} \tag{8}$$

where $|\rho|$ is the size of ρ and ρ^{rev} denotes the its reverse. For example, at $n=8$, we have

$$\begin{aligned}W'(231645)&=-W'(1[23][45]6),\\ W'(126345)&=-W'(12[543]6),\end{aligned} \tag{9}$$

and an independent basis is given by $W'(1i_2i_3i_4i_56)$. The next section is devoted to deriving properties of W'

functions that follow from factorization, which will aid in finding the explicit form of these functions. In the process, we will also find an alternative and more efficient route to obtaining BCJ numerators away from the HEFT limit.

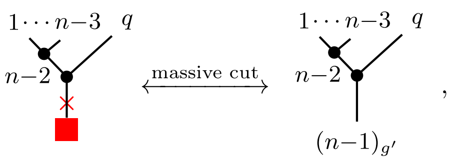
General properties of the W' function from factorization. We will first consider $\mathrm{DF}^2 + \mathrm{YM}$ theory [41], which contains a massless gluon, as well as a tachyon and a massive gluon (both with $m^2 = -1/\alpha'$).

In order to determine the corresponding W' function for BCJ numerators in $\mathrm{DF}^2 + \mathrm{YM}$ theory we will next study its properties following from consistent factorization. As mentioned above, we assume W' functions contain the massive propagators of $\mathrm{DF}^2 + \mathrm{YM}$ theory. Importantly, each function $W'(i_1 \ldots i_r)$ should have an overall propagator factor

$$\frac{\alpha'}{1 - \alpha' p^2_{i_1 \ldots i_r}}, \tag{10}$$

besides containing other similar propagators corresponding strictly to subsets of $(i_1 \ldots i_r)$.

We begin by considering the HEFT BCJ numerator $\mathcal{N}([1 \ldots (n-3)q], v)$ with a cut on a massive propagator,



(11)

where the "solid box" represents the heavy particles and g' is the massive gluon. The label $(n-2)$ on an internal line will be promoted to an external massless gluon after a series of cuts. Using the expression for the HEFT BCJ numerator in (3) and assuming the form in (10), the residue on this cut is given by

$$(1 - \alpha' p^2_{1\ldots(n-3)q})(-1)^{n-3} W'(1 \ldots n-3, q) p_q \cdot v. \tag{12}$$

By replacing v with the polarization vector $\varepsilon^{\perp}_{n-1}$, which is the transverse mode of the massive gluon, we obtain the numerator for $(n-2)$ massless gluons and one massive gluon, $\mathcal{N}(1 \ldots n-3, q, (n-1)_{g'})$ (see more details in the Supplemental Material [53]).

The BCJ numerator $\mathcal{N}(1 \ldots (n-3), q, (n-1)_{g'})$ also factorizes on the massless cut on the pole $1/p^2_{12\ldots n-3}$. Consider the amplitude with $(n-2)$ massless gluons and one massive gluon. On this massless cut we have

$$\sum_{\text{states } I} A(12 \ldots n-3, I^*) A(I, q, (n-1)_{g'}), \tag{13}$$

where "I" denotes the internal state. The three-point amplitude of two massless gluons and one massive gluon is given by $A(I, q, (n-1)_{g'}) = \mathrm{tr}(F_I \cdot F_q) p_q \cdot \varepsilon^{\perp}_{n-1}$. We now choose the polarization vector ε_q to be orthogonal to the cut momentum p_I, so the trace factor becomes

$$2\alpha' \varepsilon_I \cdot \varepsilon_q p_I \cdot p_q = \varepsilon_I \cdot \varepsilon_q, \tag{14}$$

where we used the massive on-shell condition of the external massive gluon. So on the cut and with the extra constraint on the polarization vector ε_q, the summation over the intermediate state gives a massless gluon amplitude,

$$\begin{aligned} &\sum_{\text{states } I} A(1 \ldots n-3, \varepsilon^*_I) \varepsilon_I \cdot \varepsilon_q p_q \cdot \varepsilon^{\perp}_{n-1} \\ &\quad = A(1 \ldots n-3, \varepsilon_q) p_q \cdot \varepsilon^{\perp}_{n-1}. \end{aligned} \tag{15}$$

