Letter

Spinning waveforms from the Kosower-Maybee-O'Connell formalism at leading order

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We provide the analytic waveform in time domain for the scattering of two Kerr black holes at leading order in the post-Minkowskian (weak field, but generic velocity) expansion and up to fourth order in both spins. The result is obtained by the generalization of the Kosower-Maybee-O'Connell formalism to radiative observables, combined with the analytic continuation of the five-point scattering amplitude to complex kinematics. We use analyticity arguments to express the waveform directly in terms of the three-point coupling of the graviton to the spinning particles and the gravitational Compton amplitudes, completely bypassing the need to compute and integrate the five-point amplitude. In particular, this allows us to easily include higher-order spin contributions for any spinning compact body. Finally, in the spinless case, we find a new compact and gauge-invariant representation of the Kovacs-Thorne waveform.

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Introduction. We are now living in the exciting era of gravitational-wave (GW) astronomy, with over 90 compact binary merger events detected to date by the LIGO-Virgo-KAGRA Collaboration [1]. Thanks to the improved sensitivity of the current and future GW detectors, many more events will be discovered in upcoming years. In particular, even hyperbolic encounters with aperiodic and low-intensity signals might represent interesting targets for future searches [1], as shown by the recent analysis of GW190521 [2]. These ongoing searches for signals rely on analytical or numerical template banks both for detection and for parameter estimation, calling for a better theoretical understanding of gravitational waveforms.

Focusing on the inspiral phase, we can treat compact objects at large distances as point particles using an effective field theory approach to general relativity [3]. Quantum field theory and amplitude-inspired techniques offer an analytic and efficient toolkit to perform classical calculations in the post-Minkowskian (PM) expansion [4], which is formally a weak field expansion in powers of the Newton's constant G valid for generic velocities. The leading PM contribution corresponds to tree-level diagrams, while loop diagrams become relevant at higher orders in the PM expansion. For example, see the remarkable progress for the calculation of the Hamiltonian and related scattering observables at 3PM and 4PM [5]. Motivated by this and the previous discussion, a framework to compute waveforms has been developed within the Kosower-Maybee-O'Connell (KMOC) formalism [6,7] and the worldline approach [8], albeit restricted so far to scattering configurations. Tree-level waveforms for spinless particles have been computed in Ref. [9] using the five-point amplitude [10], making contact with the Kovacs and Thorne result obtained with traditional methods [11,12] (see also the earlier result by Peters [13]). Recently, these calculations have been extended to one-loop order by combining KMOC with the heavy particle effective theory [14–17], whose results have been recently compared with post-Newtonian waveforms [18] finding disagreement at higher order. The latter may be related to a classical part of the KMOC subtraction term, as stressed in Ref. [19].

Since astrophysical black holes are always spinning, the inclusion of spin effects is an important milestone in this program. A first step in this direction has been taken in Ref. [20], where effects quadratic in spin have been included in the tree-level PM waveform. To include higher-order spin effects, we need a full description of

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Kerr black holes in terms of spinning point particles. Such remarkable correspondence has been established first at the level of the three-point amplitude for the Kerr multipoles [21]. It is still under development for the four-point Compton amplitude [22], which should match the conservative piece of the solution of the Teukolsky equation [23]. With such identification, many conservative and radiative spinning amplitudes have been recently computed [24].

In this Letter, we develop a new method to perform the phase-space integration for KMOC observables by making use of the analytic properties of amplitudes in the complex plane. Focusing on the tree level, we compute for the first time the time-domain waveform for Kerr black holes up to fourth order in both spins, using only the factorization channels of the five-point amplitude. The method presented here applies to generic field theories and beyond the classical limit.

We work in the signature $(+---)$, using relativistic units $c=1$ and with $\kappa := \sqrt{32\pi G}$. For convenience, we adopt the notation $\hat{\delta}^n(\cdot) \equiv (2\pi)^n \delta^n(\cdot)$, $\hat{\delta}^+(k^2) = \Theta(k^0)\hat{\delta}(k^2)$, and $\hat{\mathrm{d}}^n q \equiv \mathrm{d}^n q/(2\pi)^n$. We use the notation $\mathcal{M}^{(0)}_{n,\mathrm{cl}}$ to denote the classical tree-level n-point amplitude with external (outgoing) gravitons and one or two pairs of massive particles, depending on the context.

