Thermodynamics of the Einstein-Euler-Heisenberg rotating black hole

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We study the thermodynamic properties of a rotating electrically charged Einstein-Euler-Heisenberg black hole solution. This describes a QED generalization of the Kerr-Newman solution, which endows the vacuum with an effective dielectric constant. The QED-induced modifications to the thermodynamic quantities are presented and discussed. The Penrose process and the extraction of energy from the black hole at the event horizon are reviewed. We also analyze the Euler-Heisenberg nonlinear effects in the cases of supermassive, stellar, and primordial black holes. The evaporation of the black hole is analyzed as blackbody radiation.

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I. INTRODUCTION

Quantum electrodynamical (QED) vacuum corrections to the Maxwell-Lorentz theory, after one loop of quantization, can be accounted for by an effective nonlinear theory of the electromagnetic field derived by Euler and Heisenberg [1,2], using the Dirac electron-positron theory. Schwinger reformulated this result within a gauge-invariant formulation of QED [3]. When the electric fields are stronger than the critical value, $D_c \equiv m^2c^3/(e\hbar)$, spontaneous electron-positron pair production takes place, lowering the vacuum energy. The vacuum is treated as a specific type of medium, the polarizability and magnetizability properties of which are determined by clouds of virtual charges surrounding the real currents and charges [4]. This effect can be interpreted as an effective dielectric constant of the vacuum, and the Euler-Heisenberg (EH) effective Lagrangian is only valid for constant fields. This theory is a valid physical theory [5].

Ruffini *et al.* [6] considered Einstein-Euler-Heisenberg (EEH) static spherically symmetric black hole solutions endowed with electric, magnetic monopole, and dyonic charges, and they reduced the solutions to screened Reissner-Nordström (RN) ones. The nonlinear effects act only in the screening of the electric charge generating clouds of virtual charges surrounding the real charges and currents and affect the geometry only through the screened values of the real charges, i.e., these are QED corrections to the black hole horizon, the thermodynamic quantities, entropy, total energy, and maximally extractable energy. Therefore, the Euler-Heisenberg theory is considered as a screened Maxwell theory [7–11]. Another viewpoint is the work of Yajima *et al.* [12], who obtained, numerically or analytically, electrically, magnetically, and dyonically charged static black hole solutions. They treated the nonlinearity parameters as free parameters and studied the effective Euler-Heisenberg Lagrangian as a low-energy limit of the Born-Infeld theory. This standard way to consider the nonlinear contribution of the Euler-Heisenberg electrodynamics adds a term Q^4/r^6 in the mass-energy function $f(r)$ of the Maxwell linear electrodynamics, as can be seen in Refs. [12–19], which cannot be interpreted as screened Maxwell solutions. This additional term modifies the geometrical structure and the thermodynamics of the Maxwell theory [20–22].

On the other hand, Plebański introduced a class of nonlinear electrodynamic (NLED) theories [23], which contains EH as a special case [4]. In the framework of the EEH theory, Amaro *et al.* [7] derived an electrically charged static black hole solution in terms of the Plebański dual variables, and its shadow was also studied. Furthermore, Breton *et al.* [9] followed the QED interpretation of Ruffini *et al.* [6] for obtaining a screened Kerr-Newman (KN) black hole solution. They used the ansatz for a Kerr-like metric and the electromagnetic Plebański dual variables, considered symmetries for Petrov type-D metrics, and solved the Einstein equations. The nonlinearity introduces virtual charges, which lead to a screening of the real charges, not directly affecting the geometry of the underlying space-time.

Furthermore, the mechanism for the extraction of rotational energy from a Kerr black hole was first proposed in

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1971 by Penrose *et al.* [24], the so-called Penrose process. Due to the existence of the ergoregion, particles with a negative axial component of the angular momentum could be absorbed by the black hole along a negative energy orbit, which would reduce the angular momentum of the hole. In the meantime, particles will carry the reduced angular momentum away from the ergoregion and propagate to infinity. A stationary observer at infinity can regard this total process as the black hole doing work on the outgoing particles. The mass of the black hole decreases in this process, but such a decrease in the mass is not indefinite. Christodoulou showed that only up to around 29% of the mass of an extreme Kerr black hole could be extracted as rotational energy [25]. Additionally, electromagnetic energy can also be extracted from a black hole [26]. One could extract up to 50% of the mass of a hypothetical extreme RN black hole. Moreover, for charged black holes, the generalized ergoregion includes the electromagnetic contribution of the charges of the black hole and the captured particle [27,28].

The Penrose process allowed an analogy between the black hole mechanics and thermodynamics. The four laws of black hole thermodynamics were proposed in 1973 by Bardeen *et al.* [29]. In this analogy, the mass of the black hole corresponds to the energy of a thermodynamic system, the surface gravity to the temperature, and the area of the event horizon to the entropy, as previously proposed by Bekenstein [30–32]. Additionally, in 1975 Hawking [33] published the semiclassical derivation of black hole radiation.

The gravitational collapse of a star to a KN black hole, with all of the aspects of nuclear physics and electrodynamics involved, is a complex problem in astrophysics [34]. Astrophysical black holes are more likely to be neutral, but during gravitational collapse a process of charge separation is expected when the gravitational energy of the collapsing core is transformed into electromagnetic energy, and eventually the creation of electron-positron pairs by vacuum polarization. Such QED effects have been studied by Ruffini *et al.* [35,36] in the powering mechanism of gamma-ray bursts by means of black hole energy extraction. The black hole solution, which takes into account such vacuum polarization effects, is the EEH one. In this paper, we consider the EEH rotating black hole as a thermodynamic system, which incorporates EH corrections to the thermodynamic properties of the KN black hole.

The outline of the paper is as follows. In Sec. II an EEH rotating electrically charged black hole solution is revisited. In Sec. III the Penrose process and the extraction of energy from the black hole at the event horizon are reviewed. In Sec. IV we explore the black hole thermodynamics. In Sec. V we analyze the case of supermassive, stellar, and primordial black holes. In Sec. VI the summary and conclusions of the work are presented. In Appendix A the evaporation of the black hole is analyzed as blackbody radiation. In Appendix B the hypothetical θ dependence is considered.