Comparing (15) with (12) and taking $\varepsilon_{n-2} = \varepsilon_q$, we obtain an important relation between the function $W'(1 \ldots (n-3)q)$ and the BCJ numerator of massless gluons,

$$\begin{aligned} \mathcal{N}(12 \ldots n-2) = &(-1)^{n-3} \\ &\times \left((1 - \alpha' p^2_{1\ldots(n-3)q}) W'(1 \ldots (n-3)q) \right) \Big|_{\text{cuts}}, \end{aligned} \tag{16}$$

where the label "cuts" denotes all the above constraints,

$$p^2_{1\ldots(n-3)q} - 1/\alpha' = 0, \qquad p^2_{1\ldots n-3} = p_{1\ldots n-3} \cdot \varepsilon_q = 0. \tag{17}$$

Note that in (16) the BCJ numerator on the lhs is independent of p_q, which manifestly appears on the rhs, and therefore leads to spurious poles. With an appropriate choice for p_q, which we can treat as a reference momenta, we will be able to eliminate all spurious poles and obtain local numerators.

With the BCJ numerator, the amplitude is then given by

$$A(1 \ldots n-2) = \sum_{\beta \in S_{n-4}} \mathrm{m}(1 \ldots n-3, 1\beta) \mathcal{N}(1\beta n-2). \tag{18}$$

Importantly, the amplitude $A(1 \ldots n-3, n-2)$ can also be obtained by decoupling the heavy particles via a factorization limit [31],

$$A(1 \ldots n-2) = A(1 \ldots n-3, v)\Big|^{p^2_{1\ldots n-3}=0}_{v \to \varepsilon_{n-2}}, \tag{19}$$

where $A(1 \ldots n-3, v)$ depends on W' functions with $n-3$ or less gluons. In this way, we derive recursive relations of W' functions.

Combinatorial solution of the W' function. We will now use the recursive relations discussed above and relabeling symmetry (8) to determine the W' functions. We further impose manifest gauge invariance as well as factorization properties on massive particle cuts.

We first write down the general solution to the relabeling symmetry (8). A particular solution of $W'(i_1 \ldots i_{r-1} i_r)$ that obeys (8) was already constructed in [38],

$$W_0(i_1 \ldots i_r) \equiv \mathrm{tr}(F_{[i_1 \ldots i_{r-1}]} F_{i_r}), \tag{20}$$

which is precisely the leading-α' correction term. Two key properties of this particular solution are the left nested commutator of the indices except the last one and cyclic permutation invariance of the trace function. This observation leads to the general formal solution of (8):

$$\mathbb{O}_{\mathrm{cyc}([i_1 \ldots i_{r-1}] i_r)} \circ f(i_1, i_2, \ldots i_r) \equiv \sum_{\sigma \in [i_1 \ldots i_{r-1}] i_r} \sum_{\rho_1 \rho_2 \ldots \rho_r = \mathrm{cyc}(\sigma)} f(\rho_1, \rho_2, \ldots, \rho_r). \tag{21}$$

For example,

$$\begin{aligned} &\mathbb{O}_{\mathrm{cyc}([12]3)} \circ f(123) \\ &= \left(\sum_{\rho_1 \rho_2 \rho_3 = \mathrm{cyc}(123)} - \sum_{\rho_1 \rho_2 \rho_3 = \mathrm{cyc}(213)} \right) f(\rho_1 \rho_2 \rho_3) \\ &= f(123) + f(231) + f(312) - f(213) - f(132) \\ &\quad - f(321). \end{aligned} \tag{22}$$

We then impose (16). We proceed by defining partitions of ordered gluon indices $\{i_1, i_2, \ldots, i_{r-1}, i_r\}$ in $f(i_1, \ldots, i_r)$:

$$\mathbf{P}_{i_1 \ldots i_r}(x_1, x_2, \ldots, x_s), \quad \text{with} \quad \sum_{i=1}^{s} x_i = r. \tag{23}$$

