Letter

Possible ${}^{3}_{\phi}$H hypernucleus with the HAL QCD interaction

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Within the framework of the Faddeev formalism in configuration space, we investigate bound states in the ϕNN system with total isospin $T=0$ and $T=1$. The recently proposed lattice HAL QCD ϕN potential in the ${}^4S_{3/2}$ channel does not support either ϕN or ϕNN bound states. The HAL QCD ϕN potential in the ${}^2S_{1/2}$ channel suggests the bound states for ϕN and $\phi NN(S=0)$ systems. However, the binding energies are highly sensitive to variations of the enhancement factor β, and the ϕNN system is extremely strongly bound in the state $S=0$. Considering a spin-averaged potential for the state $S=1$ yields a bound state for the ${}^{3}_{\phi}$H ($S=1$) hypernucleus with the binding energy (BE) 14.9 MeV when $\beta=6.9$. The evaluation of the BE for the $S=1$, $T=1$ three-body state results in 5.47 MeV. Additionally, calculations using our approach confirm the bound states for the ϕNN ($S=2$, $T=0$ and $S=1$, $T=1$) system previously predicted with the Yukawa-type potential motivated by the QCD van der Waals attractive force, mediated by multigluon exchanges.

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Since the beginning of the new millennium, studying the composite system from two nucleons and Λ-, Ξ-, Ω-hyperon or ϕ-meson has attracted intense research interest in many theoretical works [1–19]. Unlike the case of the NN interactions, a ϕ-meson nucleon interaction is not well determined due to an insufficient number of scattering data. It is one of the open and debated questions in the strangeness sector of nuclear physics concerning the possible existence of a ϕN bound state.

The recent ALICE Collaboration measurement of the ϕN correlation function [20] led to the determination of the ϕN channel scattering length with a large real part corresponding to an attractive interaction. This represents the first experimental evidence of the attractive strong interaction between a proton and a ϕ-meson.

It has been suggested by Brodsky, Schmidt, and de Teramond [21] that the QCD van der Waals interaction, mediated by multigluon exchanges, is dominant when the interacting two color singlet hadrons have no common quarks. Assuming that the attractive QCD van der Waals force dominates the ϕN interaction since the ϕ-meson is almost a pure $s\bar{s}$ state, following [21], Gao *et al.* [22] suggested a Yukawa-type attractive potential. Using the variational method, they predicted a binding energy (BE) of 1.8 MeV for the ϕ-N system. In [20], the data are employed to constrain the parameters of phenomenological Yukawa-type potentials. The resulting values for the Yukawa-type potential, $V_{\phi N}(r)=-Ae^{-\alpha r}/r$, yields $A=0.021\pm0.009(\mathrm{stat})\pm0.006(\mathrm{syst})$ and $\alpha=65.9\pm38.0(\mathrm{stat})\pm17.5(\mathrm{syst})$ MeV. Predictions of possible ϕN bound states employing the same kind of potential with parameters $A=1.25$ and $\alpha=600$ MeV [22] are therefore incompatible with measurement [20].

Recently, Lyu *et al.* [23] presented the first results on the interaction between the ϕ-meson and the nucleon based on the $(2+1)$-flavor lattice QCD simulations with nearly physical quark masses. The HAL QCD potential is obtained from first principles $(2+1)$-flavor lattice QCD simulations in a large spacetime volume, $L^4=(8.1\ \mathrm{fm})^4$, with the isospin-averaged masses of π, K, ϕ, and N as 146, 525, 1048, and 954 MeV, respectively, at a lattice spacing of $a=0.0846$ fm. Let us mention that such simulations together with the HAL QCD method enable one to extract the YN and YY interactions with multiple strangeness, e.g., ΛΛ, ΞN [24], ΩN [25], ΩΩ [26], and ΞN [13]. Using the HAL QCD method, based on the spacetime correlation of the ϕN system in the spin $3/2$ channel, the authors suggested fits of the lattice QCD potential in the ${}^4S_{3/2}$ channel. In the following, we employ the spectroscopic notation ${}^{2s+1}S_J$ to classify the S-wave ϕN interaction, where s and J stand for total spin and total angular momentum, respectively. It was found that simple fitting functions such as the Yukawa form cannot reproduce the lattice data [23]. The lattice calculations for the ϕN interaction in the ${}^4S_{3/2}$ channel are used in [27] to constrain the spin $1/2$ counterpart (${}^2S_{1/2}$) from the fit of the

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experimental ϕN correlation function measured by the ALICE Collaboration [20].

