Celestial eikonal amplitudes in the near-horizon region

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We construct a celestial conformal field theory on the horizon corresponding to a nonperturbative eikonal scattering amplitude involving two massless scalars mediated by soft gravitons in the near-horizon region of a large eternal Schwarzschild black hole. From the *known* two-dimensional near-horizon scattering amplitude computed within the effective field theory framework, we first construct a four-dimensional amplitude involving two external s-wave legs in a flat spacetime frame around the bifurcation sphere strictly in a small angle approximation limit by resumming over the spherical harmonics. While the kinematics of external particles in this frame at leading order are analogous to a Minkowski spacetime, the eikonal amplitude differs from those about flat spacetime due to the near-horizon scattering potential. We derive a celestial correlator following a Mellin transform that provides an all loop order result, with a universal leading UV soft scaling behavior of the conformally invariant cross-ratio, and an IR pole for the scaling dimension at each loop order. We argue these properties manifest soft graviton exchanges in the near-horizon region and, consequently, the soft UV behavior of the amplitude.

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I. INTRODUCTION

Aspects of a holographic correspondence relating scattering amplitudes with correlation functions of a dual conformal field theory have been recently realized on the celestial sphere at null infinity of asymptotically flat spacetimes [1–37]. This correspondence follows from the isomorphism between the four-dimensional (4D) Lorentz group and those of the Mobius group for conformal transformations on the two-dimensional (2D) celestial sphere [38–43]. In the case of asymptotic plane wave states, the boundary conformal primary wave function generally follows from the Fourier transform of the bulk-to-boundary propagator defined on hyperbolic foliations of Minkowski spacetime. The massless limit is realized as a Mellin transform of the plane wave, resulting in the energy dependence of bulk fields being traded for a scaling dimension dependence in the corresponding boundary operators. As a consequence, the resulting correlation functions of celestial conformal operators are manifestly $SL(2,\mathbb{C})$ invariant observables in a boost eigenbasis, which have been proposed as duals of S-matrix elements in an energy-momentum eigenbasis [2–4].

Celestial conformal field theories (CCFT) have been investigated primarily from perturbative flat-spacetime amplitudes and are associated with infinite-dimensional asymptotic symmetry algebras [22–25]. Soft theorems for scattering amplitudes are realized through conformal soft theorems in CCFT [6,9,10] that constrain the operator product expansions of celestial correlation functions [11]. A remarkable property of CCFTs, on account of their involvement of boost scattering states, is that they invoke the entire energy spectrum of a theory and thus access their infrared (IR) and ultraviolet (UV) properties [20,33]. CCFTs possess several properties similar to conformal field theories, including a conformal block expansion [23] and state-operator correspondence [29]. However, due to their correspondence with scattering amplitudes on flat spacetime, they differ from conformal field theories in certain respects. This includes the presence of complex scaling dimensions for normalizable states and a delta function over the 2D cross ratio in CCFT correlation functions, with the latter due to the translation invariance of scattering amplitudes in momentum space [20,23,27,28]. More recent developments include investigations on CCFTs to leading loop orders [16,37,44], leading backreaction effects [32], and their formulation on nontrivial asymptotically flat spacetimes [18,19]. The celestial description of nonperturbative eikonal amplitudes was also initiated in [31], which

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further demonstrates a correspondence with eikonal amplitudes in AdS/CFT. The relationship between CCFT and CFT_2 correlation functions have also been explored in [34,45–47]. We also note other potentially complementary approaches to holographic descriptions of scattering amplitudes on flat spacetime, which includes Carrolian holography defined along the codimension-one null boundary of flat spacetime [30,48–52], and derivations from the large anti–de Sitter (AdS) radius limit in AdS/CFT [53–58]

In this paper, we extend the analysis of [31] on eikonal amplitudes in flat spacetime to ones in the near-horizon region of a Schwarzschild black hole, with an impact parameter $x_\perp$ comparable to the Schwarzschild radius. This amplitude has been investigated in detail over recent years [59–63]. The motivation for such an amplitude can be traced back to eikonal amplitudes defined on flat spacetimes, which address trans-Planckian scattering processes with center of mass energies far greater than the Planck mass, i.e., $\sqrt{s} \gg M_{\mathrm{Pl}}$ and correspondingly large impact parameters, i.e., $x_\perp \gg G_N\sqrt{s}$ with G_N Newton's constant. On the other hand, flat spacetime eikonal amplitudes are expected to diverge in the regime of $x_\perp \sim G_N\sqrt{s}$ due to strong gravitational effects associated with the formation of a Schwarzschild black hole with a radius of $G_N\sqrt{s}$. This can be interpreted as an IR divergence associated with absorptive soft graviton exchanges, thereby reflecting the need for soft graviton bremsstrahlung [64] to yield IR-finite results in accordance with Weinberg's approach to IR divergences [65]. These properties motivate a possible eikonal description of the scattering process in the near-horizon region with a different kinematic regime, i.e., $x_\perp \sim G_N M$ with M the mass of a background black hole, as considered in [60–63]. The resultant near-horizon eikonal amplitude then provides a description in the case of impact parameters comparable to a Schwarzschild radius, with the eikonal phase dominated by soft graviton exchanges.

This provides an interesting setting for a CCFT investigation on two grounds. First, since the near-horizon region of a Schwarzschild black hole around the bifurcation sphere can be well approximated by a flat spacetime frame in the small angle approximation, translation symmetry of massless particles can be restored in this frame. High energy massless states in an eikonal scattering process near the horizon can thus be investigated using known CCFT approaches on flat spacetime. The underlying global conformal symmetries of CCFTs originate from the isometries of the flat spacetime. This then generalizes the known CCFTs originally defined in the full flat spacetime to those defined only in the near-horizon region of a Schwarzschild black hole. Second, due to the gravitational effects of the background black hole that manifest in the phase of near-horizon eikonal amplitude, we expect the resulting CCFT to be quite different from those for scattering on flat spacetime. The detailed dynamics for the near-horizon eikonal scattering have been studied in the aforementioned works [60–63].

In a boost basis, the corresponding CCFT amplitudes can be expressed as the product of a universal conformal block for external state conformal primaries with large conformal weights (inclusive of intermediate exchanges) and a conformally invariant function of cross-ratio $z = \frac{-t}{s} \ll 1$. The former provides a universal kinematic factor, while the latter captures the underlying dynamics of CCFTs or its parent theory in the momentum basis.

As we will show, the resultant eikonal phase obtained in [60–63] will manifest a soft UV behavior in the corresponding CCFT, suggesting a possible UV completion in the near-horizon regime. We find a closed-form result for the near-horizon CCFT amplitudes, which, to all loop orders, has a leading scaling behavior of z^{-1}. This can be noted as being softer than those for celestial eikonal amplitudes on flat spacetime with massive mediating particles, which at tree-level scales like $(\sqrt{z})^{-\beta}$ [31]. Here, $\beta = \sum_{i=1}^{4} \Delta_i - 4 \gg 1$, with Δ_i being the scaling dimensions of the external boosted eigenstates. More significantly, the near-horizon CCFT has poles at $\beta = -2n$ with $n \in \mathbf{N}$ labeling the loop order. Since the loop order corresponds to the number of exchanged soft gravitons in the ladder diagrams, this implies these poles are IR divergences due to the exchange of soft gravitons in the eikonal limit. Interestingly, the near-horizon CCFT is free from any poles for $\mathrm{Re}\beta > 0$, as might be expected from a generic UV complete field theory, with an expansion for the amplitude $\sum_{n=0}^{\infty} a_n^{\mathrm{UV}} \omega^{-2n}$. These results are further consistent with the observation of [20,33] that CCFT amplitudes for UV soft theories, such as those with a stringy Hagedorn spectrum, only have negative integer poles and correspond to the production of microscopic black holes [66–68]. They may likewise be realized in a theory with only an IR soft expansion, with the amplitude going as $\sum_{n=0}^{\infty} a_n^{\mathrm{IR}} \omega^{2n}$. Thus, our results imply that the $\beta = -2n$ poles are more or less universal for strong gravity regimes, which can manifest either through black hole production or the existence of an event horizon.

The rest of our paper is organized as follows. In the next section, we review the derivation of the 2D near-horizon eikonal amplitude from a perturbative analysis on the Schwarzschild background through a spherical harmonic decomposition of the fields. In Sec. III, we proceed to derive the 4D near-horizon celestial eikonal amplitude. We first uplift the 2D amplitude to a 4D partial sum amplitude in a near-horizon region about the bifurcation sphere. The spacetime considered is a nearly flat region that arises in a small angle and large black hole limit of the near-horizon background. We then carry out the partial resummation over small angles to derive the eikonal phase. As this amplitude involves massless external states, following the prescription in [31], we derive the near-horizon celestial eikonal amplitude from the Mellin transform. In Sec. IV, we study properties of the near-horizon celestial eikonal amplitude. This involves its exact evaluation to all loop orders. The

CCFT result further provide IR poles and an overall z dependence and we discuss their physical implications. We conclude with a discussion of our results and future directions in Sec. V.