Waveforms from the analytic properties of the $\mathcal{S}$-matrix. In quantum mechanics, the change in some observable $\mathcal{O}$ from the far past to the far future is given by

$$\Delta\langle\mathcal{O}\rangle = {}_{\mathrm{out}}\langle\psi|\mathcal{O}|\psi\rangle_{\mathrm{out}} - {}_{\mathrm{in}}\langle\psi|\mathcal{O}|\psi\rangle_{\mathrm{in}}, \tag{1}$$

where $|\psi\rangle_{\mathrm{in}}$ is the initial state of the system. We take $|\psi\rangle_{\mathrm{in}}$ to be a wave packet describing two sharply localized particles with classical momenta $p_i = m_i v_i$ well separated from each other [6]. The operator under consideration is the linearized metric $h_{\mu\nu}$ at future null infinity [7],

$$\mathcal{O} = \varepsilon^{\mu\nu}_{\lambda} h_{\mu\nu} = \int \hat{\mathrm{d}}^4 k \hat{\delta}^+(k^2)[e^{-ik\cdot x} a_{-\lambda}(k) + \mathrm{c.c.}], \tag{2}$$

where $\varepsilon^{\mu\nu}_{\lambda}$ is the polarization tensor corresponding to a helicity-λ state and $a_{-\lambda}(k)$ is the annihilation operator for the gravitons. The out state $|\psi\rangle_{\mathrm{out}}$ is related to the in state by the $\mathcal{S}$-matrix

$$|\psi\rangle_{\mathrm{out}} = \mathcal{S}|\psi\rangle_{\mathrm{in}}. \tag{3}$$

After taking the classical limit and considering the leading term in the large-distance expansion (see Refs. [7,15] for more details), the *strain* can be written as

$$h(x) = \frac{\kappa}{4\pi|\vec{x}|}\int_0^\infty \hat{\mathrm{d}}\omega[W(b;k^-)e^{-i\omega u} + [W(b;k^+)]^* e^{i\omega u}], \tag{4}$$

where $k^\mu = \omega n^\mu = \omega(1,\hat{x})$ with $\hat{x} = \vec{x}/|\vec{x}|$, $u = x^0 - |\vec{x}|$ is the retarded time, $W(b;k^\pm)$ is the helicity-dependent *spectral waveform* of the emitted gravitational wave, and b is the impact parameter. For the classical scattering of two massive particles, the spectral waveform can be computed from the $\mathcal{S}$-matrix,

$$iW(b;k^\lambda) = \left\langle\!\!\left\langle \int \mathrm{d}\mu e^{i(q_1\cdot b_1 + q_2\cdot b_2)}\mathcal{I}_{a_\lambda} \right\rangle\!\!\right\rangle, \tag{5}$$

where the double-angle brackets are understood as the classical limit of the expression inside,

$$\hat{\delta}^4(q_1+q_2-k)\mathcal{I}_{a_\lambda} = \langle p_1' p_2'|\mathcal{S}^\dagger a_\lambda(\vec{k})\mathcal{S}|p_1 p_2\rangle, \tag{6}$$

and the measure is defined by

$$\mathrm{d}\mu = \left[\prod_{i=1,2}\hat{\mathrm{d}}^4 q_i \hat{\delta}(-2p_i\cdot q_i + q_i^2)\right]\hat{\delta}^4(q_1+q_2-k), \tag{7}$$

with momentum mismatches $q_i^\mu = p_i^\mu - p_i'^\mu$. At leading order in perturbation theory, this simplifies to

$$W^{(0)}(b;k^\lambda) = \int \mathrm{d}\mu e^{i(q_1\cdot b_1 + q_2\cdot b_2)}\mathcal{M}^{(0)}_{5,\mathrm{cl}}(q_1,q_2,k^\lambda), \tag{8}$$