The mesonic ϕNN system is considered in the framework of Faddeev equations in the differential form [3], using the variational folding method [4] and a two-variable integro-differential equation describing bound systems of unequal mass particles [5]. Calculations were employed with the ϕN potential from [22]. The binding energy of the ϕd hypernucleus was calculated by employing the HAL QCD potential [23] using the Schrödinger equation for Faddeev components expanded in terms of hyperspherical functions [18]. The binding energies reported in Refs. [3,4,18] are in the range of ∼6–39 MeV.

Motivated by the above discussion and the availability of newly suggested HAL QCD potentials in the $^2S_{1/2}$ and $^4S_{3/2}$ channels with a minimal and maximal spin, respectively, we present calculations for the binding energy for the ϕN and ϕNN systems in the framework of the Faddeev equations in configuration space. We compare our results with other calculations as well.

The ϕNN system represents a three-particle system. The three-body problem can be solved in the framework of the Schrödinger equation or using the Faddeev approach in the momentum [28,29] or configuration [30–34] spaces. With regard to the Faddeev equations in the configuration space, Jacobi coordinates are introduced to describe the ϕNN system. The mass-scaled Jacobi coordinates $\mathbf{x}_i$ and $\mathbf{y}_i$ are expressed via the particle coordinates $\mathbf{r}_i$ and masses m_i in the following form:

$$\mathbf{x}_i = \sqrt{\frac{2m_k m_l}{m_k + m_l}}(\mathbf{r}_k - \mathbf{r}_l),$$
$$\mathbf{y}_i = \sqrt{\frac{2m_i(m_k + m_l)}{m_i + m_k + m_l}}\left(\mathbf{r}_i - \frac{m_k\mathbf{r}_k + m_l\mathbf{r}_l}{m_k + m_l}\right). \quad (1)$$

The orthogonal transformation between three different sets of the Jacobi coordinates has the form:

$$\begin{pmatrix}\mathbf{x}_i\\ \mathbf{y}_i\end{pmatrix} = \begin{pmatrix}C_{ik} & S_{ik}\\ -S_{ik} & C_{ik}\end{pmatrix}\begin{pmatrix}\mathbf{x}_k\\ \mathbf{y}_k\end{pmatrix}, \qquad C_{ik}^2 + S_{ik}^2 = 1,$$
$$k \neq i, \quad C_{ii} = 1, \quad (2)$$

where

$$C_{ik} = -\sqrt{\frac{m_i m_k}{(M - m_i)(M - m_k)}},$$
$$S_{ik} = (-1)^{k-i}\mathrm{sign}(k - i)\sqrt{1 - C_{ik}^2}.$$

Here, M is the total mass of the system. Let us definite the transformation $h_{ik}(\mathbf{x}, \mathbf{y})$ based on Eq. (2) as

$$h_{ik}(\mathbf{x}, \mathbf{y}) = (C_{ik}\mathbf{x} + S_{ik}\mathbf{y}, -S_{ik}\mathbf{x} + C_{ik}\mathbf{y}). \quad (3)$$

In the Faddeev method in configuration space, alternatively to finding the wave function of the three-body system using the Schrödinger equation, the total wave function is decomposed into three components [30,32,34]: $\Psi(\mathbf{x}_1, \mathbf{y}_1) = \Phi_1(\mathbf{x}_1, \mathbf{y}_1) + \Phi_2(\mathbf{x}_2, \mathbf{y}_2) + \Phi_3(\mathbf{x}_3, \mathbf{y}_3)$. Each component depends on the corresponding coordinate set, which is expressed in terms of the chosen set of mass-scaled Jacobi coordinates. The transformation equation, Eq. (3), allows us to write the Faddeev equations as a system of differential equations for each $\Phi_i(\mathbf{x}_i, \mathbf{y}_i)$ component in compact form. The components $\Phi_i(\mathbf{x}_i, \mathbf{y}_i)$ satisfy the Faddeev equations [32] that can be written in the coordinate representation as

$$(H_0 + V_i(C_{ik}\mathbf{x}) - E)\Phi_i(\mathbf{x}, \mathbf{y}) = -V_i(C_{ik}\mathbf{x})\sum_{l\neq i}\Phi_l(h_{il}(\mathbf{x}, \mathbf{y})). \quad (4)$$

Here $H_0 = -(\Delta_{\mathbf{x}} + \Delta_{\mathbf{y}})$ is the kinetic energy operator with $\hbar^2 = 1$ and $V_i(\mathbf{x})$ is the interaction potential between the pair of particles (kl), where $k, l \neq i$.