II. REVIEW OF 2D BLACK HOLE EIKONAL SCATTERING AMPLITUDE

In this section, we provide a *detailed review* of the near-horizon eikonal scattering amplitude considered in [60–63]. The conventional eikonal limit of trans-Planckian 2-2 scattering in flat spacetime, with a center-of-mass energy $\sqrt{s}$ far larger than the Planckian mass M_{PL} so that graviton exchanges dominate, requires a large impact parameter to suppress the transverse momentum transfer $q_\perp$. Additionally, to avoid divergent results caused by gravitational collapse near the scattering center, the impact parameter $x_\perp \sim \mathcal{O}(\hbar/q_\perp)$ should also be far larger than the Schwarzschild radius associated with the center-of-mass energy, i.e., $x_\perp \gg G_N\sqrt{s}$ [69,70]. The resultant eikonal amplitude (for massless particles) is [71–74]

$$i\mathcal{M} = \frac{i\kappa^2 s^2}{q_\perp^2}\frac{\Gamma(1-iG_N s)}{\Gamma(1+iG_N s)}\left(\frac{4\mu^2}{q_\perp^2}\right)^{-iG_N s} \tag{1}$$

with μ denoting an infrared cutoff. This amplitude has a semiclassical interpretation as a 1-1 scattering of an ultra-high energy massless particle against a null-like shockwave background, which incorporates the backreaction [71]. This is consistent with the expectation of an eikonal limit as a resummation over ladder graviton exchanges in a coherent background. Decomposing this eikonal amplitude in a partial wave basis yields a unitary S matrix for each mode represented by an eikonal phase,

$$\delta_\ell(s) = \frac{s}{2}\log\frac{\ell(\ell+1)}{s}. \tag{2}$$

This phase encodes the peculiar dynamics from the dominant soft graviton contributions in the ladder diagrams. Later, we will compare this phase to the one from eikonal scattering in the near-horizon region.

The metric of the near-horizon region of the Schwarzschild black hole is approximately a flat metric with an implicit horizon scale in relation to the Rindler metric. This raises the possibility of formulating the eikonal scattering in the near-horizon region in a similar fashion to the approach on flat spacetime. Indeed, this idea had been proposed long ago [75,76], and has been recently refined with further details [60–63,77]. Due to being restricted to the near-horizon region, the kinematic constraints for eikonal scattering are quite different from those on flat spacetime. This especially concerns the impact parameter, which is restricted to be $\ell_{PL} \ll x_\perp \approx R$, where ℓ_{PL} is the Planck length and $R = 2G_N M$ is the Schwarzschild radius. As shown in [60,61], eikonal scattering (small angle scattering) in this regime requires $\sqrt{s} \gg \gamma M_{PL}$ with M_{PL} the Planck mass and $\gamma = \frac{M_{PL}}{M}$ an emerging dimensionless coupling between matter and gravitons of the effective theory that results from integrating out the transverse directions. Due to the smallness of γ for a typical macroscopic black hole, the new constraint on s implies that the eikonal scattering can be non-Planckian in the near-horizon region. Consequently, this enables us to circumvent the breakdown of conventional eikonal amplitudes when dealing with scattering at small impact parameters.

The Schwarzschild spacetime has the following metric in static coordinates,

$$ds^2_{\text{Schwarzschild}} = -\left(1-\frac{R}{r}\right)dt^2 + \left(1-\frac{R}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2, \tag{3}$$

where $R = 2G_N M$ is the Schwarzschild radius and M is the black hole mass. To consider the near-horizon geometry, we perform a transformation to Kruskal coordinates, which is regular at the horizon and describes the maximally extended spacetime. This can be derived from the following definitions for x^- and x^+,

$$\begin{aligned} x^- x^+ &= 2R^2\left(1-\frac{r}{R}\right)e^{\frac{r}{R}-1}; \\ \frac{x^-}{x^+} &= e^{\frac{t}{2R}} \quad \text{regions I and III}, \\ &= -e^{\frac{t}{2R}} \quad \text{regions II and IV}, \end{aligned} \tag{4}$$

with the event horizons located at $x^- x^+ = 0$ (Fig. 1).

This leads to Eq. (3) taking the form

$$\begin{aligned} ds^2_{\mathrm{NH}} = g_{\mu\nu}dx^\mu dx^\nu &= -2A(x^-,x^+)dx^- dx^+ \\ &+ r^2(x^-,x^+)d\Omega_2^2, \end{aligned} \tag{5}$$

with

$$A(x^-,x^+) = \frac{R}{r(x^-,x^+)}e^{1-\frac{r(x^-,x^+)}{R}}. \tag{6}$$

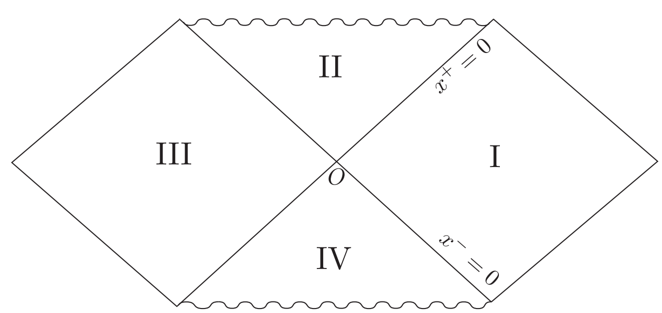

FIG. 1. Kruskal spacetime with regions I, …IV; bifurcation sphere O and $x^\pm = 0$ lines indicated.

We will consider the theory of linearized Einstein gravity minimally coupled with a massless scalar ψ in this background,

$$S[h_{\mu\nu},\psi] = \int d^4x\sqrt{-g}\left[\frac{1}{4}R^{(1)}[h_{\mu\nu};g_{\mu\nu}] + \frac{1}{2}\psi\Box\psi + \frac{1}{2}\kappa h^{\mu\nu}T_{\mu\nu}\right] \tag{7}$$

where $\frac{1}{4}\sqrt{-g}R^{(1)}[h_{\mu\nu};g_{\mu\nu}]$ is the $h_{\mu\nu}$-quadratic part of $\frac{1}{2\kappa^2}\sqrt{-(g+\kappa h)}R[g_{\mu\nu}+\kappa h_{\mu\nu}]$, $\kappa=\sqrt{8\pi G_N}$, and $T_{\mu\nu}$ is the stress tensor of the scalar ψ

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\psi\partial_\sigma\psi. \tag{8}$$

Exploiting the background spherical symmetry, one can decompose the metric and scalar fields in a spherical harmonic basis Y_ℓ^m, i.e.,

$$h_{\mu\nu} = \sum_{\ell,m} h^{\rm odd}_{\ell m,\mu\nu}Y_\ell^m + \sum_{\ell,m} h^{\rm even}_{\ell m,\mu\nu}Y_\ell^m,$$
$$\psi = \sum_{\ell,m}\psi_{\ell m}Y_\ell^m \tag{9}$$

with the additional choice of the usual Regge-Wheeler gauge,[1]

$$h^{\rm odd}_{aA} = -h_a\epsilon_A{}^B\partial_B Y_\ell^m, \qquad h^{\rm even}_{ab} = H_{ab}Y_\ell^m,$$
$$h^{\rm even}_{AB} = K\gamma_{AB}Y_\ell^m,$$

where γ_{AB} and ϵ_{AB} are, respectively, the metric and Levi-Civita tensor on the 2-sphere, while a, b are indices for the longitudinal null coordinates $x^\pm$. Spherical symmetry further ensures the decoupling of the even and odd modes. Moreover, the even parity nature of the scalar field yields no coupling to the odd modes from the interaction vertex. Therefore, only even parity graviton modes K and H_{ab} will be involved in the scattering of the massless scalar in the reduced theory.

Since the longitudinal part of the near-horizon metric is conformal to a flat metric, we can perform the following Weyl transformation and field redefinitions to yield canonical kinetic terms in the reduced theory

$$g_{ab}\to A(x^-,x^+)\eta_{ab}, \qquad H_{ab}\to \frac{1}{r}A(x^-,x^+)\mathbf{h}_{ab},$$
$$K\to\frac{1}{r}\mathbf{K}, \qquad \psi\to\frac{1}{r}\phi. \tag{10}$$

[1]We will suppress ℓ, m in scalar and graviton modes from this point onwards.

This can be used to obtain a 2D effective theory by integrating out the transverse degrees of freedom. By introducing a single traceless tensor mode, $\tilde{\mathbf{h}}^{ab} = \mathbf{h}^{ab} - \eta^{ab}(\frac{1}{2}\mathbf{h}+\mathbf{K})$, for the 3-vertex coupling to two scalars, and carrying out a field redefinition $\tilde{\mathbf{K}} = \mathbf{h} + \frac{2R^2}{\ell(\ell+1)+2}(\eta^{ab}\partial_a\partial_b - \frac{1}{R^2}\ell(\ell+1))\mathbf{K}$, we can also remove the mixed contribution between $\mathbf{K}$ and $\mathbf{h}^{ab}$. The corresponding graviton propagators are complicated due to the potentials arising from the Weyl scaling and field redefinitions Eq. (10). However, if we focus on the near-horizon region so that the metric becomes that of flat spacetime

$$A(x^-,x^+)\approx 1, \quad \text{if } r = R + \mathcal{O}(r-R), \tag{11}$$

the resulting 2D effective theory has considerably simpler properties.