For example,

$$\mathbf{P}_{1234}(2,1,1) \equiv (12)|(3)|(4), \tag{24}$$

$$\mathbf{P}_{123456}(2,1,2,1) \equiv (12)|(3)|(45)|(6). \tag{25}$$

To maintain manifest gauge invariance, each single-index subset, such as "(3)" and "(4)" in (24), is mapped to its corresponding strength tensor; in order to preserve the factorization behavior on the massive cuts, each multi-index subset, such as "(12)" in (24) and "(45)" in (25), is mapped to a lower-point W' function. Power counting considerations then dictate that all single-index subsets between two multi-index subsets, for example, $(i_L, \ldots, j_L)|(j), \ldots, (k)|(i_R, \ldots, j_R)$, must be mapped to $p_X \cdot F_j \cdot \cdots \cdot F_k \cdot p_Y$, or mapped to $p_X \cdot p_Y$ if there are no single-index subsets. To ensure the ordering of gluon indices, the sequence of dot products and traces involving field strengths must mirror the ordering of gluons. So, the labels X and Y can only correspond to adjacent indices in the lower-point W' functions, i.e., $X = j_L, Y = i_R$. These rules establish a one-to-one map from each partition to a gauge-invariant function, e.g., for the partition given in (25), we have

$$(12)|(3)|(45)|(6) = W'(12) p_2 \cdot F_3 \cdot p_4 W'(45) p_5 \cdot F_6 \cdot p_1. \tag{26}$$

The single set partition and the maximal set partition are special, and are directly fixed by power counting

$$\begin{aligned} (i_1 i_2 \ldots i_r) &= \mathrm{tr}(F_{i_1 i_2 \ldots i_r}), \\ (i_1)|(i_2)|\ldots|(i_r) &= 0. \end{aligned} \tag{27}$$

We are thus led to a simple and well-structured solution for the massless factorization behavior outlined in Eq. (16). To incorporate the relabeling symmetry (8), we sum over all partitions using the operator $\mathbb{O}$ defined in (21). In conclusion, we have

$$\begin{aligned} W'(i_1 \ldots i_{r-1} i_r) = \frac{\alpha'}{1 - \alpha' p^2_{i_1 \ldots i_r}} \Bigg[& W_0(i_1 \ldots i_{r-1} i_r) \\ & + \mathbb{O}_{\mathrm{cyc}([i_1 \ldots i_{r-1}] i_r)} \circ \Bigg(\sum_{s=1}^{r-1} \sum_{x_1 + \cdots + x_s = r}^{x_i \in \mathbb{Z}^+} \\ & \times \frac{\mathbf{P}_{i_1 \ldots i_r}(x_1, \ldots, x_s)}{s} \Bigg) \Bigg], \end{aligned} \tag{28}$$

where each partition is weighted by its corresponding overcounting number [55], and we have identified $\mathbf{P}_{i_1 \ldots i_r}(x_1, \ldots, x_s)$ with lower-point W' functions according to the rules we just discussed,

$$\begin{aligned} &\mathbf{P}_{i_1 \ldots i_r}(x_1, \ldots, x_s) \\ &= \left(\prod_{k=1}^{t-1} W'(i_{a(k)} \ldots i_{b(k)}) p_{i_{b(k)}} \cdot F_{i_{b(k)+1} \ldots i_{a(k)-1}} \cdot p_{i_{a(k+1)}} \right) \\ &\quad \times W'(i_{a(t)} \ldots i_{b(t)}) p_{i_{b(t)}} \cdot F_{i_{b(t)+1} \ldots r1 \cdots i_{a(1)-1}} \cdot p_{i_{a(1)}}, \end{aligned} \tag{29}$$

where t is the number of sets with multigluon indices. We have checked this general solution up to the eight-point HEFT amplitude (six gluons). Below are some simple examples of W' functions (more examples can be found in the Supplemental Material [53]),

$$\begin{aligned} W'(12) &= \frac{\alpha'}{1 - \alpha' p_{12}^2} W_0(12), \\ W'(123) &= \frac{\alpha'}{1 - \alpha' p_{123}^2} \big(W_0(123) - 2W'(23) p_2 \cdot F_1 \cdot p_3 \\ &\quad - 2W'(12) p_1 \cdot F_3 \cdot p_2 + 2W'(13) p_1 \cdot F_2 \cdot p_3 \big), \end{aligned} \tag{30}$$

with W_0 given in (20). Note that the poles are the physical massive propagators in the $\mathrm{DF}^2+\mathrm{YM}$ theory. The number of terms, in terms of W_0 and strength tensor products, is given in the table below.