where $\mathcal{M}^{(0)}_{5,\mathrm{cl}}(q_1,q_2,k^\lambda)$ denotes the two-to-three classical amplitude with two incoming and outgoing massive particles emitting a graviton during the scattering. The Fourier transform to impact-parameter space (IPS) reads

$$W^{(0)}(b;k^\lambda) = \int \mathrm{d}z_v \mathrm{d}z_b e^{-iz_b\sqrt{-b^2}}\frac{\hat{\mathcal{M}}^{(0)}_{5,\mathrm{cl}}}{(4\pi)^2 m_1 m_2\sqrt{\gamma^2-1}}, \tag{9}$$

where $\gamma = v_1\cdot v_2$, $\hat{\mathcal{M}}^{(0)}_{5,\mathrm{cl}}$ is the five-point classical amplitude evaluated on the support of the delta distributions in the measure (7) and z_b and z_v are the components of the momentum mismatch q_1^μ (see Sec. I of the Supplemental Material [25]).

The evaluation of these Fourier integrals has shown to be challenging [9] already at tree level as the waveform in the frequency domain involves iterated integrals of Bessel functions. The result is greatly simplified in the time domain [20], where only square roots appear. Here, we argue that these computations are further simplified by combining the analytic structures of scattering amplitudes with basic properties of Fourier transforms.

We are going to evaluate the z_v integral in (9) first, by deforming the integration contour from the real axis to infinity through the upper half-plane (UHP) (or equivalently the LHP) as in Fig. 1. The resulting integral

$$I^\lambda_{\mathrm{UHP}} = \int_{\mathcal{C}^{(+)}} \mathrm{d}z_v \hat{\mathcal{M}}^{(0)}_{5,\mathrm{cl}}(q_1,q_2,k^\lambda) \tag{10}$$

receives two types of contributions from the residue theorem, which we now discuss.

The first contribution is related to the simple poles of the tree-level five-point amplitude in the complex z_v plane at $q_1^2 = 0$ and $q_2^2 = 0$ (the massive eikonal propagators take the form $v_i\cdot k$ and do not depend on z_v). These poles correspond to factorization channels, and their residues are

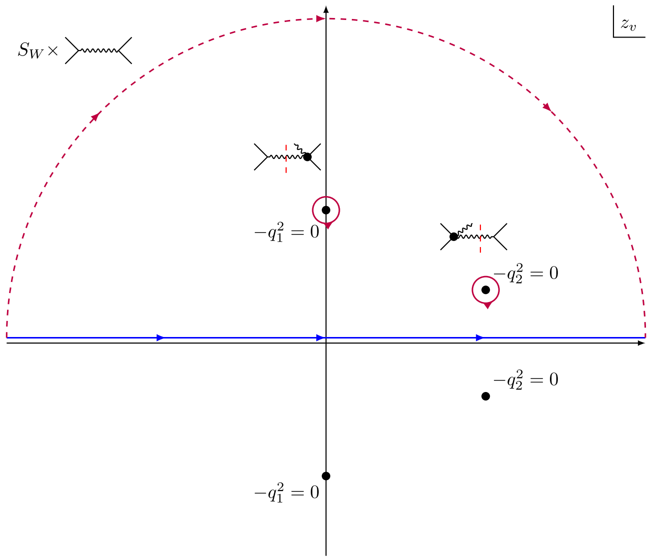

FIG. 1. The deformation of the contour $\mathcal{C}^{(+)}$ (in red) in the complex z_v plane allows us to evaluate the integral of the five-point tree-level amplitude directly in terms of the factorization channels.