Upon Fourier transforming all the fields and taking the $r\to R$ limit of Eq. (11), the 2D effective action takes on the following simple form [60–63]:

$$S[\tilde{\mathbf{h}}^{ab},\tilde{\mathbf{K}},\phi] = \frac{1}{4}\int d^2k(\tilde{\mathbf{h}}^{ab}\mathbf{P}^{-1}_{abcd}(k)\tilde{\mathbf{h}}^{cd} + \tilde{\mathbf{K}}\mathbf{P}^{-1}_{\mathbf{K}}\tilde{\mathbf{K}}) + \frac{1}{2}\int d^2p\phi\mathbf{P}^{-1}_\phi(p)\phi + \gamma\int d^2\Pi\tilde{\mathbf{h}}^{ab}(k)p_{1a}p_{2b}\phi_0(p_1)\phi(p_2), \tag{12}$$

where $d^2\Pi$ is a shorthand for $d^2kd^2p_1d^2p_2\delta^{(2)}(k+p_1+p_2)$, and the dimensionless coupling for the 3-vertex is given by

$$\gamma \coloneqq \frac{\kappa}{R} = \frac{M_{PL}}{M}. \tag{13}$$

The propagators have the expressions

$$\mathbf{P}_\phi(p) = \frac{1}{p^2+\mu^2-i\epsilon}, \qquad \mathbf{P}_\mathbf{K} = \frac{4R^2}{\ell(\ell+1)+2},$$
$$\mathbf{P}^{abcd}(k) = \mathbf{P}^{abcd}_{\rm soft} + \mathbf{P}^{abcd}_{\rm hard}(k), \tag{14}$$

where we decompose the tensor mode propagator into its soft (k-independent) and hard (k-dependent) parts as follows [63]:

$$\mathbf{P}^{abcd}_{\rm soft} = \frac{R^2}{\ell(\ell+1)+2}(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}), \tag{15}$$

$$\mathbf{P}^{abcd}_{\rm hard}(k) = -\frac{\ell(\ell+1)+2}{\ell(\ell+1)-2}\frac{1}{k^2+\mu^2}(\eta^{ab}+k^{ab})(\eta^{cd}+k^{cd}), \tag{16}$$

with

$$k^{ab} := \frac{2R^2}{\ell(\ell+1)+2}\left(k^a k^b - \frac{1}{2}k^2\eta^{ab}\right). \tag{17}$$

The soft graviton exchange associated with $\mathbf{P}^{abcd}_{\text{soft}}$ will give a leading contribution to scattering amplitudes. While integrating out the transverse part by using the orthogonality relations between spherical harmonics, all the fields in the effective field theory acquire an effective mass

$$\mu^2 := \frac{\ell(\ell+1)+1}{R^2}, \tag{18}$$

which can be thought of as an infrared regulator.

We summarize an important assumption used in the derivation of the interaction vertex in Eq. (12), namely the absence of partial wave mixing. Apart from the interaction vertex, all other terms in the effective action up to quadratic order involve decoupled partial waves due to the spherical symmetry of the background. To preserve this property for the interaction vertex, it was argued in [60,61] that scattering processes that do not distribute angular momenta across the external legs through Clebsch-Gordan coefficients are those that preserve the background spherical symmetry. This can be satisfied by fixing one of the scalar particles in the interaction 3-vertex, ϕ_0, to have no angular momentum [as in the last line of Eq. (12)]. We now elaborate more on this point. The reduced action for the interaction vertex between the graviton and external scalars will, in general, involve exchanges of angular momenta so that it takes the following form:

$$S_{\text{vertex}} = \frac{\gamma}{2}\sum_{\ell,m}\sum_{\ell_1,m_1}\sum_{\ell_2,m_2}\int d\Omega Y^m_\ell(\Omega)Y^{m_1}_{\ell_1}(\Omega)Y^{m_2}_{\ell_2}(\Omega) \int d^2x h^{ab}_{\ell m}\partial_a\phi_{\ell_1 m_1}\partial_b\phi_{\ell_2 m_2}, \tag{19}$$

which evaluates to involve the sum over Clebsch-Gordan (CG) coefficients.[2] As a result, the general interaction vertex Eq. (19) involves partial wave mode mixings. In [60,61], it was argued that such mode mixings are associated with large transverse momenta exchanges that introduce nonspherical corrections of the background geometry. To suppress such nonspherical backreaction as the semiclassical approximation requires, we must lift the mixing of partial waves of different ℓ and m in Eq. (19). This can be implemented by fixing one of the external particles to be a s wave ($\ell_2 = 0$ or $\ell_1 = 0$). This consequently simplifies the vertex action Eq. (19) to

$$S_{\text{vertex}} = \gamma\sum_{\ell,m}\int d^2x h^{ab}_{\ell m}\partial_a\phi_{\ell m}\partial_b\phi_0, \tag{21}$$

using $\int d\Omega Y^m_\ell(\Omega)Y^{m_1}_{\ell_1}(\Omega) = \delta_{\ell\ell_1}\delta_{mm_1}$, with the overall factor of 2 accounting for either scalar particle being considered in the $\ell = 0$ state. The Fourier transform of Eq. (21) is what appears as the interaction term in Eq. (12). The above assumption of no partial wave mixing would always hold at tree level. However in general partial wave mixing will provide subleading eikonal corrections from internal exchanges of the eikonal amplitude upon summing over the Clebsch-Gordon coefficients. While these corrections are beyond the scope of our paper, this is an important problem that warrants further investigation.

We also note that transverse exchanges are realized through the ℓ dependent effective mass term in Eq. (18). This will have a role in the description of external states and the eikonal approximation for the reduced theory in the following. From the effective theory in Eq. (12), we can obtain the Feynman rules to compute the amplitudes for the soft/hard graviton exchanges in the 2-2 scattering. In the vanishing effective mass limit, the external states are massless scalar particles described by longitudinal plane waves, with incoming momenta $p_1 = (p_{1+}, 0)$ and $p_2 = (0, p_{2-})$, which define the Mandelstam $\mathbf{s}$ (center-of-mass energy squared) in terms of 2D momenta as

$$\mathbf{s} = -(p_1 + p_2)^2 = 2p_{1+}p_{2-}. \tag{22}$$

Since only the tensor mode is coupled to the scalars via the interaction vertex, we just need to consider 2-2 scattering amplitude involving the exchange of tensor modes associated with graviton propagators $\mathbf{P}^{abcd}_{\text{hard}}$ and $\mathbf{P}^{abcd}_{\text{soft}}$. We denote the corresponding amplitudes as M_{hard} and M_{soft}. Using the symmetry property: $k^{ab} = k^{ba}$ and $k^{ab}p_{1a}p_{2b} = 0$, the hard graviton exchanges contribute to

$$M_{\text{hard}} \propto \frac{\gamma^2\mathbf{s}^2}{\mathbf{s}+\mu^2}, \tag{23}$$

while from the soft graviton exchange one can obtain

$$M_{\text{soft}} = (i\gamma p_{1a}p_{1b})(2\mathbf{P}^{abcd}_{\text{soft}})(i\gamma p_{2c}p_{2d}) = \frac{\gamma^2R^2\mathbf{s}^2}{\ell^2+\ell+2}. \tag{24}$$

Note that this 2D soft amplitude is suppressed for large ℓ, contrary to the large ℓ dominance of 4D eikonal amplitude in flat space. Two important properties can be inferred from the above results. Due to the effective mass involving the Schwarzschild radius, we have a modified regime for eikonal scattering in the near-horizon region

[2] The CG coupling is related to the CG coefficients $\langle \ell_1, \ell_2; m_1, m_2 | \ell_1, \ell_2; \ell, m\rangle$ via [78]

$$\int d\Omega Y^m_\ell(\Omega)Y^{m_1}_{\ell_1}(\Omega)Y^{m_2}_{\ell_2}(\Omega) = \sqrt{\frac{(2\ell_1+1)(2\ell_1+1)}{4\pi(2\ell+1)}} \langle \ell_1, \ell_2; 0, 0 | \ell_1, \ell_2; \ell; 0\rangle\langle \ell_1, \ell_2; m_1, m_2 | \ell_1, \ell_2; \ell, m\rangle. \tag{20}$$

$$\mathbf{s} \gg \mu^2 \quad \text{or} \quad \sqrt{\mathbf{s}} \gg \gamma M_{PL}. \tag{25}$$

In addition, in the large $\mathbf{s}$ limit we always have

$$\frac{M_{\text{hard}}}{M_{\text{soft}}} \sim \mathcal{O}(\mathbf{s}^{-1}). \tag{26}$$

Thus, M_{hard} is a subleading contribution to M_{soft} in the large $\mathbf{s}$ limit. As the soft graviton exchange dominates to all loop orders, one can resum the corresponding ladder diagrams to derive the leading contribution to the near-horizon eikonal amplitude [61]

$$iA^{\ell}_{\text{NH eikonal}}(\mathbf{s}) = 2\mathbf{s}(e^{i\chi_\ell(\mathbf{s})} - 1), \tag{27}$$

where the associated eikonal phase is

$$\chi_\ell(\mathbf{s}) = \frac{\kappa^2 \mathbf{s}}{\ell^2 + \ell + 2}. \tag{28}$$

with $\kappa = \gamma R$. The $\mathbf{s}$ and ℓ dependences of this phase differ from the ones in eikonal amplitudes on flat spacetimes Eq. (2). In the following section, we generalize the eikonal amplitude in Eq. (27) to a 4D amplitude *within a small angle approximation near the bifurcation sphere* and subsequently provide its CCFT description in a boost eigenbasis.