Gluons	2	3	4	5	6	7
Number of terms in W'	1	4	45	921	30485	1539170

The W' function can now be plugged in (3) to obtain BCJ numerators in $\mathrm{DF}^2+\mathrm{YM}$ theory in the HEFT limit. Numerators for the pure theory, away from HEFT, can be obtained by taking the decoupling limit (19). However, this leads to spurious poles. In the next section we describe an alternative approach to obtain manifestly local numerators in the $\alpha'\to 0$ expansion.

Local BCJ numerator for the massless gluon amplitude. As shown in (16), the function $W'(1\ldots(n-3)q)$ automatically generates the BCJ numerators for the massless gluons. However, the numerators given in (16) in general contain spurious poles, due to the auxiliary momentum p_q. We denote this BCJ numerator as $\mathcal{N}_{\mathrm{NL}}(12\ldots n-2)$, where "NL" stands for "nonlocal." Expanding in terms of the α', we have

$$\mathcal{N}_{\mathrm{NL}}(12\ldots n-2)=\mathcal{N}_{\mathrm{NL}}^{\mathrm{YM}}(12\ldots n-2)+\sum_{\mathrm{i}=1}^{\infty}(\alpha')^{\mathrm{i}}\mathcal{N}_{\mathrm{NL}}^{(\mathrm{i})}(12\ldots n-2). \quad (31)$$

where the expanded terms are the nonlocal BCJ numerator for each order of α'.

We will now exploit the fact that result is independent of the auxiliary momentum p_q. To do so, we first impose the massive on-shell condition and massless cut condition in (17), which we can solve as

$$p_1\cdot p_q=1/(2\alpha')-p_{2\ldots n-3}\cdot p_q, \quad (32)$$

allowing us to remove $p_1\cdot p_q$, leaving only independent kinematic variables. The spurious poles in $\mathcal{N}_{\mathrm{NL}}$ originate from the massive poles, and have the form

$$\frac{1}{(\sum p_X\cdot p_Y)+p_Z\cdot p_q}. \quad (33)$$

Take, for example, a numerator $\mathcal{N}(1234)$ which according to (16) is given by applying the corresponding cuts to $W'(123q)$, i.e., $p_1\cdot p_q=1/2\alpha'-p_2\cdot p_q-p_3\cdot p_q$. The recursive form of $W'(123q)$ in turn contains $W'(12q)$, whose massive propagator becomes a spurious pole of the form described above. Since p_q is auxiliary, the spurious pole is conveniently removed by setting

$$p_i\cdot p_q=z\to\infty \quad \text{for } i>1, \quad (34)$$

and keeping only the $\mathcal{O}(z^0)$ pieces. Any other orders in z must cancel out in the amplitude and can be ignored in the final BCJ numerator, since the amplitude cannot be a function of the auxiliary p_q. Importantly, this choice preserves the relabeling symmetry of the BCJ numerator for $\{2,3,\ldots,n-3\}$, but spoils its gauge invariance. This is an unavoidable price to pay for having local YM numerators [56,57]. Applying this procedure to the α' expansion (31), we have

$$\mathcal{N}^{(\mathrm{i})}(12\ldots n-2)=\mathcal{N}_{\mathrm{NL}}^{(\mathrm{i})}(12\ldots n-2)|_{z\to\infty}^{p_i\cdot p_q=z}, \quad (35)$$

where the superscript index $\mathrm{i}=\mathrm{YM},1,2,\ldots$. Now, the BCJ numerator does not contain any denominators and is purely local. At four points, the local YM BCJ numerator is