The system of equations, Eq. (4), written for three nonidentical particles can be reduced to a simpler form for a case of two identical particles. The Faddeev equations in configuration space for a three-particle system with two identical particles are given in our previous studies [35–37]. In the case of the ϕNN system, the total wave function of the system is decomposed into the sum of the Faddeev components Φ_1 and Φ_2 corresponding to the $(NN)\phi$ and $(\phi N)N$ types of rearrangements: $\Psi = \Phi_1 + \Phi_2 - P\Phi_2$, where P is the permutation operator for two identical particles. Therefore, the set of the Faddeev equations, Eq. (4), is rewritten as follows [34]:

$$(H_0 + V_{NN} - E)\Phi_1 = -V_{NN}(\Phi_2 - P\Phi_2),$$
$$(H_0 + V_{\phi N} - E)\Phi_2 = -V_{\phi N}(\Phi_1 - P\Phi_2). \quad (5)$$

In Eqs. (5), V_{NN} and $V_{\phi N}$ are the interaction potentials between two nucleons and the ϕ meson and nucleon, respectively. The spin-isospin variables of the system can be represented by the corresponding basis elements. After the separation of the variables, one can define the coordinate part Ψ^R of the wave function $\Psi = \xi_{\mathrm{isospin}} \otimes \eta_{\mathrm{isospin}} \otimes \Psi^R$. The details of our method for the solution of the system of differential equations, Eqs. (5), are given in [35,38,39].

In Ref. [23], the interaction between the ϕ meson and the nucleon is studied based on the (2 + 1)-flavor lattice QCD simulations with nearly physical quark masses. The authors found that the ϕN correlation function is mostly dominated by the elastic scattering states in the $^4S_{3/2}$ channel without significant effects from the two-body $\Lambda K(^2D_{3/2})$ and $\Sigma K(^2D_{3/2})$ and the three-body open channels including

$\phi N \to \Sigma^* K, \Lambda(1405)K \to \Lambda\pi K, \Sigma\pi K$. The fit of the lattice QCD potential by the sum of two Gaussian functions for an attractive short-range part and a two-pion exchange tail at long distances with an overall strength proportional to m_π^{4n} [40] has the following functional form in the ${}^4S_{3/2}$ channel with the maximum spin 3/2 [23]:

$$V^{3/2}_{\phi N}(r) = \sum_{j=1}^{2} a_j \exp\left[-\left(\frac{r}{b_j}\right)^2\right] + a_3 m_\pi^4 F(r, b_3)\left(\frac{e^{-m_\pi r}}{r}\right)^2, \tag{6}$$

with the Argonne-type form factor [41]

$$F(r, b_3) = (1 - e^{-r^2/b_3^2})^2. \tag{7}$$

For comparison, the lattice QCD ϕN potential is also parametrized using three Gaussian functions [23]:

$$V^{3/2}{}_{G\phi N}(r) = \sum_{j=1}^{3} a_j \exp\left[-\left(\frac{r}{b_j}\right)^2\right]. \tag{8}$$

The HAL QCD potential in the ${}^2S_{1/2}$ channel with a minimum spin of 1/2 [27] has a much stronger attractive β-enhanced short-range part and the same two-pion exchange long-range tail as in the ${}^4S_{3/2}$ channel. The real part of the potential in the ${}^2S_{1/2}$ channel reads [27] as follows:

$$V^{1/2}_{\phi N}(r) = \beta\left(a_1 e^{-r^2/b_1^2} + a_2 e^{-r^2/b_2^2}\right) + a_3 m_\pi^4 F(r, b_3)\left(\frac{e^{-m_\pi r}}{r}\right)^2, \tag{9}$$

where the factor $\beta = 6.9^{+0.9}_{-0.5}(\text{stat})^{+0.2}_{-0.1}(\text{syst})$. The other values of the parameters are common in both ${}^4S_{3/2}$ and ${}^2S_{1/2}$ channels [27]. The imaginary part of the ϕN potential is related to the second-order kaon exchange and corresponds to absorption processes. A proportionality coefficient for this part is $\gamma = 0.0^{+0.0}_{-3.6}(\text{stat})^{+0.0}_{-0.18}(\text{syst})$ [27].