III. CONSTRUCTION OF NEAR-HORIZON CELESTIAL EIKONAL AMPLITUDE

We first provide detailed arguments for uplifting the 2D *black hole eikonal amplitude* to a 4D partial wave amplitude in a near-horizon region considered for a large black hole in the small angle approximation. We then accordingly resum the partial wave result Eq. (27) using known techniques in the small angle approximation about flat spacetimes [79,80] to derive an amplitude defined in terms of four-momenta and the impact parameter. The 4D eikonal amplitude which follows from the partial wave amplitude in the small scattering angle limit is hence only defined in a near-horizon Minkowski frame around the bifurcation sphere of the maximally extended Schwarzschild spacetime, which will be shown to be consistent with approximations used in the derivation of the 2D near-horizon eikonal amplitude reviewed in the previous section. We lastly perform the Mellin transform on this momentum space amplitude to derive a celestial correlator on the horizon.

A. Uplifting the partial wave eikonal amplitude to four dimensions

To uplift Eq. (27) to a 4D amplitude two key issues need to be addressed. The first concerns the kinematic constraint for the "eikonal limit" in the effective 2D theory, which will differ in a 4D spacetime. Thus, it is *a priori* unclear if one can promote the 2D black hole eikonal amplitude to a 4D one, which can also allow for a CCFT description. For general amplitudes, such kinematic lifting could be difficult to realize. Thus, we need to consider a particular class of amplitudes for our purpose. On a related note, we would also need to address the status of momentum conservation for scattering involving 4D momenta since translation is, in general, broken on black hole spacetimes. The second issue concerns the possible mixing between partial wave modes due to introducing transverse exchanges. In the presence of partial wave mode mixing, the resummation of eikonal amplitudes will need to generalize the contribution from the interaction vertex to the eikonal phase in Eq. (27). Since both issues are closely associated with the possible difference in kinematic symmetries between Minkowski and Schwarzschild spacetimes, the best way to resolve them is to discuss the recovery of Minkowski isometries in the near-horizon region. In the following, we argue that this can be achieved in a small angle and large black hole radius approximation, and is of particular relevance for near-horizon amplitudes satisfying the eikonal approximation and spherical symmetry in the Minkowski frame, as we now explain.

We will be interested in the leading contribution of the metric Eq. (5) in the $r \to R$ limit[3]

$$ds^2_{\text{NH}} = -2dx^- dx^+ + R^2 d\Omega_2^2 + \mathcal{O}(R^{-1}). \tag{29}$$

In further considering a small angle approximation, i.e., considering the leading planar approximation to the angular coordinates in a region far smaller than R, the spacetime can be transformed to a flat spacetime metric, noted as a "Minkowski coordinate frame" in [59], around the bifurcation sphere in the maximally extended Schwarzschild spacetime. More specifically if we consider $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, and assume that the transverse directions X and Y are related to the angles θ and ϕ via [59][4]

$$X = R\left(\theta - \frac{\pi}{2}\right), \qquad Y = R\phi; \tag{30}$$

Equation (29) gives the Minkowski coordinate frame metric

[3]One may also be interested in the near-horizon metric up to $\mathcal{O}(r - R)$,

$$ds^2 = -2dx^- dx^+ + R^2 d\Omega_2^2 + \left[4\left(\frac{r}{R} - 1\right)dx^- dx^+ + 2R^2\left(\frac{r}{R} - 1\right)d\Omega_2^2\right].$$

[4]If instead we considered $d\Omega_2^2 = \frac{4}{(1+z\bar{z})^2} dz d\bar{z}$, then the transformations $z = \frac{X}{2R} + i\frac{Y}{2R}$ and $\bar{z} = \frac{X}{2R} - i\frac{Y}{2R}$ would also recover the flat spacetime metric.

$$ds^2_{\text{Minkowski frame}} = -dx^+dx^- + dx^2_\perp, \tag{31}$$

where we have replaced $R^2 d\Omega_2^2 = dX^2 + dY^2 \coloneqq dx^2_\perp$, and have rescaled $x^\pm \to x^\pm/\sqrt{2}$ for convenience. The small angle approximation is implemented above by considering the leading order terms in a Taylor series expansion of the transverse metric with θ small.[5] Therefore, Minkowski isometries are formally recovered in this approximation. While translation invariance is generally broken on a black hole spacetime, it follows from the isometries of Eq. (31) that translation invariance, and consequently momentum conservation, are satisfied by scattering processes within the Minkowski coordinate frame. We stress that the Minkowski coordinate frame metric Eq. (31) for the near-horizon geometry holds exactly only in the large R limit, so that the subleading $\mathcal{O}(R^{-1})$ terms are negligible. An important consequence of the large R limit is that we have a geometry with large transverse directions. As a consequence, we may consider forward scattering processes with small transverse momenta exchanges in the Minkowski coordinate frame, which we consider in the following.

Based on the Minkowski frame metric in the large R limit of the near-horizon geometry, which is nothing but Minkowski spacetime, we may directly apply the CCFT formalism for flat spacetime to our near-horizon eikonal amplitude by performing a Mellin transformation. However, we note that 2D kinematic variables discussed in the previous section, such as Mandelstam variables, differ from the 4D ones by lack of transverse directions. Hence, for general scattering states, the 2D Mandelstam $\mathbf{s}$ variable in Eq. (22) will be incompatible with the 4D Mandelstam s defined in the Minkowski coordinate frame. To bypass this difficulty, we will restrict our consideration to *forward scattering amplitudes with small transverse momenta exchange*. Therefore for the 2-2 scattering process we define

$$\begin{aligned}
&p_i^+ = p_i^0 + p_i^3 \gg p_{i,\perp}, \quad p_i^- \simeq 0 \quad \text{for } i = 1,3;\\
&p_i^- = p_i^0 - p_i^3 \gg p_{i,\perp}, \quad p_i^+ \simeq 0 \quad \text{for } i = 2,4;\\
&\quad t = -(p_1+p_3)^2, \quad s = -(p_1+p_2)^2 \approx 2p_1^+ p_2^- \approx \mathbf{s},
\end{aligned} \tag{32}$$

where $i = 1, 2$ labels the incoming particles, and $i = 3, 4$ the outgoing particles. We note from the last line of Eq. (32) that we now have small nonvanishing exchange momenta, and that the 4D Mandelstam variable in the Minkowski coordinate frame approximately agrees with the corresponding 2D Mandelstam in Eq. (22) up to $p^2_{i,\perp}$ corrections. Our consideration of $p_{i,\perp}$ small is consistent with $p_A \simeq 0$ as adopted in [60–63], and such states can also be realized naturally in the context of trans-Planckian scattering [75,81,82].

[5]More specifically, we have $\sin\theta = \sin(\frac{X}{R} + \frac{\pi}{2}) = 1 + \mathcal{O}(\frac{X}{R})$.

In short, to promote the 2D kinematic relations to the 4D ones in the near-horizon Minkowski coordinate frame, and with the purpose of subsequently deriving a CCFT description, we will only consider forward scattering amplitudes for the 2-2 process with external states satisfying Eq. (32). This subset of amplitudes can be lifted from 2D to 4D while respecting eikonal kinematics. For more general scattering amplitudes, the kinematic lifting will be more nontrivial.

We will now provide the upliftment of the near-horizon eikonal amplitude. A general 4D N-particle scattering amplitude can be formally obtained from the partial wave analysis as follows [83,84]:

$$A^N_{4D} = \mathcal{N} \sum_{\{\ell_i, m_i\}} A^{\{\ell_i, m_i\}}_{\text{p.w.}} \prod_i Y_{\ell_i, m_i}(\Omega_i), \tag{33}$$

where $i = 1, \ldots, N$ label the external particles, $\mathcal{N}$ is a normalization constant, $Y_{\ell_i, m_i}(\Omega_i)$ are the spherical harmonics for the external particles defined on the 2-sphere at Ω_i, and $A^{\{\ell_i, m_i\}}_{\text{p.w.}}$ is the partial wave amplitude equipped with the constraints of angular momentum conservation. The product in Eq. (33) is over the spherical harmonics for each external particle of the amplitude. Hence the formal sum in Eq. (33) generally leads to a complicated kernel for transforming the partial wave amplitudes to the corresponding 4D amplitude. However, as we have discussed, for our purpose of constructing the CCFT dual of uplifted eikonal amplitudes in the near-horizon regime, we will only consider forward scattering semiclassical amplitudes. Due to small transverse momenta exchanges in these amplitudes, additional partial wave mode mixings are not introduced. This is consistent with the absence of partial wave mode mixing to prevent nonspherical backreaction in the semiclassical analysis of [60–63].[6] For the 2-2 scattering, this reduces the multisums over (ℓ_i, m_i) into a single sum of the overall (ℓ, m), i.e., this is reflected in the fact that the partial wave amplitude Eq. (27) depends only on the overall (ℓ, m). We further recall that the label ℓ in Eq. (27) refers to the partial wave of one of the external states in the 2-2 process, with the other particle fixed to be a $\ell = 0$ state. Thus, by the aforementioned assumption, the above formal partial wave summation for the 2-2 eikonal scattering in the near-horizon Minkowski frame simplifies to

$$A_{\text{NH eikonal}} = \frac{\mathcal{N}}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\theta) A^\ell_{\text{NH eikonal}}(s), \tag{34}$$

where $A^\ell_{\text{NH eikonal}}(s)$ is given by Eq. (27) with the argument in terms of the 2D Mandelstam $\mathbf{s}$ now replaced with the 4D

[6]Here we have adopted the same spherical symmetry approximation used in [60–63] to remove the mixing of partial waves. This approximation is valid for the tree-level amplitude. However, its validity for the internal legs in the eikonal amplitudes remains to be clarified.