$$\begin{aligned}\mathcal{N}^{\mathrm{YM}}(1234)=&\frac{1}{2}\varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot p_1p_{12}\cdot\varepsilon_3-\frac{1}{2}\varepsilon_1\cdot\varepsilon_4p_1\cdot F_2\cdot\varepsilon_3\\&-\varepsilon_2\cdot p_1\varepsilon_1\cdot F_3\cdot\varepsilon_4+\frac{1}{2}\varepsilon_1\cdot\varepsilon_4\varepsilon_3\cdot p_1\varepsilon_2\cdot p_1\\&-p_{12}\cdot\varepsilon_3\varepsilon_1\cdot F_2\cdot\varepsilon_4+\varepsilon_1\cdot F_2\cdot F_3\cdot\varepsilon_4. \quad (36)\end{aligned}$$

This surprisingly matches the numerators in [58,59], but at higher points we obtain expressions that differ by generalized gauge transformations. More examples are included in the Supplemental Material [53] and KiHA5.0 [60].

Kinematic algebra for local BCJ numerators in Yang-Mills theory. We will show that, for pure YM theory, the local BCJ numerators can also be obtained directly from the kinematic Hopf algebra, with a corresponding evaluation map,

$$\mathcal{N}^{\mathrm{YM}}(12\ldots n-2)=\langle T_{(1)}\star T_{(2)}\cdots\star T_{(n-3)}\rangle_{\mathrm{YM}}. \quad (37)$$

In this construction, the gluon $(n-2)$ will enter the mapping rule differently from the others, and does not have an associated generator. The evaluation map can be deduced from the W' function when it is truncated to leading orders in the α' expansion. When expanding in α', the on-shell condition (16) for the massive gluon can reduce the α' order by one power. Since we are interested in pure YM, this implies we can focus purely on terms up to order linear in α', which are single trace terms of the form

$$\alpha' W_0(1\ldots(n-3)q)+\sum\alpha' p_X\cdot F_{\tau_b}\cdot p_Y W'(i_1\tau_a i_r). \quad (38)$$

On the massive gluon on-shell condition (32), the first term contributes to leading order as

$$\varepsilon_1\cdot F_{2\ldots n-3}\cdot\varepsilon_q. \quad (39)$$

The second term contributes to the leading order only when $i_1\tau_2 i_r$ contains gluon indices 1 and q, since other terms are independent under the on-shell condition (32). Then, according to the W'-function relations in Eq. (8), the second term can be written as

$$\sum \alpha' p_X \cdot F_{\tau_b} \cdot p_{qY} W'(1\tau_a q). \tag{40}$$

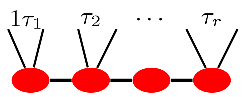
The propagator in the W' function is also simplified for the leading order contribution in the limit $p_i \cdot p_q = z \to \infty$,

$$\frac{\alpha'}{1-\alpha' p^2_{1q\tau_a}} \to \frac{1}{2p_q \cdot p_{\tau_b}}, \tag{41}$$

where we used the on-shell condition $p^2_{1q\tau_a\tau_b} = 1/\alpha'$.

In the limit $z \to \infty$, the parameter z in (40) and (41) cancels, which ensures that the BCJ numerator retains its local form. We can now identify the specific terms within the W' function that contribute to local BCJ numerators, finally leading to the evaluation map for pure YM:

$$\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)}\rangle_{\mathrm{YM}} :=$$



$$= \overline{G}_{1\tau_1} \frac{\overline{G}_{\tau_2}(p_{\Theta(\tau_2)})}{n-3-|\tau_1|} \cdots \frac{\overline{G}_{\tau_r}(p_{\Theta(\tau_r)})}{n-3-|\tau_1\ldots\tau_{r-1}|}, \tag{42}$$

with

$$\langle T_{(j\tau_1),(\tau_2),\ldots,(\tau_r)}\rangle_{\mathrm{YM}} := 0, \quad \text{if } j > 1, \tag{43}$$

and

$$\overline{G}_{1\tau_1} = \varepsilon_1 \cdot F_{\tau_1} \cdot \varepsilon_{n-2},$$
$$\overline{G}_{\tau_i j}(p_{\Theta(\tau_i j)}) = p_{\Theta(\tau_i j)} \cdot F_{\tau_i} \cdot \varepsilon_j. \tag{44}$$

The set $\Theta(\tau_i)$ consists of all indices to the left of τ_i and smaller than the first index in τ_i, as in (3). We can check this reproduces, for example, Eq. (36).