We present the results of calculations for the feasibility of expected bound states for ϕN and ϕNN systems. For calculations of the BEs of these systems, we use the HAL QCD ϕN potential in the ${}^4S_{3/2}$ and ${}^2S_{1/2}$ channels with the maximum and minimum spins, respectively. We employ the same NN MT-I-III potential [42,43] as in [3–5,18] for the comparison of the results. The input parameters for potentials are listed in Table I. For comparison, we also perform BE calculations for ϕN and ϕNN systems with a previously suggested Yukawa-type ϕN potential with parameters from [20,22].

The spin configurations of the ϕNN system are illustrated in Fig. 1(a). Here, we present two configurations for the isospin state $T = 0$ which means that the considered system includes the deuteron, d, which corresponds to the $NN(s = 1)$ state. There are two different components of the ϕN potential. For calculations for the $S = 1$ state, we used an averaged over spin variables potential. To acquire the overall ϕN potential, the spin-averaged interaction for the ${}^4S_{3/2}$ and ${}^2S_{1/2}$ channel potentials is defined as [27]

$$\bar{V}_{\phi N} = \frac{1}{3} V^{1/2}_{\phi N} + \frac{2}{3} V^{3/2}_{\phi N}. \tag{10}$$

According to Eq. (10), the configuration $S = 1$ becomes $S = 2$ when components of the ϕN potential are equal. For example, it can be the 3/2 ϕN component. The configurations for the $S = 0$ and $S = 1$, $T = 1$ states are presented in Fig. 1(b).

First, let us consider the ϕN system. Results of calculations for the two-body binding energy, B_2, scattering length, $a_{\phi N}$, and effective radius, $r_{\phi N}$, for ϕN are presented in Table II for the ${}^4S_{3/2}$ and ${}^2S_{1/2}$ channels. Although the

TABLE I. The parameters for the ϕN potential in the ${}^4S_{3/2}$ channel with statistical errors are quoted in parentheses. For the $a_3 m_\pi^{4n}$ column, $n = 1$ and $n = 0$ for $V^{3/2}_{\phi N}$ and $V^{3/2}_{G\phi N}$, respectively [23]. The attractive β-enhanced short-range part and the two-pion exchange long-range tail for the ϕN potential (9) in the ${}^2S_{1/2}$ channel have the same parameters as in the ${}^4S_{3/2}$ channel. The parameters for the singlet and triplet NN interactions for MT potential [42,43] are shown.

ϕN potential in the ${}^4S_{3/2}$ channel [23] and in the ${}^2S_{1/2}$ channel [27]						
	a_1 (MeV)	a_2 (MeV)	$a_3 m_\pi^{4n}$ (MeV fm^{2n})	b_1 (fm)	b_2 (fm)	b_3 (fm)
$V^{3/2}_{\phi N}$	−371(27)	−119(39)	−97(14)	0.13(1)	0.30(5)	0.63(4)
$V^{3/2}{}_{G\phi N}$	−371(19)	−50(35)	−31(53)	0.15(3)	0.66(61)	1.09(41)

Singlet ${}^1S_0$ and triplet ${}^3S_1$ NN potential [42,43]				
I, J	V_r (MeV)	V_a (MeV)	μ_1 (fm^{-1})	μ_2 (fm^{-1})
1,0	−521.959	1438.72	1.55	3.11
0,1	−626.885	1438.72	1.55	3.11