Mandelstam s, and θ is the angle between the interacting particles. In Eq. (34), we have used the sum over (ℓ, m) which can be further simplified by the formula

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y^{*}_{\ell m}(\Omega') = \frac{2\ell+1}{4\pi} P_{\ell}(\cos\theta) \tag{35}$$

because the 2-2 partial wave amplitude $A^{\ell}_{\mathrm{NH\ eikonal}}$ is m independent. Note that our assumption of small transverse exchanges implies that Eq. (34) holds for small-angle scattering.

Setting $\mathcal{N} = 4\pi$, we arrive at our final expression for the Minkowski frame partial wave amplitude

$$A_{\mathrm{NH\ eikonal}} = 2s \sum_{\ell} (2\ell+1) \left[\exp\left(\frac{i\kappa^2 s}{\ell^2+\ell+2} \right) - 1 \right] \times P_{\ell}(\cos\theta). \tag{36}$$

The normalization has been chosen to provide an overall scaling consistent with graviton mediated eikonal amplitudes and does not affect the main results in our analysis to follow.

In summary, the 4D amplitude $A_{\mathrm{NH\ eikonal}}$ given in Eq. (36) provides a consistent uplifting of the 2D partial wave eikonal amplitude given in Eq. (27), defined in Minkowski frame about the bifurcation sphere. We caution the reader that this does not apply for general scattering amplitudes in the near-horizon Minkowski frame. It holds for semiclassical forward eikonal amplitudes, which is what we consider for the dual CCFT description. Equation (36) bears the usual form for eikonal scattering with $\chi_l = \frac{\kappa^2 s}{\ell^2+\ell+2}$ appearing in Eq. (27) playing the role of eikonal phase. However, it differs from its counterpart Eq. (2) for eikonal scattering in flat spacetime due to different underlying dynamics. While we kinematically go over to a flat spacetime Minkowski frame in the small angle and large Schwarzschild radius limit, the phase (resulting from nonvanishing curvature contributions in this limit) provides a different eikonal resummation than on flat spacetimes Eq. (2) and captures near-horizon effects of the Schwarzschild spacetime on the scattering. In the following subsection, we proceed to evaluate Eq. (36) using known techniques in flat spacetime for small angle scattering.

For scattering processes in the near-horizon region to be consistent with the small angle approximation and involving plane wave states, we may thus adopt the usual approach for CCFTs on flat spacetimes. As the near-horizon eikonal amplitude is a high energy forward scattering process involving massless plane waves as external states, a dual celestial description can be derived using the Mellin transform on the external states. We will return to a more detailed discussion of these properties in Sec. III C. For the moment, we point out three key differences with flat spacetime CCFT constructions:

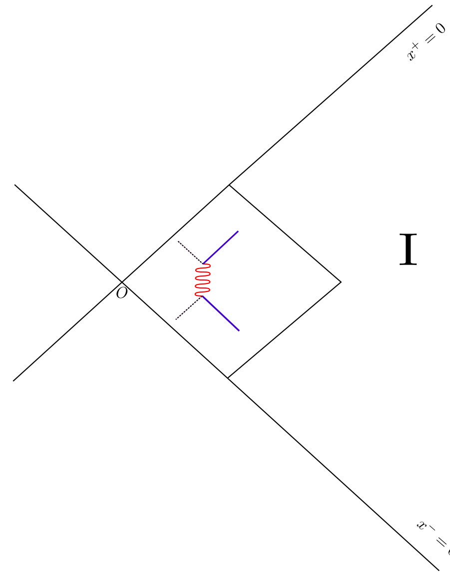

FIG. 2. Minkowski frame defined in an $\mathcal{O}(R)$ region about the bifurcation sphere O in the exterior region I of the global Kruskal spacetime. Also indicated is a representative 2-2 scattering process between particle with label ℓ (bold) and $\ell = 0$ (dashed) mediated by a soft graviton (wavy).

(1) Time is rescaled by a factor of $\frac{1}{2R}$ (and a constant) relative to the "global time coordinate" t in Eq. (3). As such, while we will still denote the frequency as ω in the Mellin transform, it is related to the frequency at null infinity by a factor of $2R$.
(2) As evident from Eq. (29), the asymptotic conformal boundary of the spacetime is entirely a portion of the past and future event horizons about the bifurcation sphere and not null infinity.
(3) Dual celestial correlators will be constructed only for eikonal scattering processes (with small transverse momentum exchange) respecting background spherical symmetry in the *Minkowski frame* (see Fig. 2) near the bifurcation sphere.

We also note there exist earlier holographic proposals based on AdS_3 spacelike foliations of the near-horizon geometry in the limit of approaching the nondegenerate horizon [85,86] which realize the above features. In particular, the Schwarzschild spacetime is conformal to an optical metric with time rescaled by $\frac{1}{2R}$ and with the conformal boundary located at the event horizon. However, this formalism provides no particular advantage over known approaches in flat spacetime for investigating scattering processes in the near-horizon region involving plane wave states (Fig. 2).

B. 4D near-horizon eikonal amplitude from partial sum

In this subsection, we will carry out the explicit sum of Eq. (36). We recall that in Eq. (36) ℓ labels the total angular momentum for one of the partial waves in the scattering amplitude, s is the square of the center of mass energy in 4D momentum space, and θ is the small angle between the incoming and scattered particles. The sum over ℓ in Eq. (36) can be traded for a 2D integral over the transverse directions. This follows from the relation between the transverse direction and angular momentum mode ℓ arising from the definition of angular momentum (squared). From the angular momentum operator [74,81,82], we have

$$J_\mu = \frac{1}{\sqrt{2p_1.p_2}}\epsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\rho x^\sigma, \tag{37}$$

with x^σ the displacement between the two particles. On squaring Eq. (37) and using $J^2 = J_\mu J^\mu = l(l+1)$, along with the leading approximations in Eq. (32), we then arrive at [81,82]

$$\ell(\ell+1) \simeq s|x_\perp|^2, \tag{38}$$

This turns the eikonal phase χ_ℓ given in Eq. (36) to

$$\chi_\ell = \frac{\kappa^2 s}{s|x_\perp|^2+2} = \frac{\kappa^2}{|x_\perp|^2} + \mathcal{O}(s^{-1}), \tag{39}$$

which receives its dominant contribution for large ℓ (or equivalently large s) and small $x_\perp$, and is different from the corresponding graviton mediated eikonal amplitude on flat spacetime that grows with s [73,81]. In the following, we retain the complete expression for the eikonal phase in Eq. (39), which captures all $\mathcal{O}(s^{-1})$ properties. However, as in all eikonal amplitudes, all corrections in the more subleading $\mathcal{O}(\sqrt{-\frac{t}{s}})$ are ignored.

Another ingredient is an integral representation of the Legendre polynomials for small-angle scattering, for which we have

$$P_\ell(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left(i2\ell\sin\left(\frac{\theta}{2}\right)\cos\phi\right), \tag{40}$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{-t}}{2\sqrt{s}} = \frac{|p_\perp|}{2\sqrt{s}}, \tag{41}$$

where we used the known relation between Legendre polynomials and Bessel functions in the small angle approximation in Eq. (40), while Eq. (41) is the relation between the exchanged momentum and center of mass energy for small angle scattering.

Substituting Eqs. (38), (40), and (41) in Eq. (36), we find the following result for the 4D near-horizon eikonal amplitude [by ignoring all the subleading $\mathcal{O}(\sqrt{-\frac{t}{s}})$ corrections]

$$A_{\text{NH eikonal}} = 2s\int d^2x_\perp\left[\exp\left(\frac{i\kappa^2 s}{s|x_\perp|^2+2}\right)-1\right]e^{i\vec{p}_\perp.\vec{x}_\perp}. \tag{42}$$

This differs from the expression for the eikonal amplitude on asymptotically flat spacetimes through the eikonal phase. The two main differences lie in the dependence of the eikonal phase on s and the impact parameter $x_\perp$. The eikonal phase for graviton mediated scattering on asymptotically flat spacetimes grows with s and holds for large impact parameters $x_\perp$, which is evident from substituting Eq. (2) in Eq. (38) [81]. In contrast, the subleading terms in the eikonal phase of the near-horizon scattering process decay with large s [as indicated in Eq. (39)] with the dominant contribution from $x_\perp \ll 1$ in units of the Schwarzschild radius. In other words, one can see from Eq. (39) that while we consider a high energy scattering process with $\frac{-t}{s} \ll 1$ near the horizon, the eikonal phase grows more dominant as we reduce the impact parameter. In the following subsection, we determine how this manifests in a celestial description and compare our result with the celestial eikonal amplitude on asymptotically flat spacetimes.

C. Near-horizon celestial eikonal amplitude

We have noted that the near-horizon geometry in the limit of approaching the horizon can be described by a flat spacetime metric. The near-horizon eikonal amplitude is a scattering process restricted to this region involving external massless plane wave states. Hence a near-horizon celestial description of this amplitude can result from a Mellin transform of the near-horizon eikonal amplitude through its action on the external states, following the same arguments as recently provided for flat spacetime eikonal amplitudes in [31].