II. ROTATING EINSTEIN-EULER-HEISENBERG BLACK HOLE

Using the ansatz for a Kerr-like space-time with the EH NLED as a source, Bretón *et al.* [9] derived a rotating electrically charged EEH black hole space-time. The resulting solution to the Einstein field equations appears as a screened KN solution, which in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 = & -\left(1-\frac{2Mr-\tilde{Q}^2}{\Sigma}\right)dt^2+\frac{\Sigma}{\Delta}dr^2 \\ & -\frac{(2Mr-\tilde{Q}^2)2a\sin^2\theta}{\Sigma}dtd\phi+\Sigma d\theta^2 \\ & +\left(r^2+a^2+\frac{(2Mr-\tilde{Q}^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \\ \Sigma = & \, r^2+a^2\cos^2\theta, \\ \Delta = & \, r^2-2Mr+a^2+\tilde{Q}^2=(r-r_+)(r-r_-). \end{aligned} \tag{1}$$

It is characterized by three parameters: the mass M, the angular momentum a, and the screened electric charge $\tilde{Q}$, given by

$$\begin{aligned} \tilde{Q}^2 = Q^2\Bigg\{1-\frac{5\alpha}{225\pi}\Bigg[D_Q^2-4H_Q^2\left(1-\frac{a^2\cos^2\theta}{\Sigma}\right) \\ \times\left(7-12\frac{a^2\cos^2\theta}{\Sigma}+12\frac{a^4\cos^4\theta}{\Sigma^2}\right)\Bigg]\Bigg\}, \end{aligned} \tag{2}$$

where the square of the radial components of the electromagnetic fields read

$$D_Q^2=\frac{Q^2}{\Sigma^2 D_c^2}, \qquad H_Q^2=\frac{\mathcal{M}^2\cos^2\theta}{\Sigma^3 D_c^2}. \tag{3}$$

Both fields D_Q and H_Q depend on the charge of the black hole Q and critical field $D_c \equiv m^2c^3/(e\hbar)$ at which electron-positron pairs are created. The magnetic field H_Q is induced by the rotation a and charge Q of the black hole since it arises from the induced magnetic moment $\mathcal{M}=Qa$. Additionally, the screened induced magnetic dipole moment now reads $\tilde{\mathcal{M}}=\tilde{Q}a$. It is important to note that both fields D_Q and H_Q would also depend on r and θ through the metric function $\Sigma=r^2+a^2\cos^2\theta$. However, as mentioned above, the EH effective Lagrangian is only valid for constant fields. Therefore, according to the QED interpretation of the EH NLED [6], the vacuum polarization acts as clouds of virtual charges surrounding the real electric charge and thus the rotationally induced magnetic moment and affecting the geometry only through the screened values of the real charges, as it happens in flat space-time [34]. Therefore, the effects of the vacuum polarization are nearly constant and affect only the electric charge of the EH NLED. This means that $\tilde{Q}(r,\theta)$ [Eq. (2)] has to be evaluated at constants r and θ, depending on the point at which one aims to analyze the effect of the vacuum polarization on the charge of the black hole. For example, when analyzing black hole thermodynamics, one analyzes the effects near the event horizon; or, when studying the trajectories of test particles, one can restrict the motion to the equatorial plane.

Throughout the article, we consider the EEH black hole as a thermodynamic system affected by captured matter at the equatorial plane, i.e., we consider $\tilde{Q}$ as evaluated at the event horizon radius of the KN black hole,

$$R_{\mathrm{H}} = M + \sqrt{M^2 - a^2 - Q^2}, \tag{4}$$

and perpendicular to the rotation axis, i.e., $\theta = \pi/2$. Then, the screened charge reads

$$\tilde{Q} = Q\left\{1 - \frac{\alpha}{90\pi}\frac{Q^2}{R_{\mathrm{H}}^4 D_c^2}\right\}, \tag{5}$$

which only depends on the black hole parameters a, Q, and M, the fine-structure constant α, and the critical field D_c. In this case, the charge (5) is always screened, i.e., $\tilde{Q} < Q$. As a or Q increases, R_{H} decreases, and then the EH effect grows since it is proportional to R_{H}^{-4}. Figure 1 presents the screened charge $\tilde{Q}$ as a function of the angular momentum a of the black hole, while Fig. 2 displays $\tilde{Q}$ as a function of its real charge Q.

The size of the EH effects also depends on the mass M of the black hole. It reads $\frac{\alpha}{90\pi D_c^2}\frac{(Q/M)^2}{(R_{\mathrm{H}}/M)^4}\frac{1}{M^2}$, with $1 \le R_{\mathrm{H}}/M \le 2$, and where the charge is restricted by the event horizon condition, $0 \le |Q|/M \le 1$. The remaining factor $1/M^2$ is responsible for the size of the quantum

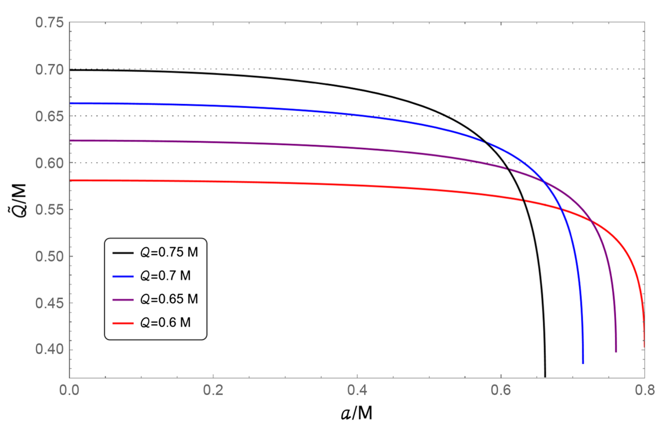

FIG. 1. Screened charge $\tilde{Q}$ (5) as a function of the angular momentum a for different values of the charge Q and a fixed mass, $M = 1 \times 10^4 M_{\odot}$.