The above map demonstrates that indeed kinematic Hopf algebras apply directly to pure YM, without relying on factorization limits of HEFT numerators. We note the number of terms in the local BCJ numerators are twice that in the nonlocal BCJ numerator [31] (i.e., Fubini numbers [61]), the same as what was obtained in [52] using different methods. This approach can be further extended to higher α' corrections of the local BCJ numerators by generalizing the above analysis.

Conclusion and outlook. This Letter starts by investigating the construction of the BCJ numerators for the HEFT limit of $\mathrm{DF}^2 + \mathrm{YM}$ theory, whose α' expansion generates higher derivative corrections to YM theory, compatible with the color-kinematics duality. The approach is based on a kinematic Hopf algebra, where a fusion product of generators ensures the color-kinematics duality holds, with the full α' dependence contained in the mapping rule from abstract generators to kinematic functions. Our proposed mapping rule contains a new object we term the W' function, which can be used to directly obtain local BCJ numerators in YM, demonstrating that the HEFT limit is not required for the Hopf algebra construction to work. These results are strong evidence that kinematic Hopf algebras universally underpin the color-kinematics duality.

To further test this fascinating possibility of universality, one could also explore the appearance of Hopf algebras in scalar theories by using transmutation operators [62] or dimensional reduction [63]. It would also be interesting to connect to approaches such as [64–67], as well as the recently discovered family of theories containing higher derivative corrections to YM that obey the color-kinematics duality, besides $\mathrm{DF}^2 + \mathrm{YM}$ [48]. Interestingly, such theories are order $\mathcal{O}(m^3)$ or higher in the HEFT limit, whereas $\mathrm{DF}^2 + \mathrm{YM}$ theory is order $\mathcal{O}(m)$.

The BCJ numerators presented in this Letter serve as a direct means to construct gravitational amplitudes extending beyond Einstein gravity [68–72]. By incorporating them into the classical HEFT expansion graphs [73–75] of binary black hole scattering, our approach facilitates the calculation of classical observables such as bending angles or waveforms within the binary black hole system, enabling the study of potential physical effects beyond pure Einstein gravity.

More generally, a better understanding of the kinematic Hopf algebras in such varied contexts could help reveal new physical implications of this structure, and elucidate further aspects of the color-kinematics duality.

Acknowledgments. We would like to thank Andreas Brandhuber and Gabriele Travaglini for stimulating discussions. G. C. has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 847523 "INTERACTIONS." C. W. is supported by a Royal Society University Research Fellowship URF\R\221015 and a STFC Consolidated Grant No. ST\T000686\1 "Amplitudes, strings & duality".

[1] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys. Rev. D **78**, 085011 (2008).

[2] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys. Rev. Lett. **105**, 061602 (2010).

[3] R. Monteiro and D. O'Connell, J. High Energy Phys. 07 (2011) 007.

[4] N. E. J. Bjerrum-Bohr, P. H. Damgaard, R. Monteiro, and D. O'Connell, J. High Energy Phys. 06 (2012) 061.

[5] C. Cheung and C.-H. Shen, Phys. Rev. Lett. **118**, 121601 (2017).

[6] G. Chen, H. Johansson, F. Teng, and T. Wang, J. High Energy Phys. 11 (2019) 055.

[7] S. Mizera, Phys. Rev. Lett. **124**, 141601 (2020).

[8] G. Chen, H. Johansson, F. Teng, and T. Wang, J. High Energy Phys. 10 (2021) 042.

[9] M. Ben-Shahar and H. Johansson, J. High Energy Phys. 08 (2022) 035.