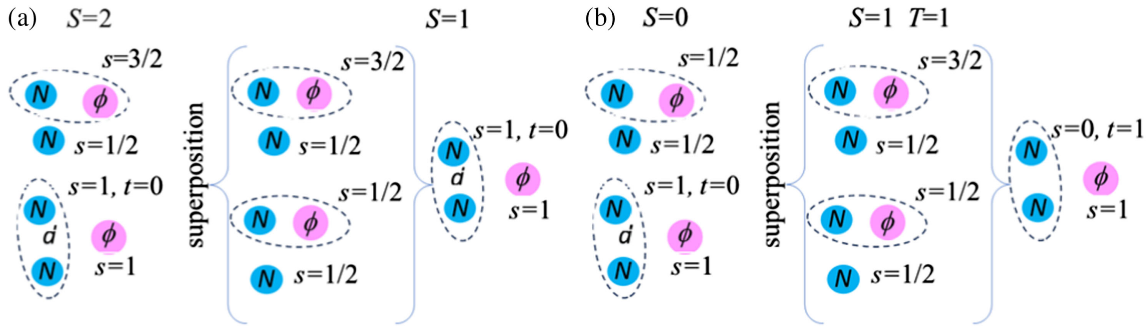

FIG. 1. Spin-isospin configurations in the ϕNN system: (a) $S = 2$, $T = 0$ and $S = 1$, $T = 0$; (b) $S = 0$, $T = 0$ and $S = 1$, $T = 1$. The channels $(\phi N)N$ and $\phi(NN)$ are shown.

HAL QCD ϕN potential in the $^4S_{3/2}$ channel is found to be attractive for all distances and reproduces a two-pion exchange tail at long distances, no bound ϕN state is found with this interaction. The ϕN system is strongly bound with the HAL QCD potential in the $^2S_{1/2}$ channel with a reasonable scattering length when the short-range attractive part is enchanted with the factor $\beta = 6.9$ suggested in [27]. Let us mention that the $^2S_{1/2}$ state binding energy is very sensitive to the variation of β within the statistical and systematic error margins reported in [27].

In Table II, we present the numerical results for the ϕNN system obtained with the HAL QCD interactions and a Yukawa-type potential with parametrizations from [20,22]. The calculations of the BEs with the Yukawa-type potential motivated by the QCD van der Waals attractive force mediated by multigluon exchanges led to the same results as previously reported in [3–5]. Our calculations indicate that neither the HAL QCD interaction in the $^4S_{4/2}$ channel nor the Yukawa-type interaction with parameters [20] supports the existence of the $S = 2$ bound state. Thus, the HAL QCD interaction in the $^4S_{3/2}$ channel with the maximum spin $3/2$ suggests no bound state for the $^3_\phi\mathrm{H}$ hypernucleus, in contrast to the binding energy range reported in [18], which is 6.7–7.3 MeV. Results obtained for the BEs of ϕNN of 22.42 and 38.04 MeV ($t = 0$) in the framework of our approach utilizing the Yukawa-type ϕN potential [22] and the singlet and triplet spin NN interaction [42], respectively, confirm calculations [3,5] and are in good agreement within ± 1.5 MeV.

Based on our calculations, the HAL QCD interaction in the $^4S_{3/2}$ channel does not provide enough attractiveness to bind a ϕ meson onto a nucleon or deuteron to form a bound

TABLE II. The scattering lengths $a^{3/2}_{\phi N}$ and $a^{1/2}_{\phi N}$, effective radii $r^{3/2}_{\phi N}$ and $r^{1/2}_{\phi N}$ in fm, and binding energies $B^{3/2}_2$ and $B^{1/2}_2$ in MeV for ϕN in the $s = 3/2$ and $s = 1/2$ spin states, respectively, and B_3 in MeV is the binding energy of ϕd or ϕNN. β is the scaling factor for the attractive short-range part of $V^{1/2}_{\phi N}$ potential [see Eq. (9)]. "UNB" indicates that no bound state is found. The bound energy $B^{\phi NN}_3$ of the ϕNN system ($S = 1$, $T = 1$) is shown in parentheses.