products of lower-point tree-level amplitudes summed over helicities σ of the internal state,

$$\frac{I^{\lambda}_{\mathcal{C}^{(+)}_{q_1}}}{2\pi i}=\sum_{\sigma}\mathcal{M}^{(0)}_{4,\mathrm{cl}}(p_2,k^{\lambda},-q_1^{\sigma})\mathcal{M}^{(0)}_{3,\mathrm{cl}}(p_1,q_1^{\sigma})\operatorname*{Res}_{z_v=\hat{z}_1}\frac{-1}{q_1^2},$$
$$\frac{I^{\lambda}_{\mathcal{C}^{(+)}_{q_2}}}{2\pi i}=\sum_{\sigma}\mathcal{M}^{(0)}_{4,\mathrm{cl}}(p_1,k^{\lambda},-q_2^{\sigma})\mathcal{M}^{(0)}_{3,\mathrm{cl}}(p_2,q_2^{\sigma})\operatorname*{Res}_{z_v=\hat{z}_2}\frac{-1}{q_2^2},\qquad(11)$$

where $\hat{z}_1$ and $\hat{z}_2$ are the UHP solutions of the pole constraints $-q_i^2=(z_v-\hat{z}_i)(z_v-\hat{z}_i^*)=0$. The second type of contribution is due to the arc at infinity. We show in Secs. I and II of the Supplemental Material [25] that, if we choose to split the z_v contour into two equal pieces in the UHP and LHP, such contribution is not classical and therefore can be ignored.

The waveform computation in the time domain then becomes straightforward. Indeed, it is convenient to perform the ω integration before taking the Fourier transform to IPS. After rescaling all the dimensionful variables by appropriate powers of ω, the ω dependence of the KMOC integrand can be factored out, and the integration to the time domain evaluates to a delta distribution $\delta(u+z_b\sqrt{-b^2})$. Finally, the z_b integration is trivialized, and the strain at leading order can be written as

$$h^{(0)}(x)=\frac{\kappa}{(4\pi)^3|\vec{x}|\sqrt{-b^2}}\frac{(I^{\lambda}_{\mathrm{UHP}}-I^{\lambda}_{\mathrm{LHP}})}{4m_1m_2\sqrt{\gamma^2-1}}\bigg|_{z_b=-u/\sqrt{-b^2}}.\qquad(12)$$

We expect this method to bring several advantages at a higher orders in perturbation theory. Indeed, evaluating the Fourier transform to IPS of the one-loop five-point amplitude in gravity is notoriously complicated [14,15]. There are two main obstacles, the set of functions appearing at one loop is largely more complex, and each appears multiplied by complicated rational functions with spurious singularities. Applying our method to higher orders will allow us to bypass this problem completely, as the contour deformation is by definition insensitive to such unphysical singularities, selecting only those terms which give nonanalyticities—hence, long-range contributions—in the momentum mismatches q_i^2. The main technical obstacle at loop level is the evaluation of the integral along the discontinuities of the KMOC integrand. Such discussion is left for future works.

Tree-level waveform for schwarzschild black holes. In this section, we consider scalar fields minimally coupled to gravity, mimicking macroscopic objects like black holes or neutron stars. Their intrinsic scales (spin and finite-size effects) induce corrections which can be neglected to leading order at large distances.

To compute the tree-level scattering waveform for spinless particles using (12), we need the three-point and four-point amplitudes [26]

$$\mathcal{M}^{(0)}_{3,\mathrm{cl}}(p,k)=-\kappa m^2(\varepsilon\cdot v)^2,$$
$$\mathcal{M}^{(0)}_{4,\mathrm{cl}}(p,k_1,k_2)=\frac{\kappa^2m^2}{q^2}\left(\frac{v\cdot F_1\cdot F_2\cdot v}{v\cdot k_1}\right)^2,\qquad(13)$$

where $F_i^{\mu\nu}=k_i^{\mu}\varepsilon_i^{\nu}-k^{\nu}\varepsilon_i^{\mu}$ are the (linearized) field strengths of the gravitons. These amplitudes can be obtained, for example, using the Feynman rules from Einstein-Hilbert action with a minimally coupled scalar and then taking the classical limit (see Ref. [14], for example).