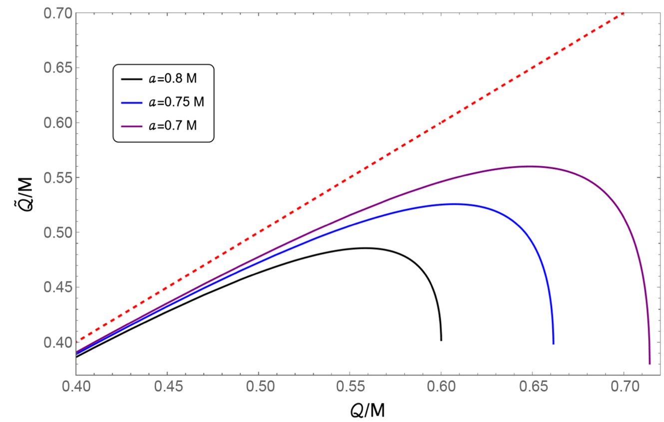

FIG. 2. Screened charge $\tilde{Q}$ (5) as a function of the real charge Q for different values of the angular momentum a and a fixed mass, $M = 1 \times 10^4 M_{\odot}$. The dashed line corresponds to the KN case.

corrections. Bigger masses and bigger charges are required in order to visualize the EH effects. The geometric properties of the black hole, like the event horizon, as well as the thermodynamics are modified by the presence of clouds of virtual charges surrounding real charges of the black hole.

A. The event horizon

The event horizon r_+ and inner horizon r_- of the EEH rotating black hole are obtained from the condition $\Delta = 0$, i.e.,

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - \tilde{Q}^2}. \tag{6}$$

The event horizon of the KN black hole is a sphere with radius R_{H} and its size is modified by the screening of its charge to a bigger radius $r_+ \ge R_{\mathrm{H}}$ (see Fig. 3). Figure 4 displays r_+ as a function of the black hole real charge Q and angular momentum a.

B. Event horizon area

The area of the event horizon hypersurface, A, with $t = \mathrm{const}$ and $r = r_+$, reads

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{\pi}\sqrt{|g_{\theta\theta}g_{\phi\phi}|}\,d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi}(r_+^2 + a^2)\sin\theta d\theta d\phi = 4\pi(r_+^2 + a^2) \\ &= 4\pi\left[2M(M + \sqrt{M^2 - a^2 - \tilde{Q}^2}) - \tilde{Q}^2\right], \end{aligned} \tag{7}$$

which can be rewritten, up to first order in α, as

$$A = A^{\mathrm{KN}} + \frac{4\alpha}{45D_c^2}\frac{Q^4}{R_{\mathrm{H}}^3}\left(\frac{1}{\sqrt{M^2 - a^2 - Q^2}}\right), \tag{8}$$

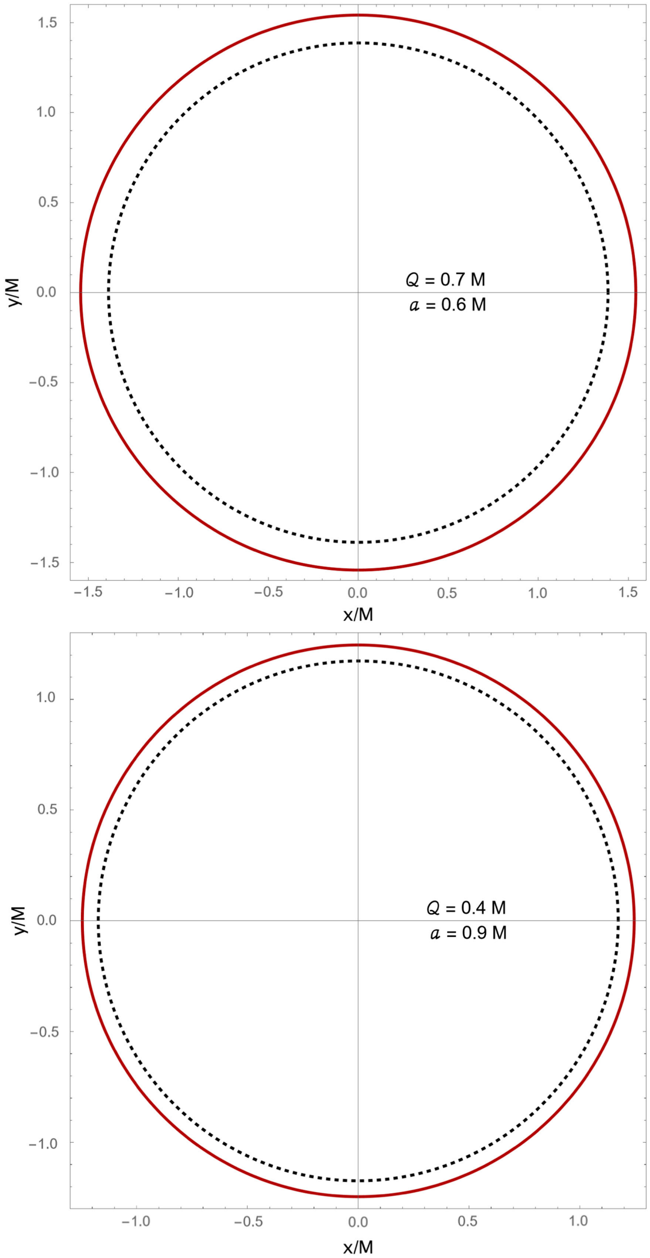

FIG. 3. EEH event horizon for different values of the black hole parameters and a fixed mass, $M = 1 \times 10^4 M_{\odot}$. The dashed line corresponds to the KN case and the continuous line to the EEH one. The horizon is enlarged.

with the event horizon area of the KN black hole $A^{\mathrm{KN}} = 4\pi(R_{\mathrm{H}}^2 + a^2)$. The values of the parameters M, a, and Q vary if the black hole absorbs particles with mass, angular momentum, or charge and, consequently, the area A varies. In this case, the area increases due to the EH effect. In particular, as Q increases, the area grows. The extreme case $M^2 = a^2 + Q^2$ is restricted.

Throughout the article, the quantities corresponding to the KN case are labeled with superscripts KN, while those corresponding to the EEH case have no labels.