[10] R. Monteiro, R. Stark-Muchão, and S. Wikeley, J. High Energy Phys. 09 (2023) 030.

[11] R. Monteiro, J. High Energy Phys. 01 (2023) 092.

[12] A. Lipstein and S. Nagy, Phys. Rev. Lett. **131**, 081501 (2023).

[13] P. Ferrero and D. Francia, J. High Energy Phys. 02 (2021) 213.

[14] M. Ben-Shahar, L. Garozzo, and H. Johansson, J. High Energy Phys. 08 (2023) 222.

[15] C. R. Mafra and O. Schlotterer, Phys. Rep. **1020**, 1 (2023).

[16] C.-H. Fu, P. Vanhove, and Y. Wang, J. High Energy Phys. 09 (2018) 141.

[17] M. Ben-Shahar and M. Guillen, J. High Energy Phys. 12 (2021) 014.

[18] L. Borsten, B. Jurčo, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Phys. Rev. Lett. **126**, 191601 (2021).

[19] L. Borsten and S. Nagy, J. High Energy Phys. 07 (2020) 093.

[20] R. Bonezzi, F. Diaz-Jaramillo, and S. Nagy, Phys. Rev. D **108**, 065007 (2023).

[21] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Phys. Rev. Lett. **131**, 041603 (2023).

[22] K. Armstrong-Williams, S. Nagy, C. D. White, and S. Wikeley, arXiv:2401.10750.

[23] G. R. Brown, J. Gowdy, and B. Spence, Phys. Rev. D **109**, 026009 (2024).

[24] C.-H. Fu and Y. Wang, Springer Proc. Math. Stat. **396**, 287 (2022).

[25] M. E. Hoffman, J. Algebraic Combinatorics **11**, 49 (2000).

[26] J. Blumlein, Comput. Phys. Commun. **159**, 19 (2004).

[27] M. Aguiar and S. A. Mahajan, *Monoidal Functors, Species and Hopf Algebras* (American Mathematical Society, Providence, RI, 2010), Vol. 29.

[28] M. E. Hoffman and K. Ihara, J. Algebra **481**, 293 (2017).

[29] F. Fauvet, L. Foissy, and D. Manchon, Ann. l'Inst. Fourier **67**, 911 (2017).

[30] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, J. High Energy Phys. 07 (2021) 047.

[31] A. Brandhuber, G. Chen, H. Johansson, G. Travaglini, and C. Wen, Phys. Rev. Lett. **128**, 121601 (2022).

[32] A. Brandhuber, G. R. Brown, G. Chen, J. Gowdy, G. Travaglini, and C. Wen, J. High Energy Phys. 12 (2022) 101.

[33] G. Chen, G. Lin, and C. Wen, Phys. Rev. D **107**, L081701 (2023).

[34] G. Lin and G. Yang, J. High Energy Phys. 02 (2024) 012.

[35] G. Lin and G. Yang, J. High Energy Phys. 02 (2024) 013.

[36] N. E. J. Bjerrum-Bohr, G. Chen, Y. Miao, and M. Skowronek, J. High Energy Phys. 07 (2024) 242.

[37] Q. Cao, J. Dong, S. He, and Y.-Q. Zhang, Phys. Rev. D **107**, 026022 (2023).

[38] G. Chen, L. Rodina, and C. Wen, J. High Energy Phys. 02 (2024) 096.

[39] See also [40] for a different approach studying the color-kinematics duality for $\alpha' F^3 + \alpha'^2 F^4$ theory.

[40] Q. Bonnefoy, G. Durieux, and J. Roosmale Nepveu, arXiv:2310.13041.

[41] H. Johansson and J. Nohle, arXiv:1707.02965.

[42] Y.-t. Huang, O. Schlotterer, and C. Wen, J. High Energy Phys. 09 (2016) 155.

[43] T. Azevedo, M. Chiodaroli, H. Johansson, and O. Schlotterer, J. High Energy Phys. 10 (2018) 012.

[44] T. Azevedo, R. L. Jusinskas, and M. Lize, J. High Energy Phys. 01 (2020) 082.