The scalar waveform at *leading order* is then derived from (12). We notice that the residues in (11) keep the physical poles at $v_i\cdot k=0$ and introduce three additional singularities (in complex time) given by $1/\sqrt{1+T_i^2}$ and $1/\mathsf{S}^2$ factors, where we used the variables introduced in the seminal papers of Kovacs and Thorne [11,12]:

$$T_i:-\frac{\sqrt{\gamma^2-1}(u-b_i\cdot n)}{\sqrt{-(b_1-b_2)^2}(v_i\cdot n)},\qquad(14)$$

$$\mathsf{S}^2:-\frac{T_1^2-2\gamma T_1T_2+T_2^2}{\gamma^2-1}-1.\qquad(15)$$

The time variables T_1 and T_2 are the characteristic timescales of the acceleration of the particles, while S encodes the relative spacetime difference of the two bodies in relation to the light cone of the observer. The former singularities are already introduced by the $1/q_i^2$ poles, while the latter requires the factor $1/q_1^2q_2^2$, and it is a signature of gravitational nonlinearities. As already noticed in the original paper, there might be physical points for which $\mathsf{S}^2=0$, but such singularity is spurious. Indeed, we have

$$(1-\gamma^2)\mathsf{S}^2 = \prod_{\Sigma=\pm}\left[\gamma + T_1T_2 - \Sigma\sqrt{(T_1^2+1)(T_2^2+1)}\right], \tag{16}$$

but it is worth noticing that we can always factorize the term $(\gamma + T_1T_2 - \sqrt{(T_1^2+1)(T_2^2+1)})$ in the numerator, so the spurious singularity vanishes manifestly.

In performing our phase space integration, we implicitly changed the $i\epsilon$ prescription of the ω integration [27] because we are interested in finite energy contributions to the tree-level waveform in the Bondi-Metzner-Sachs frame where

$$\lim_{u\to+\infty} h^{(0)}(x) = -\lim_{u\to-\infty} h^{(0)}(x). \tag{17}$$

With such prescription, we find then a new compact and manifestly gauge-invariant expression for the strain,

$$\begin{aligned} h^{(0)}(x) = {}& \frac{G^2 m_1 m_2}{|\vec{x}|\sqrt{-b^2}}\frac{1}{\bar{w}_1^2\bar{w}_2^2\sqrt{1+T_2^2}(\gamma+\sqrt{(1+T_1^2)(1+T_2^2)}+T_1T_2)}\left(\frac{3\bar{w}_1 + 2\gamma(2T_1T_2\bar{w}_1 - T_2^2\bar{w}_2 + \bar{w}_2) - (2\gamma^2-1)\bar{w}_1}{\gamma^2-1}f_{1,2}^2\right. \\ & - \frac{4\gamma T_2\bar{w}_2 f_1 + 2(2\gamma^2-1)[T_1(1+T_2^2)\bar{w}_2 f_1 + T_2(T_1T_2\bar{w}_1+\bar{w}_2)f_2]}{\sqrt{\gamma^2-1}}f_{1,2} \\ & \left. + 4(1+T_2^2)\bar{w}_2 f_1 f_2 - 4\gamma(1+T_2^2)\bar{w}_2(f_1^2+f_2^2) + 2(2\gamma^2-1)(1+2T_2^2)\bar{w}_2 f_1 f_2\right) + (1\leftrightarrow 2), \end{aligned} \tag{18}$$

where $\bar{w}_i = v_i\cdot n$ and we have defined [28]

$$f_{1,2} = v_1\cdot\varepsilon_k v_2\cdot n - v_1\cdot n v_2\cdot\varepsilon_k, \tag{19}$$

$$f_i = \tilde{b}\cdot\varepsilon_k v_i\cdot n - \tilde{b}\cdot n v_i\cdot\varepsilon_k. \tag{20}$$

Our result agrees with Kovacs and Thorne [12] (see also Ref. [9]) when restricted to their chosen frame [29] and the sum of the two terms in (17) matches the linear memory in Eq. (27) of Ref. [20].

Tree-level waveform for kerr black holes. In this section, we discuss the calculation of the scattering waveform for two Kerr black holes, which are identified with spinning point particles. The essential ingredients in the computation are the three-point and the four-point (Compton) amplitudes; although they are typically presented in the literature as helicity amplitudes [21,30], we propose here a new equivalent gauge-invariant representation in terms of polarization vectors.