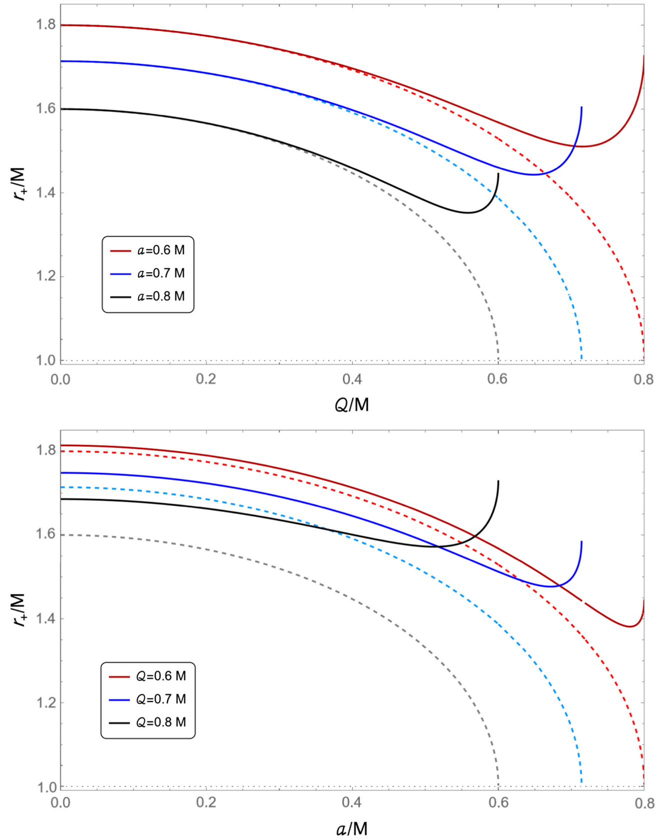

FIG. 4. Event horizon radius r_+ (6) of the EEH rotating black hole (continuous line) compared to the KN one (dashed line) as a function of the real charge of the black hole Q and the angular momentum a and with a fixed mass, $M = 1 \times 10^4 M_{\odot}$.

C. Surface gravity

The surface gravity of a general black hole is not well defined. However, one can define the surface gravity for a black hole whose event horizon is a Killing horizon. The surface gravity κ_+ of a static Killing horizon is the acceleration, as exerted at infinity, needed to keep an object at the event horizon. Mathematically, if a Killing vector k^a is normalized $k^a k_a = -1$, then the surface gravity is defined by

$$k^a \nabla_a k^b = \kappa_+ k^b, \tag{9}$$

where the equation is evaluated at the event horizon.

For a static and asymptotically flat space-time like, for instance, the Schwarzschild solution, k^a is a time translation Killing vector, which becomes null at the horizon, i.e., $k^a \partial_a = \frac{\partial}{\partial t}$. A stationary space-time is axially symmetric with a rotational Killing vector field $k^b \partial_b = \frac{\partial}{\partial \phi}$ in addition to the time translation one. Then, $(k^a \partial_a)$ will be a linear combination of both,

$$(k^a\partial_a)_\pm = \frac{\partial}{\partial t} \pm \Omega\frac{\partial}{\partial\phi}. \tag{10}$$

The linear combination of the time-translation and axial-symmetry Killing vectors is null at the horizon, where Ω is the angular velocity of the black hole, which is constant at the event horizon and is given by (16).

Therefore, the surface gravity, at the event horizon, for a screened KN black hole is

$$\kappa_+ = \frac{r_+ - r_-}{2(r_+^2+a^2)} = \frac{\sqrt{M^2-a^2-\tilde{Q}^2}}{2M^2-\tilde{Q}^2+2M\sqrt{M^2-a^2-\tilde{Q}^2}}, \tag{11}$$

which up to first order in α can be rewritten as

$$\kappa_+ = \kappa_+^{\mathrm{KN}} + \frac{\alpha}{90\pi}\frac{(2a^2+Q^2)}{D_c^2(R_\mathrm{H}^2+a^2)^2}\left(\frac{Q^4}{R_\mathrm{H}^4\sqrt{M^2-a^2-Q^2}}\right), \tag{12}$$

where $\kappa_+^{\mathrm{KN}} = \sqrt{M^2-a^2-Q^2}/(R_\mathrm{H}^2+a^2)$ is the surface gravity of the KN black hole. The extreme case is once again restricted.

D. Ergoregion and other properties

The ergoregion is the region $r_+ < r < r_{\mathrm{st}}$ between the event horizon and the static limit surface r_{st}, defined as

$$r_{\mathrm{st}} = M + \sqrt{M^2 - \tilde{Q}^2 - a^2\cos^2\theta}. \tag{13}$$

Figure 5 shows the EH effect on the ergoregion of the black hole. The static limit surface is enlarged, as it is the event horizon. Nevertheless, since the EH effect is bigger at r_+ than at r_{st}, the ergoregion of the EEH black hole is smaller than that of the KN black hole. Consequently, less energy could be extracted from the EEH black hole.

Additional quantities are the mass-energy parameter

$$M_E = \frac{r_+^2+a^2}{2r_+} = \frac{\left(2M\left[M+\sqrt{M^2-a^2-\tilde{Q}^2}\right]-\tilde{Q}^2\right)}{2\left(M+\sqrt{M^2-a^2-\tilde{Q}^2}\right)}, \tag{14}$$

the angular momentum of the black hole

$$J = a\left[\frac{r_+^2+a^2}{2r_+}\right] = \frac{a\left(2M\left[M+\sqrt{M^2-a^2-\tilde{Q}^2}\right]-\tilde{Q}^2\right)}{2\left(M+\sqrt{M^2-a^2-\tilde{Q}^2}\right)}, \tag{15}$$

the angular velocity of the event horizon

$$\Omega_{r_+} = \frac{a}{r_+^2+a^2} = \frac{a}{2M^2+2M\sqrt{M^2-a^2-\tilde{Q}^2}-\tilde{Q}^2}, \tag{16}$$