We accordingly define the 4D 2-2 near-horizon celestial eikonal amplitude as the Mellin transform of the near-horizon eikonal amplitude given in Eq. (42) including the momentum conserving delta function

$$\tilde{A}_{\text{NH eikonal}} = (2\pi)^4\left(\prod_{i=1}^4\int_0^\infty d\omega_i\omega_i^{\Delta_i-1}\right)A_{\text{NH eikonal}}\delta^{(4)}\left(\sum_{i=1}^4 p_i\right). \tag{43}$$

We consider the momenta of the external states as considered in [31]. This involves an all-outgoing convention with

$$p_i = \eta_i\omega_i\hat{q}_i; \qquad i = 1,\cdots 4 \tag{44}$$

with $\eta_i = +1$ for the outgoing states $i = 3, 4$, $\eta_i = -1$ for the incoming states $i = 1, 2$, and $\hat{q}_i$ a null vector parametrized in terms of the following transverse and longitudinal components

$$\hat{q}_i = (1 + q_i, q_{i,\perp}, 1 - q_i), \\ = (1 + z_i\bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i\bar{z}_i), \quad i = 1, 3, \tag{45}$$

$$\hat{q}_i = (1 + q_i, q_{i,\perp}, -1 + q_i), \\ = \frac{1}{z_i\bar{z}_i}(1 + z_i\bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i\bar{z}_i), \quad i = 2, 4, \tag{46}$$

where $(z, \bar{z})$ is a point on the celestial sphere at the horizon. The massless condition $\hat{q}_i^2 = 0$ imposes $4q_i = |q_{i,\perp}|^2$, which relates the longitudinal component q_i with the transverse component $q_{i,\perp}$, a two component vector. The constraints then turn into the expressions of $\hat{q}_i$s in terms of z_is. Hence for an eikonal scattering process with $s \gg -t$, we have $q_i \sim |q_{i,\perp}|^2 \ll 1$, with $\omega_1 \simeq \omega_3$ and $\omega_2 \simeq \omega_4$. We use $\simeq$ to indicate an equivalence up to corrections subleading in $\mathcal{O}(q_\perp^2)$.

The above considerations for the external states further imply that for $i = 1, 3$ we have $p_i^+ = p_i^0 + p_i^3 = 2\eta_i\omega_i \gg p_{i,\perp}$ and $p_i^- = p_i^0 - p_i^3 \simeq 0$, while for $i = 2, 4$ we have $p_i^- = 2\eta_i\omega_i \gg p_{i,\perp}$ and $p_i^+ \simeq 0$. With the above definitions, the delta function in Eq. (43) takes the form

$$\delta^{(4)}\left(\sum_{i=1}^4 p_i\right) = 2\delta(p_1^+ + p_3^+)\delta(p_2^- + p_4^-)\delta^{(2)}\left(\sum_{i=1}^4 p_{i,\perp}\right), \\ = \frac{1}{2}\delta(\omega_1 - \omega_3)\delta(\omega_2 - \omega_4)\delta^{(2)}\left(\sum_{i=1}^4 \eta_i\omega_i q_{i,\perp}\right). \tag{47}$$

Likewise, we have for the Mandelstam variables s and t

$$s \simeq -2p_1^+ p_2^- = 4\omega_1\omega_2, \\ -t \simeq (p_{1,\perp} + p_{3,\perp})^2 = (\omega_3 q_{3,\perp} - \omega_1 q_{1,\perp})^2. \tag{48}$$

Substituting Eqs. (47) and (48) in Eq. (43), we get the following expression:

$$\tilde{A}_{\text{NH Eikonal}} = 4(2\pi)^4 \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \omega_1^{\Delta_1+\Delta_3-1}\omega_2^{\Delta_2+\Delta_4-1} \\ \times \int d^2x_\perp \sum_{n=1}^\infty \frac{1}{n!}\left(\frac{i\kappa^2}{|x_\perp|^2 + \frac{1}{2\omega_1\omega_2}}\right)^n e^{-i\omega_1 q_{13,\perp}\cdot x_\perp} \\ \times \delta^{(2)}(\omega_1 q_{13,\perp} + \omega_2 q_{24,\perp}), \tag{49}$$

where we have written the eikonal phase as a series expansion and used the notation $q_{i,\perp} - q_{j,\perp} = q_{ij,\perp}$.

With the exponential representation for the 2D delta function

$$\int d^2\bar{x}_\perp \exp(-ik\cdot\bar{x}_\perp) = (2\pi)^2\delta^{(2)}(k), \tag{50}$$

we may express Eq. (49) as

$$\tilde{A}_{\text{NH Eikonal}} = 4(2\pi)^2 \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \omega_1^{\Delta_1+\Delta_3-1}\omega_2^{\Delta_2+\Delta_4-1} \\ \times \int d^2x_\perp \int d^2\bar{x}_\perp \sum_{n=1}^\infty \frac{1}{n!}\left(\frac{i\kappa^2}{|x_\perp|^2 + \frac{1}{2\omega_1\omega_2}}\right)^n \\ \times e^{-i\omega_1 q_{13,\perp}\cdot(x_\perp + \bar{x}_\perp)} e^{-i\omega_2 q_{24,\perp}\cdot\bar{x}_\perp}. \tag{51}$$

The second line in Eq. (51) can be further simplified. We first express all frequencies appearing in the eikonal phase as resulting from the action of celestial momentum operators on $\omega_i^{\Delta_i-1}$ for the incoming particles, $P_i^\mu = -\hat{q}_i^\mu e^{\partial_{\Delta_i}}$ for $i = 1, 2$ [8]. We can expand the eikonal phase as an infinite series

$$\left(|x_\perp|^2 + \frac{1}{2\omega_1\omega_2}\right)^{-n} = \sum_{k=0}^\infty C_{n-1}^{k+n-1} x_\perp^{-2(n+k)}\left(-\frac{1}{2\omega_1\omega_2}\right)^k \tag{52}$$

with $C_{n-1}^{k+n-1} = \frac{(k+n-1)!}{k!(n-1)!}$. From the standard identity of the celestial momentum operator acting on the integrand of the Mellin transform [8,31]

$$\omega_1^{\Delta_1+\Delta_3-1-k} = e^{-k\partial_{\Delta_1}}\omega_1^{\Delta_1+\Delta_3-1}, \\ \omega_2^{\Delta_2+\Delta_4-1-k} = e^{-k\partial_{\Delta_2}}\omega_2^{\Delta_2+\Delta_4-1}, \tag{53}$$

we have the relation

$$\left(\frac{i\kappa^2}{|x_\perp|^2 + \frac{1}{2\omega_1\omega_2}}\right)^n \omega_1^{\Delta_1+\Delta_3-1}\omega_2^{\Delta_2+\Delta_4-1} \\ = \left(\frac{i\kappa^2}{|x_\perp|^2 + \frac{1}{2}\exp(-\partial_{\Delta_1})\exp(-\partial_{\Delta_2})}\right)^n \\ \times \omega_1^{\Delta_1+\Delta_3-1}\omega_2^{\Delta_2+\Delta_4-1}. \tag{54}$$

We hence see that a factor of $s = 4\omega_1\omega_2$ can be interpreted as arising from the action of a shifting operator $4e^{\partial_{\Delta_1}}e^{\partial_{\Delta_2}}$ in the celestial basis following Eq. (53). We in addition consider the transformation $x_\perp + \bar{x}_\perp \to x_\perp$ resulting in an eikonal phase that depends on the transverse distance $x_\perp - \bar{x}_\perp$. By performing this transformation and using Eq. (54) in Eq. (51), we get the result

$$\tilde{A}_{\text{NH Eikonal}} = 4(2\pi)^2 \int d^2x_\perp \int d^2\bar{x}_\perp \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i\kappa^2}{|x_\perp - \bar{x}_\perp|^2 + \frac{1}{2}\exp(-\partial_{\Delta_1})\exp(-\partial_{\Delta_2})} \right)^n$$
$$\times \int_0^\infty d\omega_1 \omega_1^{\Delta_1+\Delta_3-1} e^{-i\omega_1 q_{13,\perp}.x_\perp} \int_0^\infty d\omega_2 \omega_2^{\Delta_2+\Delta_4-1} e^{-i\omega_2 q_{24,\perp}.\bar{x}_\perp},$$
$$\coloneqq 4(2\pi)^2 \int d^2x_\perp \int d^2\bar{x}_\perp (e^{i\hat{\chi}_{\text{NH}}} - 1) \frac{i^{\Delta_1+\Delta_3}\Gamma(\Delta_1+\Delta_3)}{(-q_{13,\perp}\cdot x_\perp + i\epsilon)^{\Delta_1+\Delta_3}} \frac{i^{\Delta_2+\Delta_4}\Gamma(\Delta_2+\Delta_4)}{(-q_{24,\perp}\cdot \bar{x}_\perp + i\epsilon)^{\Delta_2+\Delta_4}}, \tag{55}$$

where in the last equation we defined the eikonal phase operator

$$\hat{\chi}_{\text{NH}} \coloneqq \frac{\kappa^2}{|x_\perp - \bar{x}_\perp|^2 + \frac{1}{2}\exp(-\partial_{\Delta_1})\exp(-\partial_{\Delta_2})}, \tag{56}$$

and made use of the identity

$$\int_0^\infty d\omega \omega^{\Delta-1} e^{-i\eta\omega q.x_\perp} = \frac{i^\Delta \Gamma(\Delta)}{(-q\cdot x_\perp + i\eta\epsilon)^\Delta}. \tag{57}$$

Equation (57) provides a definition of massless conformal primary wave functions for scattering on asymptotically flat spacetimes, while the form of Eq. (55) is very similar to the result for flat spacetime celestial eikonal amplitudes given in [31]. The main difference with the flat spacetime result stems from the form of the eikonal phase operator Eq. (56). In [31] the eikonal phase operator for graviton mediated scattering takes the following form[7]:

$$\hat{\chi}_{\text{flat}} \propto \kappa^2 \exp(\partial_{\Delta_1})\exp(\partial_{\Delta_2}) G_\perp(x_\perp, \bar{x}_\perp), \tag{58}$$

where $G_\perp(x_\perp, \bar{x}_\perp)$ is the transverse part of the propagator of exchange graviton. Comparing Eqs. (56) and (58), we see that one difference concerns the action of $\exp(\partial_{\Delta_1})\exp(\partial_{\Delta_2})$, with the inverse dependence appearing in near-horizon eikonal amplitudes. In the following section, we will investigate Eq. (55) in more detail and compare our results with those of celestial eikonal amplitudes on asymptotically flat spacetimes.