The gauge-invariant three-point amplitude describing the coupling of a Kerr black hole to gravity takes an exponential form [21,31]

$$\mathcal{M}_{3,\text{cl}}^{(0)} = -\kappa m^2(\varepsilon\cdot v)^2\exp\left(\frac{i\varepsilon\cdot S\cdot k}{\varepsilon\cdot v}\right), \tag{21}$$

where S is the unit mass spin tensor of the Kerr black hole, related to the Pauli–Lubanski pseudovector a^μ by

$$S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}v_\rho a_\sigma, \qquad a^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}v_\nu S_{\rho\sigma}. \tag{22}$$

The exponential features a spurious pole in the polarization vector ε starting at $\mathcal{O}(S^3)$, which makes this expression unsuitable for computing higher-point amplitudes from unitarity [21]. Interestingly, the spurious pole can be removed by exploiting the four-dimensional identity

$$\left(\frac{\varepsilon\cdot S\cdot k}{\varepsilon\cdot v}\right)^2 = k\cdot S\cdot S\cdot k = -(k\cdot a)^2. \tag{23}$$

Expanding the exponential in (21) and applying this identity, we find a new alternating structure (which resums to sine and cosine [24,32]) free from the spurious pole in ε at any order in the spin expansion [30]:

$$\begin{aligned} \mathcal{M}_{3,\text{cl}}^{(0)} = -\kappa m^2\Big[&(\varepsilon\cdot v)^2 + i(\varepsilon\cdot v)(\varepsilon\cdot S\cdot k) \\ &- \frac{1}{2}(\varepsilon\cdot v)^2(k\cdot S\cdot S\cdot k) \\ &- \frac{i}{3!}(\varepsilon\cdot v)(\varepsilon\cdot S\cdot k)(k\cdot S\cdot S\cdot k) \\ &+ \frac{1}{4!}(\varepsilon\cdot v)^2(k\cdot S\cdot S\cdot k)^2 + \ldots\Big]. \end{aligned} \tag{24}$$

We find that the Compton amplitude can be written in a compact gauge-invariant representation (in agreement with the fixed-helicity representation [23,30])

$$\mathcal{M}_{4,\text{cl}}^{(0)} = \frac{\kappa^2 m^2\omega_0^2}{8(k_1\cdot k_2)(k_1\cdot v)^2}\left(1+\frac{\omega_1}{\omega_0}+\frac{\omega_2}{\omega_0}\right) + \mathcal{O}(S^3), \tag{25}$$

with suitably contracted spin multipole coefficients ω_i,

$$\begin{aligned} \omega_0 &= -2v\cdot F_1\cdot F_2\cdot v, \\ i\omega_1 &= k_1\cdot F_2\cdot vS\cdot F_1 - \frac{(k_1-k_2)\cdot v}{2}S\cdot F_1\cdot F_2 + (1\leftrightarrow 2), \\ \omega_2 &= (k_1\cdot k_2)\left[S\cdot S\left(\frac{F_1\cdot F_2}{2} - v\cdot F_1\cdot F_2\cdot v\right) - 2S\cdot S\cdot F_1\cdot F_2\right], \\ &\quad - \frac{\omega_0}{2}(k_1+k_2)\cdot S\cdot S\cdot(k_1+k_2), \end{aligned} \tag{26}$$

where we define $A \cdot B = \eta_{\mu\nu}\eta_{\alpha\beta}A^{\alpha\mu}B^{\nu\beta}$ and $A \cdot B \cdot C \cdot D = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}\eta_{\alpha\beta}A^{\alpha\mu_1}B^{\nu_1\mu_2}C^{\nu_2\mu_3}D^{\nu_3\beta}$ for any tensors A, B, C, D. The gauge-invariant amplitude in (25) can be equivalently written as an exponential for any graviton polarization (see also Ref. [22])

$$\mathcal{M}_{4,\mathrm{cl}}^{(0)} = \frac{\kappa^2 m^2 \omega_0^2}{8(k_1 \cdot k_2)(k_1 \cdot v)^2} \exp\left(\frac{\omega_1}{\omega_0}\right) + \mathcal{O}(S^5), \tag{27}$$

and in four dimensions, one has

$$\frac{\omega_2}{\omega_0} = \frac{1}{2}\left(\frac{\omega_1}{\omega_0}\right)^2. \tag{28}$$