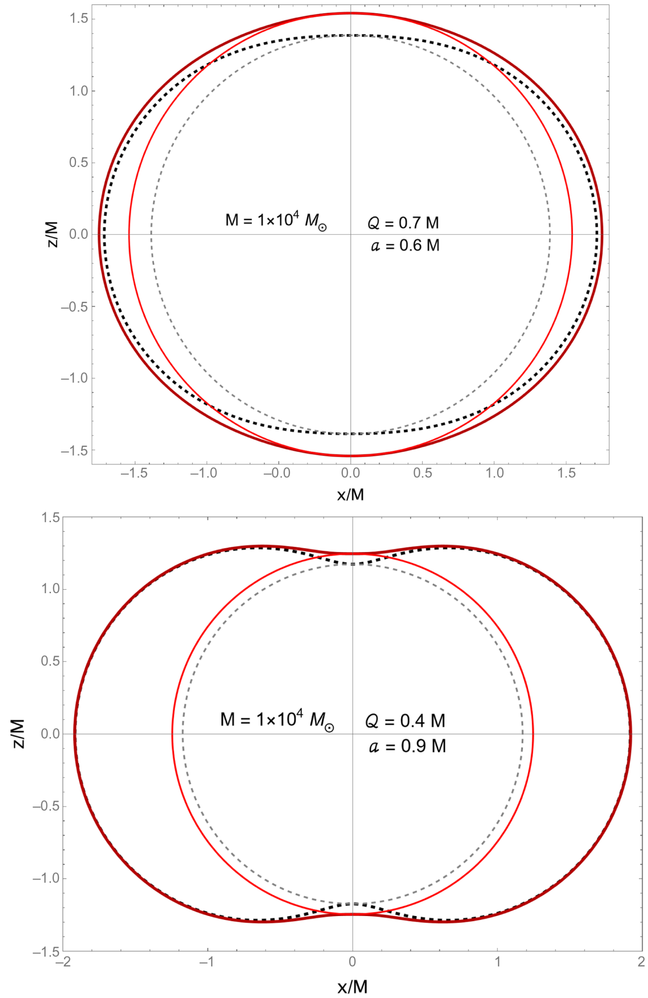

FIG. 5. EEH ergoregion (continuous line) compared to the KN one (dashed line) for different values of the black hole parameters Q and a and for a fixed $M = 1\times10^4 M_\odot$. The area of the KN ergoregion is bigger than the EEH one and thus more energy could be extracted from a KN black hole.

and the black hole's electric potential at the event horizon

$$\Phi_{r_+} = \frac{\tilde{Q}r_+}{r_+^2+a^2}. \tag{17}$$

III. THE PENROSE PROCESS AND ENERGY EXTRACTION

Penrose *et al.* [24] first suggested that rotational energy can be extracted from a rotating black hole. Christodoulou *et al.* [26] considered energy extraction from a static charged black hole and showed that electromagnetic energy can also be extracted. Energy extraction by the Penrose process can be described as follows [37]. (i) Consider a rotating black hole and a test particle A dropping from infinity with energy E_A, charge q_A, and axial component of the angular momentum L_A, and arriving following a geodesic at a point (r, θ) very close to the event horizon r_+. (ii) At this point, it "splits" into two particles, $A \to B + C$, with the corresponding parameters (E_B, q_B, L_B) and (E_C, q_C, L_C). The particle B crosses the event horizon, while C escapes to infinity. (iii) The captured particle B will then change the black hole mass, charge, and intrinsic angular momentum.

The equations of motion for a charged test particle around an EEH rotating black hole were presented in [11]. The test particle constants of motion corresponding to the Killing vectors ∂_t and ∂_ϕ are, respectively, the energy E and the axial component of the angular momentum L, given by

$$E = -g_{tt}\dot{t} - g_{t\phi}\dot{\phi} + \tilde{q}\,\tilde{Q}\,r/\Sigma, \tag{18}$$

$$L = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi} + a\sin^2\theta\tilde{q}\,\tilde{Q}\,r/\Sigma, \tag{19}$$

with the screened test particle real charge (evaluated at the event horizon and the equatorial plane) given by

$$\tilde{q} = q\left\{1 - \frac{\alpha}{30\pi}\frac{Q^2}{R_H^4 D_c^2}\right\}. \tag{20}$$

Since the screening term depends only on the black hole parameters, which are the same for the three particles at the splitting moment, it is straightforward to verify that the conservation of the charge, $\tilde{q}_A = \tilde{q}_B + \tilde{q}_C$, still holds, i.e.,

$$\tilde{q}_A = (q_B + q_C)\left\{1 - \frac{\alpha}{30\pi}\frac{Q^2}{R_H^4 D_c^2}\right\} = \tilde{q}_B + \tilde{q}_C. \tag{21}$$

Additionally, after the splitting, the total 4-momentum is conserved, $p_A^\mu = p_B^\mu + p_C^\mu$. Hence, from (18) and (19), the energy and the axial component of the angular momentum are also conserved:

$$E_A = E_B + E_C, \quad L_A = L_B + L_C. \tag{22}$$

To an observer at infinity, the change in the black hole mass δM is equal to $\delta M = E_A - E_C$, where E_A is the energy of the particle, which is split near the event horizon, and E_C is the outgoing particle energy, both measured at infinity, and similarly for δJ and for $\delta\tilde{Q}$. From (21) and (22), one obtains

$$\delta M = E_B, \quad \delta J = L_B, \quad \delta\tilde{Q} = \tilde{q}_B. \tag{23}$$

The changes in mass, charge, and angular momentum are equal to the energy, charge, and axial component of angular momentum that the captured particle B carries inside the event horizon.

In a rotating black hole, the static limit surface r_{st} is defined by the condition $g_{tt} = 0$ and only coincides with the event horizon r_+ at the poles. In the region between the two surfaces, the Killing vector ∂_t becomes spacelike, i.e., in the ergoregion, where the energy E of an uncharged test particle can be negative. If the energy of the captured particle is negative, $E_B < 0$, then the escaping particle reaching infinity would have more energy than the original one, $E_C > E_A$, and the mass of the black hole would decrease, i.e., $\delta M < 0$. This is the Penrose process of energy extraction [24].

In the case of a charged test particle, the energy (18) also includes an electromagnetic contribution. Hence, the region reached by charged particles with $E < 0$ corresponds to a "generalized ergoregion" [27,28]. For charges $\tilde{q}$ and $\tilde{Q}$ with opposite sign, the generalized ergoregion is larger than the original one, while it is smaller if the signs of the test particle charge $\tilde{q}$ and black hole charge $\tilde{Q}$ are the same.