IV. PROPERTIES OF THE NEAR-HORIZON CELESTIAL EIKONAL AMPLITUDE

Celestial amplitudes, on account of their involvement of boost eigenstates, are known to be sensitive to both UV and IR properties of scattering processes [20]. Lorentz and translation invariance on the celestial sphere further manifest in certain universal properties that celestial amplitudes must possess.

The near-horizon celestial eikonal amplitude Eq. (55) can be expanded as the sum of the Feynmann ladder diagrams, which are classified by the number of exchanged soft gravitons, i.e., denoted by n, or $(n-1)$ loops,

$$\tilde{A}_{\text{NH eikonal}} = \sum_{n=1}^{\infty} \tilde{A}^{(n)}_{\text{NH eikonal}} \tag{59}$$

with

$$\tilde{A}^{(n)}_{\text{NH eikonal}} = \frac{4(2\pi)^2}{n!} \int d^2x_\perp \int d^2\bar{x}_\perp (i\hat{\chi}_{\text{NH}})^n \times \frac{i^{\Delta_1+\Delta_3}\Gamma(\Delta_1+\Delta_3)}{(-q_{13,\perp}\cdot x_\perp + i\epsilon)^{\Delta_1+\Delta_3}} \times \frac{i^{\Delta_2+\Delta_4}\Gamma(\Delta_2+\Delta_4)}{(-q_{24,\perp}\cdot \bar{x}_\perp + i\epsilon)^{\Delta_2+\Delta_4}}. \tag{60}$$

The form of Eq. (60) is exact for all n. Below, we will argue that our result satisfies the expected universal properties of celestial amplitudes and is consistent with the defining properties of near-horizon eikonal amplitudes—that is it is mediated by soft graviton exchanges in the large s limit.

We start with the integral representation of celestial eikonal amplitude from Eq. (49), incorporating the definition Eq. (59)

$$\tilde{A}^{(n)}_{\text{NH eikonal}} = 4(2\pi)^4 \int_0^\infty d\omega_1 \omega_1^{\Delta_1+\Delta_3-1} \int_0^\infty d\omega_2 \omega_2^{\Delta_2+\Delta_4-1} \times \int d^2x_\perp \frac{1}{n!} \left(\frac{i\kappa^2}{|x_\perp|^2 + \frac{1}{2\omega_1\omega_2}} \right)^n e^{-i\omega_1 q_{13,\perp}\cdot x_\perp} \times \delta^{(2)}(\omega_1 q_{13,\perp} + \omega_2 q_{24,\perp}). \tag{61}$$

One of the properties of this celestial eikonal amplitude is that the integration over $x_\perp$ can be evaluated exactly to give the modified Bessel function of the second kind of integer order $n-1$,

[7] The form of Eq. (58) is consistent with Eq. (2) after replacing the action of shift operators by $s = 4\omega_1\omega_2$, along with the relation Eq. (38) and the explicit form of $G_\perp(x_\perp, \bar{x}_\perp) \sim \ln|x_\perp - \bar{x}_\perp|$.

$$\int d^2x_\perp \left(\frac{i\kappa^2}{|x_\perp|^2+\frac{1}{2\omega_1\omega_2}}\right)^n e^{-i\omega_1 q_{13,\perp}\cdot x_\perp} = 2\pi \frac{i^n\kappa^{2n}}{(n-1)!}\left(\omega_1|q_{13,\perp}|\sqrt{\frac{\omega_1\omega_2}{2}}\right)^{n-1} \times K_{n-1}\left(\sqrt{\frac{\omega_1}{2\omega_2}}|q_{13,\perp}|\right), \tag{62}$$

where we have used the known integral (cf. 6.565.4 of [87])

$$\int_0^\infty \frac{J_\nu(bx)x^{\nu+1}}{(x^2+a^2)^{\mu+1}}dx = \frac{a^{\nu-\mu}b^\mu}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(ab), \quad -1<\mathrm{Re}\,\nu<\mathrm{Re}\,(2\mu+3/2); a,b>0. \tag{63}$$

Our analysis will be further considered in a center-of-mass frame with

$$|q_\perp| \coloneqq |q_{13,\perp}| = |q_{24,\perp}|, \tag{64}$$

Hence, on substituting Eq. (62) in Eq. (61) and performing the rescalings $\omega_1|q_\perp|\to\omega_1$ and $\omega_2|q_\perp|\to\omega_2$, we find

$$\tilde{A}^{(n)}_{\mathrm{NH\ eikonal}} = 4(2\pi)^5\frac{i^n\kappa^{2n}}{n!(n-1)!}(|q_\perp|)^{-\beta-n-3}\int_0^\infty d\omega_1\omega_1^{\Delta_1+\Delta_3-1}\int_0^\infty d\omega_2\omega_2^{\Delta_2+\Delta_4-1} \times\left(\omega_1\sqrt{\frac{\omega_1\omega_2}{2}}\right)^{n-1}K_{n-1}\left(\sqrt{\frac{\omega_1}{2\omega_2}}|q_\perp|\right)\delta^{(2)}(\omega_1 n_{13,\perp}+\omega_2 n_{24,\perp}), \tag{65}$$

where we defined the two-dimensional vectors

$$n_{13,\perp} = (n^1_{13,\perp}, n^2_{13,\perp}) \coloneqq \frac{q_{13,\perp}}{|q_\perp|}, \quad n_{24,\perp} = (n^1_{24,\perp}, n^2_{24,\perp}) \coloneqq \frac{q_{24,\perp}}{|q_\perp|}, \tag{66}$$

and

$$\beta \coloneqq \Delta_1+\Delta_2+\Delta_3+\Delta_4-4. \tag{67}$$

To further simplify Eq. (65), we consider the 2-2 scattering in the center of mass frame with the following parametrization for the transverse momenta in terms of the cross ratio $z=\frac{-t}{s}$ [31],

$$q_{13,\perp} = (-(\sqrt{z}+\sqrt{\bar{z}}), i(\sqrt{z}-\sqrt{\bar{z}})), \quad q_{24,\perp} = (\sqrt{z}+\sqrt{\bar{z}}, i(\sqrt{z}-\sqrt{\bar{z}})), \tag{68}$$

so that

$$|q_\perp| = 2\sqrt{|z|}, \quad n^1_{24,\perp}n^2_{13,\perp} = i\frac{z-\bar{z}}{|q_\perp|^2} = -n^2_{24,\perp}n^1_{13,\perp}. \tag{69}$$

In the parametrization of $\hat{q}_i$ of Eqs. (45) and (46), this corresponds to the choice

$$z_1=0, \qquad z_2=\infty, \qquad z_3=\sqrt{z}, \qquad z_4=-\frac{1}{\sqrt{\bar{z}}}, \tag{70}$$

such that in the eikonal limit,

$$z=\frac{-t}{s}\approx\frac{z_{13}z_{24}}{z_{12}z_{34}}\ll 1. \tag{71}$$

The two-dimensional delta function $\delta^{(2)}(\omega_1 n_{13,\perp}+\omega_2 n_{24,\perp})$ can be factored into a product of delta functions [31]

$$\delta^{(2)}(\omega_1 n_{13,\perp}+\omega_2 n_{24,\perp}) = \frac{1}{\omega_1}\delta\left(\omega_2+\omega_1\frac{n^1_{24,\perp}}{n^1_{13,\perp}}\right) \times\delta(n^1_{24,\perp}n^2_{13,\perp}-n^2_{24,\perp}n^1_{13,\perp}), = \frac{|q_\perp|^2}{2\omega_1}\delta(\omega_2-\omega_1)\delta(z-\bar{z}), \tag{72}$$

where in the second line of Eq. (72) we made use of Eqs. (68) and (69), along with the standard scaling property of the Dirac delta function. On carrying out the integral over ω_2 in Eq. (65) and using $\delta(z-\bar{z})$ to write $|q_\perp|=2\sqrt{z}$, we find

$$\tilde{A}^{(n)}_{\mathrm{NH\ eikonal}} = 2(2\pi)^5\frac{i^n\kappa^{2n}2^{-\beta-2}}{n!(n-1)!}2^{\frac{3}{2}(1-n)} \times K_{n-1}(\sqrt{2z})(\sqrt{z})^{-\beta-n-1}\delta(z-\bar{z}) \times\int_0^\infty d\omega_1\omega_1^{\beta+2n-1}. \tag{73}$$