Similarly to the three-point case, the exponential (27) features a spurious pole in ω_0 starting at $\mathcal{O}(S^3)$, which we remove by means of the identity (28), thus arriving at a manifestly gauge-invariant expression for the Compton amplitude valid up to $\mathcal{O}(S^4)$ and free of spurious poles:

$$\mathcal{M}_{4,\mathrm{cl}}^{(0)} = \kappa^2 m^2 \frac{\omega_0^2 + \omega_0\omega_1 + \omega_0\omega_2 + \frac{\omega_1\omega_2}{3} + \frac{\omega_2^2}{6}}{8(k_1 \cdot k_2)(k_1 \cdot v)^2} + \mathcal{O}(S^5). \tag{29}$$

This new form of the Compton amplitude matches the solution of the Teukolsky equation [23] and has the correct factorization channels, reproducing (21).

Equipped with the amplitudes (21) and (29), we repeat the same steps as in the scalar scattering case to compute the waveform up to the fourth order in spin for both black holes, i.e., $\mathcal{O}(S_1^4, S_2^4)$. The recipe to compute the tree-level spin-multipole expansion of the waveform

$$h^{(0)}(x) = \sum_{s_1,s_2=0}^{\infty} h^{(s_1,s_2)}(x), \tag{30}$$

where s_i indicates the degree of homogeneity with respect to the spin tensor, needs to be modified because the z_b integral gives contributions of the type $\delta^{(s_1+s_2)}(\sqrt{-b^2}z_b + u)$, as explained in Sec. I of the Supplemental Material [25]. Thus, the spinning tree-level waveform is

$$h^{(s_1,s_2)}(x) = \frac{\kappa}{(4\pi)^3|\vec{x}|(\sqrt{-b^2})^{s_1+s_2+1}} \frac{(-i)^{s_1+s_2}}{4m_1m_2\sqrt{\gamma^2-1}} \times \frac{\partial^{s_1+s_2}}{\partial^{s_1+s_2} z_b}(I_{\mathrm{UHP}}^{(s_1,s_2)} - I_{\mathrm{LHP}}^{(s_1,s_2)})\Big|_{z_b=-u/\sqrt{-b^2}}, \tag{31}$$

where the residues are now computed isolating the s_1^{th} and s_2^{th} multipoles in the factorized amplitudes.

The final result can be written as

$$h^{(0)}(x) = \sum_{s_1,s_2=0}^{4} \frac{G^2 m_1 m_2}{|\vec{x}|(\sqrt{-b^2})^{s_1+s_2+1}} \mathfrak{h}_{s_1,s_2}(x) + \mathcal{O}(S_1^5, S_2^5), \tag{32}$$

where the $\mathfrak{h}_{s_1,s_2}$'s are provided in ancillary files.[1] For $s_1 + s_2 \leq 2$, our results match the ones presented in Ref. [20].

Conclusion. In this Letter, we combined the KMOC formalism with the analytic properties of the $\mathcal{S}$-matrix to develop an efficient framework for the calculation of the time-domain gravitational waveform for spinless and spinning bodies.

Observables like the waveform involve a phase-space integration over the classical amplitude, which, as we showed, can be easily evaluated by complex analysis tools once its singularity structure is understood. Focusing on the leading order, we computed the time-domain waveform directly from the factorization channels of the five-point amplitude, bypassing the complexity in its direct calculation. This is particularly convenient in the spinning case, because we only need the three-point and the four-point (Compton) amplitudes to determine the waveform solely algebraically.

Using our method, we first provided a new compact gauge-invariant expression of the leading-order waveform for spinless particles, discussing in detail its singularity structure. Our result agrees with the traditional Kovacs-Thorne result [12] and a more recent worldline calculation [20]. We then considered the spinning case, where we use a new gauge-invariant representation of the Compton amplitude which agrees with the solution of the Teukolsky equation [23] to compute analytically the leading-order waveform relevant for scattering of Kerr black holes. At quadratic order in spin this agrees with Ref. [20]. We provided an analytic expression valid up to fourth order in spin which can be directly extended to all spin orders once the full Kerr Compton amplitude is understood.