If small objects are captured by the black hole, they would make infinitesimal changes in the black hole parameters, i.e., dM, $d\tilde{Q}$, and dJ. These changes are determined from the particle motion, analyzing the values of the energy-at-infinity E that allow the test particle to reach a point near the event horizon. The equations of motion for a charged test particle in the EEH rotating black hole space-time are of the same form as the ones for the KN case [11], with the real charges replaced by the screened charges (2) and (20),

$$(\Sigma\dot{\theta})^2 = K - [L - aE]^2 - [L^2\csc^2\theta + a^2(\mu^2 - E^2)]\cos^2\theta, \tag{24}$$

$$(\Sigma\dot{r})^2 = [(r^2 + a^2)E - aL - \tilde{q}\,\tilde{Q}\,r]^2 - \Delta(\mu^2 r^2 + K), \tag{25}$$

where K is the modified Carter constant and μ is the test particle mass. These equations depend on three constants of motion, and can be combined in order to obtain a quadratic polynomial in terms of E, L, and the velocities $\dot{r}$ and $\dot{\theta}$. Then, from (24) and (25), the quadratic polynomial, $\zeta E^2 - 2\chi E + \rho = 0$, reads

$$[(r^2 + a^2)^2 - \Delta a^2\sin^2\theta]E^2 - 2[(aL + \tilde{q}\,\tilde{Q}\,r)(r^2 + a^2) - aL\Delta]E + (aL + \tilde{q}\,\tilde{Q}\,r)^2 - \Delta L^2\csc^2\theta - \mu^2\Delta\Sigma - \Sigma^2[\dot{r}^2 + \Delta\dot{\theta}^2] = 0. \tag{26}$$

The solution corresponding to the 4-momentum pointing towards the future is the one with the positive square root [37], i.e.,

$$E = \frac{\chi + \sqrt{\chi^2 - \zeta\rho}}{\zeta}, \tag{27}$$

with

$$\zeta = [(r^2 + a^2)^2 - \Delta a^2\sin^2\theta] = (r^2 + a^2)\Sigma + (2Mr - \tilde{Q}^2)a^2\sin^2\theta, \tag{28}$$

$$\chi = (aL + \tilde{q}\,\tilde{Q}\,r)(r^2 + a^2) - aL\Delta = (2Mr - \tilde{Q}^2)aL + (r^2 + a^2)\tilde{q}\,\tilde{Q}\,r, \tag{29}$$

and

$$\rho = (aL + \tilde{q}\tilde{Q}r)^2 - \Delta L^2 \csc^2\theta - \mu^2 \Delta\Sigma - \Sigma^2[\dot{r}^2 + \Delta\dot{\theta}^2]. \tag{30}$$

In order for the energy extraction process to occur, E must be negative. Outside the event horizon, $\zeta > 0$; then, in order to have $E < 0$ in (27), the function χ must be negative, i.e., $\chi < 0$. This only happens for $aL < 0$ or $\tilde{q}\tilde{Q} < 0$, which would decrease the magnitude of the angular momentum and/or the charge of the black hole.

Additionally, the contribution of $\dot{r}$ and $\dot{\theta}$ to E is always positive. Then, from (30), the minimum energy required to extract energy from the black hole through a particle crossing the event horizon can only occur when $\dot{r} = \dot{\theta} = 0$. From the condition $\dot{r} = 0$ at r_+ in (25), the minimum energy is given by

$$E_{\min} = \frac{aL + \tilde{q}\tilde{Q}r_+}{(r_+^2 + a^2)}. \tag{31}$$

From (23), with $dM \geq E_{\min}$ and $J = aM$, one obtains the inequality

$$dM \geq \frac{a}{(r_+^2 + a^2)} dJ + \frac{\tilde{Q}r_+}{(r_+^2 + a^2)} d\tilde{Q}. \tag{32}$$

In the extreme case $r_+^2 = M^2 = a^2 + \tilde{Q}^2$, the inequality

$$M dM \geq a da + \tilde{Q} d\tilde{Q} \tag{33}$$

is satisfied. This is interpreted as the preservation of the event horizon, meaning that no naked singularity can be created by means of infinitesimal processes, and corresponds to the Penrose cosmic censorship conjecture.

Rotational and electromagnetic energies can be extracted from a rotating charged black hole, which would lead to a decrease in its mass. Nevertheless, such a decrease is not indefinite [25]. The irreducible mass of a rotating charged black hole is given by [26]

$$M_{\rm ir} \equiv \frac{1}{2}\sqrt{r_+^2 + a^2} \tag{34}$$

and is used to compute the amount of energy that can be extracted from a black hole. Since the irreducible mass of the EEH black hole, $M_{\rm ir}$, is larger than that of the KN one, $M_{\rm ir}^{\rm KN}$, i.e.,

$$M_{\rm ir} \equiv \frac{1}{2}\sqrt{r_+^2 + a^2} \geq \frac{1}{2}\sqrt{R_{\rm H}^2 + a^2} \equiv M_{\rm ir}^{\rm KN}, \tag{35}$$

less energy can be extracted from an EEH black hole, as concluded from the analysis of the respective ergoregions (see Fig. 5). Moreover, from (34) one obtains

$$\frac{(r_+ - r_-)}{2M_{\rm ir}} dM_{\rm ir} = dM - \frac{r_+\tilde{Q}}{r_+^2 + a^2} d\tilde{Q} - \frac{a}{r_+^2 + a^2} dJ = dM - \Phi_{r_+} d\tilde{Q} - \Omega_{r_+} dJ, \tag{36}$$

and using (32) one gets the inequality

$$dM_{\rm ir} \geq 0. \tag{37}$$

The irreducible mass of a black hole cannot be decreased by infinitesimal processes of matter injection. Processes in which the irreducible mass remains constant are said to be reversible. Additionally, rewriting (34), the mass-energy of the EEH black hole is given by

$$M^2 = \left(M_{\rm ir} + \frac{\tilde{Q}^2}{4M_{\rm ir}}\right)^2 + \frac{J^2}{4M_{\rm ir}^2}, \tag{38}$$

which includes the contribution of the rotational and electromagnetic energies to the total black hole mass-energy. On the one hand, up to 29% of the mass-energy of an extreme Kerr black hole ($Q = 0$) can be stored in its rotational energy term [25]. On the other hand, in the case of a charged black hole, since $\tilde{Q} < Q$ and $M_{\rm ir} > M_{\rm ir}^{\rm KN}$, both the electromagnetic term $\frac{\tilde{Q}^2}{4M_{\rm ir}}$ and the rotational term $\frac{J^2}{4M_{\rm ir}^2}$ are smaller than those of the KN case, and thus less rotational and electromagnetic energy can be stored. For $J = 0$ the static result in [6] is recovered.