[45] T. Azevedo and O. T. Engelund, J. High Energy Phys. 11 (2017) 052.

[46] L. M. Garozzo, L. Queimada, and O. Schlotterer, J. High Energy Phys. 02 (2019) 078.

[47] G. Menezes, J. High Energy Phys. 03 (2022) 074.

[48] J. J. M. Carrasco, M. Lewandowski, and N. H. Pavao, J. High Energy Phys. 02 (2023) 015.

[49] J. J. M. Carrasco, M. Lewandowski, and N. H. Pavao, Phys. Rev. Lett. **131**, 051601 (2023).

[50] J. J. M. Carrasco and N. H. Pavao, Phys. Rev. D **109**, 065006 (2024).

[51] L. Garozzo and A. Guevara, J. High Energy Phys. 07 (2024) 002.

[52] C. Cheung and J. Mangan, J. High Energy Phys. 11 (2021) 069.

[53] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.110.L041902 for a brief introduction to the kinematic Hopf algebra, details on the amplitude with one massive gluon or tachyon, examples of W' functions, and examples of local BCJ numerators.

[54] D. Vaman and Y.-P. Yao, J. High Energy Phys. 11 (2010) 028.

[55] Because of the cyclic summation in the operator $\mathbb{O}$, if $\mathrm{cyc}(x_1, \ldots, x_s) = \mathrm{cyc}(x'_1, \ldots, x'_s)$, the two partitions generate identical terms under the action of $\mathbb{O}_{\mathrm{cyc}([i_1 \ldots i_{r-1}] i_r)}$. Therefore, for each partition element with s subsets, the $\mathbb{O}$ operator introduces an overcounting factor equal to s.

[56] N. Arkani-Hamed, L. Rodina, and J. Trnka, Phys. Rev. Lett. **120**, 231602 (2018).

[57] L. Rodina, J. High Energy Phys. 09 (2019) 084.

[58] Y.-J. Du and F. Teng, J. High Energy Phys. 04 (2017) 033.

[59] A. Edison and F. Teng, J. High Energy Phys. 12 (2020) 138.

[60] G. Chen, https://github.com/amplitudegravity/kinematichopfalgebra (2022).

[61] O. A. Gross, Am. Math. Mon. **69**, 4 (1962).

[62] C. Cheung, C.-H. Shen, and C. Wen, J. High Energy Phys. 02 (2018) 095.

[63] M. Chiodaroli, M. Gunaydin, H. Johansson, and R. Roiban, J. High Energy Phys. 06 (2017) 064.

[64] J. J. M. Carrasco, L. Rodina, Z. Yin, and S. Zekioglu, Phys. Rev. Lett. **125**, 251602 (2020).

[65] I. Low, L. Rodina, and Z. Yin, Phys. Rev. D **103**, 025004 (2021).

[66] J. J. M. Carrasco, L. Rodina, and S. Zekioglu, J. High Energy Phys. 06 (2021) 169.

[67] N. H. Pavao, Phys. Rev. D **107**, 065020 (2023).
[68] A. Brandhuber and G. Travaglini, J. High Energy Phys. 01 (2020) 010.
[69] M. Accettulli Huber, A. Brandhuber, S. De Angelis, and G. Travaglini, Phys. Rev. D **102**, 046014 (2020).
[70] M. Accettulli Huber, A. Brandhuber, S. De Angelis, and G. Travaglini, Phys. Rev. D **103**, 045015 (2021).
[71] W. T. Emond and N. Moynihan, J. High Energy Phys. 12 (2019) 019.
[72] N. Sennett, R. Brito, A. Buonanno, V. Gorbenko, and L. Senatore, Phys. Rev. D **102**, 044056 (2020).
[73] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, J. High Energy Phys. 10 (2021) 118.
[74] A. Brandhuber, G. R. Brown, G. Chen, S. De Angelis, J. Gowdy, and G. Travaglini, J. High Energy Phys. 06 (2023) 048.
[75] A. Herderschee, R. Roiban, and F. Teng, J. High Energy Phys. 06 (2023) 004.