We leave a number of open questions to future investigations. The first is to understand to which extent classical observables depend on the analytic structure of amplitudes (i.e., poles and branch cuts), which, as shown in this work, can help to bypass traditional techniques. A second pressing problem is to understand the analytic continuation for the waveform discussed here, building on the dictionary between scattering and bound observables [20,33]. An additional future direction is moving away from the restriction of having Kerr black holes and considering more general compact spinning objects (like neutron stars), allowing generic multipoles for the three-point coupling [21]. In such case, the four-point amplitudes can be bootstrapped imposing locality and unitarity, up to contact interactions which can be taken into account properly. Finally, it would be interesting to compare directly post-Minkowskian scattering waveforms both with analytic post-Newtonian spinning waveforms [20,34] and effective-one-body waveforms [35,36], with the idea that in the future we might be able to detect black-hole hyperbolic encounters [1,37].

[1]Ancillary files available online at https://bitbucket.org/spinning-gravitational-observables/tree-level-waveform/.

Note added. Recently, the preprints [38,39] appeared. In Ref. [39], the authors apply the integration method presented in this paper to the study of spinless-spinning scattering using the resummed-in-spin Compton amplitude presented in Ref. [22], which matches the result from black-hole perturbation theory up to $\mathcal{O}(S^4)$. In Ref. [38], the authors computed the waveform with the generic parametrization of the spin multipoles developed in Refs. [21,22,24]. Both approaches successfully reproduced our results.

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[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.110.L041502 for the detailed derivation of waveform phase space integration in terms of tree-level amplitudes.

[26] Here, these amplitudes $\mathcal{M}^{(0)}_{3,\mathrm{cl}}(p,k)$ and $\mathcal{M}^{(0)}_{4,\mathrm{cl}}(p,k_1,k_2)$ correspond to a massive particle emitting one or two gravitons, respectively.

[27] To compare with Kovacs and Thorne and the recent work [9], we need to redefine the waveform imposing that it vanishes in the far (retarded) past $\tilde{h}(x){:} - h(x) - \lim_{u\to-\infty} h(x)$. Indeed, by exchanging the order of integration, we ignored the fact that the amplitude develops a $\frac{1}{\omega}$ pole as $\omega \to 0$ (from Weinberg's soft theorem). The correct prescription for this pole is $1/(\omega + i\varepsilon)$. Moreover, we also ignored terms proportional to $\delta(\omega)$ to the waveform in frequency space, which give a time-independent contribution and set $\lim_{u\to-\infty} h(x) = 0$.

[28] It is worth emphasizing that these three structures are not independent of each other, but they are related (linearly) by the Bianchi identity $f_{1,2}(T_1\bar{w}_1 - T_2\bar{w}_2) + \sqrt{\gamma^2-1}(f_2\bar{w}_1 - f_1\bar{w}_2) = 0$ and (quadratically) by the four-dimensional identity $(f_2\bar{w}_1 - f_1\bar{w}_2)^2 = \frac{(f_1^2 - 2\gamma f_1 f_2 + f_2^2)(T_1\bar{w}_1 - T_2\bar{w}_2)^2}{\gamma^2-1}$.

[29] The comparison is performed by picking the frame $v_1^\mu = (1,0,0,0)$, $v_2^\mu = (\gamma, \sqrt{\gamma^2-1}, 0, 0)$, $\tilde{b}^\mu = (0,0,-1,0)$, $n^\mu = (1, \cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$ and using the following polarization vectors: $\epsilon^\mu = \frac{1}{\sqrt{2}}(\partial_\theta n^\mu + i\frac{\partial_\phi n^\mu}{\sin\theta})$, $\epsilon_+^{\mu\nu} = \Re(\epsilon^\mu\epsilon^\nu)$, $\epsilon_\times^{\mu\nu} = \Im(\epsilon^\mu\epsilon^\nu)$.

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