In order to interpret the fact that there is less rotational and electromagnetic energy available to be extracted from the EEH black hole than there is in the KN case, one can review the pair production in the KN space-time. By considering that both the gravitational and electromagnetic background fields of the KN black hole are stationary, i.e., the quantum field of the electron and QED phenomena such as pair production, Damour *et al.* [38] obtained the rate of pair production around a KN black hole by locally applying the Schwinger formula [3,34]

$$\frac{\Gamma}{V} = \frac{\alpha}{4\pi^2} E_{(1)} B_{(1)} \sum_{n=1}^{\infty} \frac{1}{n} \coth\left(n\pi \frac{B_{(1)}}{E_{(1)}}\right) \exp\left(-n\pi \frac{E_c}{E_{(1)}}\right), \tag{39}$$

where Γ/V is the decay rate of the vacuum per unit volume in the electromagnetic field of the KN black hole, and $E_{(1)} = Q\Sigma^{-2}[r^2 - a^2\cos^2\theta]$ and $B_{(1)} = 2\mathcal{M}\Sigma^{-2} r\cos\theta$ are the components parallel to $\omega^{(1)}$ of the electric and induced magnetic fields in the local Lorentz frame of a stationary observer defined by the following tetrad [39]:

$$\omega^{(0)} = \sqrt{\frac{\Delta}{\Sigma}}[dt - a\sin^2\theta d\phi],$$
$$\omega^{(1)} = \sqrt{\frac{\Sigma}{\Delta}}dr,$$
$$\omega^{(2)} = \frac{1}{\sqrt{\Sigma}},$$
$$\omega^{(3)} = \frac{\sin\theta}{\sqrt{\Sigma}}[(r^2 + a^2)d\phi - adt]. \tag{40}$$

Furthermore, the number of pairs produced in a region $\mathcal{D}$ of the KN space-time,

$$\mathcal{N} = \int_{\mathcal{D}} d^4x\sqrt{-g}\frac{\Gamma}{V}, \tag{41}$$

grows as either the charge or angular momentum increases. The bigger the values of Q and a, the more pairs are produced, and thus the EEH effects become more visible, i.e., less stored rotational or electromagnetic energy is available. This can be interpreted as part of the stored energy being used for the pair production.

IV. BLACK HOLE THERMODYNAMICS

The Penrose process of energy extraction and the works of Christodoulou [25,26] allowed an analogy between black hole mechanics and thermodynamics. On the one hand, from (7) and (34), the black hole area A in terms of the irreducible mass M_{ir} reads

$$A = 16\pi M_{\mathrm{ir}}^2, \tag{42}$$

and the inequality (37) becomes $dA \geq 0$. On the other hand, Bekenstein [30,32] studied the entropy of a black hole, considering it as a thermodynamic system. Near equilibrium, a thermodynamic system at temperature T changes its state, and the consequent increments of its energy and entropy are related by the first law of thermodynamics [31]:

$$TdS = dE - dW, \tag{43}$$

where dW is the work done on the system by external agents. The change dM in the black hole's mass is the change dE in its energy. Since an external agent increases the black hole's charge by $d\tilde{Q}$ and its angular momentum by dJ, the term $\frac{r_+\tilde{Q}}{r_+^2+a^2}d\tilde{Q} + \frac{a}{r_+^2+a^2}dJ$ in (36) represents the work dW done on the system. Using (42), and comparing with (43), one can obtain the relation $TdS = \frac{(r_+-r_-)}{4A}dA = \frac{\kappa_+}{8\pi}dA$, with the surface gravity (11).

In 1973, Bardeen, Carter, and Hawking formulated the four laws of classical black hole mechanics [29], which are analogous to the four laws of thermodynamics. The proposed four laws of black hole thermodynamics can be summarized as follows:

(1) The zeroth law of black hole mechanics states that the surface gravity κ_+ of a stationary black hole is constant over the event horizon, which is essentially the requirement of transitivity of the equilibrium state.

(2) The first law manifests a relation between variations of the mass M, event horizon area A, angular momentum J, and screened electric charge $\tilde{Q}$ if the black hole is perturbed,

$$dM = \frac{\kappa_+}{8\pi}dA + \Omega_{r_+}dJ + \Phi_{r_+}d\tilde{Q}, \tag{44}$$

where κ_+ is the surface gravity (11), Ω_{r_+} is the angular velocity of the event horizon (16), and Φ_{r_+} is the electrostatic potential (17) of the black hole at its event horizon.

(3) The second law of black hole mechanics is Hawking's area theorem, which states that the surface area of the event horizon never decreases with time,

$$dA \geq 0. \tag{45}$$

(4) The third law is formulated by stating that it is impossible to achieve $\kappa_+ = 0$ in a finite series of physical processes.