Lastly, we can perform the ω_1 integral by analytically continuing to the regime of large scaling dimension $\Delta_i\gg 1$ with the following result [17]

$$\int_0^\infty d\omega_1\omega_1^{\beta+2n-1} \coloneqq \delta(i(\beta+2n)). \tag{74}$$

Thus the integral in Eq. (74) is well defined when we set $\beta = -2n + ib$. Hence, on substituting Eq. (74) in Eq. (73) we arrive at the final result:

$$\tilde{A}^{(n)}_{\text{NH eikonal}} = 2(2\pi)^6 \frac{i^n \kappa^{2n} 2^{-\beta-2}}{n!(n-1)!} 2^{\frac{3}{2}(1-n)} \times K_{n-1}(\sqrt{2z})(\sqrt{z})^{-\beta-n-1} \times \delta(z-\bar{z})\delta(i(\beta+2n)). \tag{75}$$

The contribution $2^{-\beta-2}\delta(z-\bar{z})$ in Eq. (75) is part of the universal kinematic contribution in all celestial amplitudes [21,31]. The delta function over z in particular manifests translation invariance on the celestial sphere. The scaling behavior of the amplitude will follow from $(\sqrt{z})^{-\beta-n-1}K_{n-1}(\sqrt{2z})$, which, as we now show, has a universal n-independent leading scaling behavior for all $\tilde{A}^{(n\geq 2)}_{\text{NH eikonal}}$. The leading contribution of $K_{n-1}(\sqrt{2z})$ at each order n is given by

$$K_{n-1}(\sqrt{2z}) \approx -\frac{1}{2}(-\sqrt{2z})^{n-1}\partial_z^{n-1}\ln\left(\frac{z}{2}\right). \tag{76}$$

We consider Eq. (76) in Eq. (75) at tree level ($n=1$) and higher loops more generally ($n \geq 2$). For the tree-level exchange, we find

$$\begin{aligned}\tilde{A}^{(1)}_{\text{NH eikonal}} &= -(2\pi)^6 i\kappa^2 2^{-\beta-2}\ln\left(\frac{z}{2}\right)(\sqrt{z})^{-\beta-2}\delta(z-\bar{z})\delta(i(\beta+2)) + \cdots, \\ &= -(2\pi)^6 i\kappa^2 \ln\left(\frac{z}{2}\right)\delta(z-\bar{z})\delta(i(\beta+2)) + \mathcal{O}(\sqrt{z}),\end{aligned} \tag{77}$$

where we utilized $\delta(i(\beta+2))$ in the second line of Eq. (77) and $\cdots$ in the first line represent contributions subleading in small z. The tree-level exchange is hence dominated by $z^0 \ln z$. Similarly, for $n \geq 2$ by using (76) and $\boldsymbol{\delta}(i(\beta+2n))$ in Eq. (75), we find the result to be

$$\begin{aligned}\tilde{A}^{(n\geq 2)}_{\text{NH eikonal}} &= (2\pi)^6 \frac{i^n\kappa^{2n}}{(n-1)n!} 2^{-\beta-(n+3)}(\sqrt{z})^{-\beta-2(n+1)}\delta(z-\bar{z})\delta(i(\beta+2n)) + \cdots, \\ &= (2\pi)^6 \frac{i^n\kappa^{2n}}{(n-1)n!} 2^{n-3} z^{-1}\delta(z-\bar{z})\delta(i(\beta+2n)) + \mathcal{O}((\sqrt{z})^{-1}).\end{aligned} \tag{78}$$

We hence find the universal leading behavior of z^{-1} for all $n \geq 2$ loop orders in the near-horizon celestial eikonal amplitude. It is quite interesting to consider the dynamical interpretation of the factor $\delta(i(\beta+2n))$ in Eq. (75). Recall that the same pole structures are also proposed and discussed for the celestial amplitudes of IR or UV soft theories [20]. For the IR soft ones, these poles appear after Mellin transforming the IR expansion of the amplitude in the momentum basis, i.e., $\sum_{n=0}^{\infty} a_n^{\text{IR}}\omega^{2n}$. In a UV soft theory, such as string theory, the UV behavior is softened by the productions of stringy Hagedorn states, which can be understood as the microscopic black holes by the string/black hole correspondence [66–68]. This implies that the celestial amplitudes can capture both nonperturbative physics in both UV and IR sides through the characteristics of $\beta = -2n$ poles. In our case, we are considering the celestial eikonal amplitudes for which the $\beta = -2n$ appears with n labeling the order of the loop/ladder diagrams. Thus, a natural interpretation of these poles is the dominance of the soft graviton exchanges, with n corresponding to the number of soft gravitons appearing in the ladder diagrams. An additional consequence is the absence of poles for $\text{Re}\,\beta > 0$ in a manner analogous to UV soft theories. In this sense, these poles are the manifestation of the IR divergence due to soft graviton exchanges in the near-horizon region. This implies that celestial amplitudes can capture nonperturbative effects of strong gravity due to either black hole production or the existence of an event horizon.

V. SUMMARY AND CONCLUSION

Eikonal amplitudes provide an important class of nonperturbative scattering processes that have been recently investigated in the celestial basis. In this paper, we considered the celestial description of eikonal amplitudes past the critical length scale for the impact parameter, which is taken to be large in the usual eikonal approximation. In this regime, the leading approximation to the eikonal amplitude is governed by a resummation over soft graviton exchanges in the near-horizon geometry of a Schwarzschild black hole. Hence, near-horizon celestial eikonal amplitudes provide an interesting case of nonperturbative scattering processes in the boost eigenbasis beyond those on flat spacetime.

The near-horizon eikonal amplitude [60–63] accounts for the leading backreaction about a Schwarzschild background and is a two-dimensional result following the integration over spherical harmonics. The resulting eikonal phase (28) is dominated by small ℓ modes and transverse directions $x_\perp$. In Sec. III we first derived the corresponding four-dimensional eikonal amplitude through a partial sum of the two-dimensional result and subsequently performed a Mellin transform on the external states to derive the near-horizon celestial eikonal amplitude. The conformal primary wave functions describing the external states are the same as those for the celestial eikonal amplitude on flat spacetimes, which follows from the isometries of the near-horizon region in the small angle approximation being identical to those on flat spacetimes. However, the eikonal phase, which captures the interactions of the external states with the exchanged gravitons near the horizon, crucially differs from the flat spacetime eikonal phase. More specifically, the four-dimensional near-horizon eikonal amplitude is defined from impact parameters comparable to the Schwarzschild radius and a perturbative series in s^{-1} around s being infinite. This manifests the property that the near-horizon eikonal amplitude are mediated by soft graviton modes.

In Sec. IV we investigated the near-horizon celestial eikonal amplitude to derive the main results of our paper, namely an exact all-loop order result for the celestial amplitude. The result involves universal kinematic factors of celestial amplitudes on flat spacetime. This is expected from our consideration of the near-horizon region in the small angle approximation, which simply provides a flat spacetime for the scattering process. However, the dynamical content of the celestial amplitude differs considerably from celestial amplitudes on flat spacetime. One of these differences follows from the $\delta(i(\beta+2n))$ contribution. While this behavior is consistent with the expectation of soft UV behavior in CCFT, the near horizon celestial eikonal amplitude provides the specific representation of exchanged soft gravitons with loop order n. We, in addition, have a $\sqrt{z}^{-\beta-n-1}K_{n-1}(\sqrt{2z})$ term as an all-loop order contribution from the near-horizon eikonal phase. This term on expanding about $z\ll 1$, and in conjunction with $\delta(i(\beta+2n))$, provides a universal leading n-independent behaviour for $\sqrt{z}$. Thus, the leading scaling behavior of the cross-ratio z in the celestial amplitude result is independent of loop order.

There are several further avenues to explore in the context of near-horizon amplitudes. It will be interesting to go beyond the small angle approximation used for the near-horizon background. In particular, we expect a correction to the 2-sphere part of the metric in (3), which are also known to influence near-horizon symmetries [88–90]. The first nontrivial corrections to the "Minkowski coordinate frame" results can be obtained by treating $\frac{r(x^-,x^+)-R}{R}\approx f(x^-,x^+)\ll 1$ as a perturbation parameter. The metric, including the leading-order correction, takes the following form:

$$ds^2 = -(1-2f(x^-,x^+))dx^-dx^+ + R^2(1+2f(x^-,x^+))d\Omega^2 + \mathcal{O}(f(x^-,x^+)^2). \quad (79)$$

With this correction, the transverse part of the metric will now depend on the light-cone coordinates (x^-,x^+). As a result, the two-dimensional eikonal amplitude and its four-dimensional uplift will require us to account for partial wave mode mixing, which leads to angular momentum transfers and nontrivial Clebsch-Gordan coefficients. In addition, it will also be important to better understand the celestial correspondence between the 1-1 scattering of a massless scalar field on a shockwave background and the 2-2 graviton mediated eikonal amplitude for massless external scalar fields. We expect this correspondence to hold for the near horizon eikonal amplitude upon considering a shockwave in the near horizon region.

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