ϕN potential	β	$a^{3/2}_{\phi N}$	$a^{1/2}_{\phi N}$	$r^{3/2}_{\phi N}$	$r^{1/2}_{\phi N}$	$B^{3/2}_2$	$B^{1/2}_2$	$B^{\phi NN}_3(S=2)$	$B^{\phi NN}_3(S=1)$	$B^{\phi NN}_3(S=0)$
$-A\frac{e^{-\alpha r}}{r}$ [20]	···	−1.13	···	36.4	···	UNB	···	UNB	···	···
$-A\frac{e^{-\alpha r}}{r}$ [22]	···	2.38	···	0.17	···	9.40	···	38.04	··· (22.42)	···
$V^{3/2}_{\phi N}$ ($^4S_{3/2}$) [23]	···	−1.37	···	2.42	···	UNB	···	UNB	···	···
$V^{3/2}_{G\phi N}$ ($^4S_{3/2}$) [23]	···	−1.36	···	2.04	···	UNB	···	UNB	···	···
$(\frac{1}{3}V^{1/2}_{\phi N} + \frac{2}{3}V^{3/2}_{\phi N})$ [27]	6.9 [27]	−1.37	1.5	2.24	~0	UNB	27.7	···	14.90 (5.47)	···
$V^{1/2}_{\phi N}$ [27]	6.9 [27]	···	1.5	···	~0	···	27.7	···	···	64.13
$(\frac{1}{3}V^{1/2}_{\phi N} + \frac{2}{3}V^{3/2}_{\phi N})$	5.0	−1.37	8	2.24	0.7	UNB	0.7	···	11.37	···
$V^{1/2}_{\phi N}$	5.0	···	8	···	0.7	···	0.7	···	···	18.56
$(\frac{1}{3}V^{1/2}_{\phi N} + \frac{2}{3}V^{3/2}_{\phi N})$	6.0	−1.37	2.5	2.24	0.3	UNB	8.81	···	13.09	···
$V^{1/2}_{\phi N}$	6.0	···	2.5	···	0.3	···	8.81	···	···	37.11
$(\frac{1}{3}V^{1/2}_{\phi N} + \frac{2}{3}V^{3/2}_{\phi N})$	6.9	−1.37	1.5	2.24	~0	UNB	27.7	···	14.90	···
$V^{1/2}_{\phi N}$	6.9	···	1.5	···	~0	···	27.7	···	···	64.13
$(\frac{1}{3}V^{1/2}_{\phi N} + \frac{2}{3}V^{3/2}_{\phi N})$	8.0	−1.37	1	2.24	~0	UNB	69.85	···	17.52	···
$V^{1/2}_{\phi N}$	8.0	···	1	···	~0	···	69.85	···	···	113.7

state. Conversely, employing the HAL QCD ϕN interaction in the ${}^{2}S_{1/2}$ channel with minimal spin $1/2$ results in bound ϕNN, although the BE is highly sensitive to the variation of the factor β and the ϕNN system is extremely strongly bound in the state $S = 0$. Employing the spin-averaged potential Eq. (10), we consider both the HAL QCD potentials in the ${}^{2}S_{1/2}$ and ${}^{4}S_{3/2}$ channels when the factor $\beta = 6.9$. This leads to the numerical value of the binding energy 14.9 MeV for the ${}^{3}_{\phi}$H hypernucleus in the spin state $S = 1$. Changing the β factor to $\beta = 6.0$, we obtained for the ϕNN BE 13.09 MeV, albeit with a larger scattering length. It is important to note that varying the β factor within the margin of the error leads to larger and less realistic BEs, especially for the $S = 0$ state as shown in Table II.

In conclusion, we employ the HAL QCD ϕN potential in the ${}^{2}S_{1/2}$ and ${}^{4}S_{3/2}$ channels with the maximum and minimum spin, respectively, in the framework of Faddeev equations in configuration space to evaluate the binding energy of the ϕNN system. The HAL QCD ϕN potential in the ${}^{4}S_{3/2}$ channel does not support bound states for either ϕN or ϕNN, although it exhibits attraction. Conversely, employing the HAL QCD ϕN potential in the ${}^{2}S_{1/2}$ channel yields bound states for both ϕN and ϕNN. The binding energies of these systems are notably sensitive to variations in the enhancement of the short-range attractive part, parametrized by the factor β. Considering both potentials, we find binding energies of 5.47 and 14.9 MeV for the states $S = 1$, $T = 0$ and $S = 1$, $T = 1$ (with singlet and triplet components of the NN MT I-III potential), respectively, when $\beta = 6.9$. Our calculations confirm the existence of $S = 2$ bound states for the ϕNN system previously predicted within the Faddeev equations in the differential form [3] and theoretical formalism [5] where the ϕN potential was utilized [22]. The presented analysis demonstrates the possible existence of the ${}^{3}_{\phi}$H hypernucleus.

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