The close analogy between the four laws of black hole mechanics and the laws of ordinary thermodynamics is very interesting. In this analogy, the mass of the black hole mathematically corresponds to the energy of a thermodynamic system, the area of the horizon to the entropy, and the surface gravity to the temperature. While the correspondence between mass and energy is a physical identity, the other two correspondences are only analogies in classical general relativity. Classical black holes have zero temperature, and the area of the event horizon has the dimension of length squared. With the physical interpretation of the black hole entropy by Bekenstein [31] and the semiclassical derivation of black hole radiation by Hawking [33], we can treat black holes as thermodynamic systems with the Bekenstein-Hawking entropy

$$S_{\mathrm{BH}} = \frac{A}{4} = \pi(r_+^2 + a^2) = \frac{r_+ - r_-}{4T_{\mathrm{H}}} \tag{46}$$

and the Hawking temperature

$$T_{\mathrm{H}} = \frac{\kappa_+}{2\pi} = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}. \tag{47}$$

Investigations in gravitating systems have pointed out that entropy functions with a nonextensive nature tend to appear in systems with long-range interactions like gravity (e.g., [40]). The Bekenstein-Hawking black hole entropy

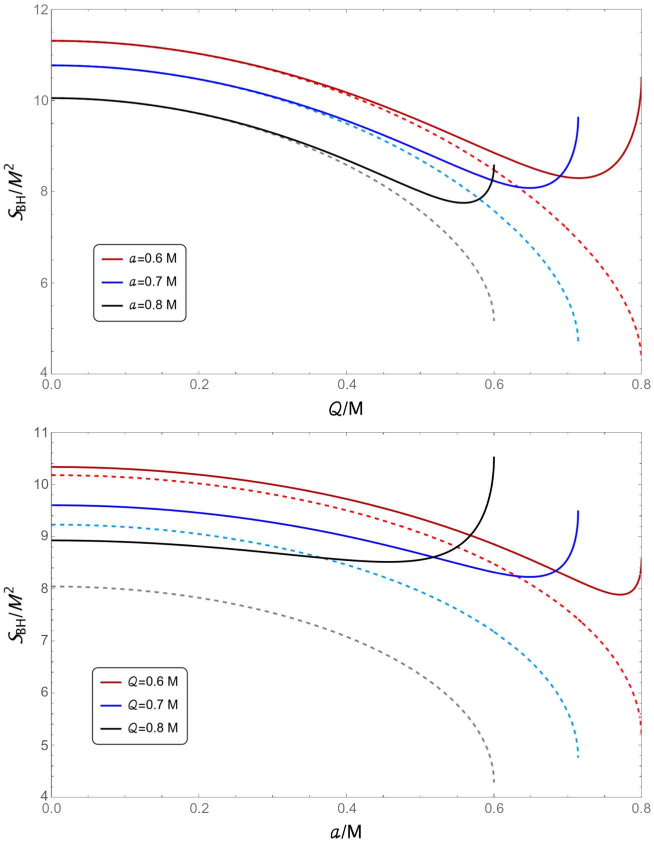

FIG. 6. Entropy of the EEH black hole as a function of the black hole parameters and for a fixed mass, $M = 1 \times 10^4 M_{\odot}$. The dashed line corresponds to the KN case and the continuous line to the EEH one.

also seems to be nonextensive since it is proportional to the area of the event horizon instead of being proportional to its volume. It is important to mention that the calculation of the black hole volume and its relation with the corresponding event horizon area constitutes a nontrivial issue [41].

The entropy of the EEH rotating black hole is given by

$$S_{\rm BH} = S_{\rm BH}^{\rm KN} + \frac{\alpha}{45 D_c^2} \frac{Q^4}{R_{\rm H}^3} \left(\frac{1}{\sqrt{M^2 - a^2 - Q^2}} \right), \quad (48)$$

with $S_{\rm BH}^{\rm KN} = \pi(R_{\rm H}^2 + a^2)$. Figure 6 displays the Bekenstein-Hawking entropy (46) as a function of the charge and angular momentum of the black hole. The black hole parameters satisfy the event horizon condition, $a^2 + \tilde{Q}^2 < M^2$, where the extreme case is forbidden according to the third law. The entropy of the EEH black hole is bigger than that of the KN one, which would imply that the EEH case corresponds to a more probable state. The EH effect on the entropy is bigger for large values of Q and a.

In order to interpret this behavior, one can consider absorbed charged particles that enter the black hole at the equatorial plane, modifying the thermodynamics quantities. Let us consider a charging process of the black hole, where Q increases due to the captured particles having the same-sign charge. According to the lhs of Fig. 6, in the KN case the entropy decreases as the charge increases. For the entropy to decrease, work has to be done on the hole by an external agent, such as in-going matter in accretion processes. Only in the EEH case, the entropy begins to increase when Q reaches a certain value. At this point, the EEH effect becomes very relevant and, since going from a low-entropy state to a higher-entropy one implies the possibility of obtaining energy from the system, one could suggest that virtual pairs would be produced at a faster rate, in accordance with (39).

On the other hand, in a discharging process Q decreases due to opposite-sign charges crossing the event horizon, and the entropy increases in both the KN case and the EEH one. The entropy of a system on which no work is done increases. In a discharging process, opposite-sign charges are additionally attracted by the electric force, and then no work has to be done against a repulsive electric force. Nevertheless, for the EEH case the entropy first has to decrease if near-extreme values of the black hole parameters were previously reached. Some work would first have to be done in order to lower the high pair production. This would be the case if a previous large charging process took place or when a charged black hole is the remnant of a charged collapsed star.

Figure 7 displays the entropy as a function of the mass of the black hole. The limit when mass tends to zero corresponds to black hole evaporation. In the KN case, the entropy tends to zero as the mass becomes smaller. In the EEH case, the entropy tends to a value different from zero. This is due to the fact that we are considering the first-order EH correction to the entropy (48), which is different from zero in that limit. It is associated with the entropy of the vacuum polarization around the black hole. However, in order to achieve the limit $M \to 0$, one would first need to extract all of the rotational and electromagnetic energy of the black hole. Moreover, the irreducible mass cannot decrease by matter injection, and the black hole can then only be evaporated by Hawking radiation. The differences between the EH process and Hawking radiation are discussed in Appendix A.

A. Zeroth law

The zeroth law of black hole mechanics, i.e., the surface gravity is constant over the event horizon of a black hole, is analogous to the zeroth law of thermodynamics, which states that the temperature is constant throughout a body in thermal equilibrium. It suggests that the surface gravity is analogous to the temperature. A constant $T_{\rm H}$ in thermal equilibrium for a normal system is analogous to a constant surface gravity κ_+ over the event horizon of a stationary black hole.

There exist two independent versions of the zeroth law of black hole mechanics. The first one, owing to Carter [42,43], states that for any black hole that is stationary, axially

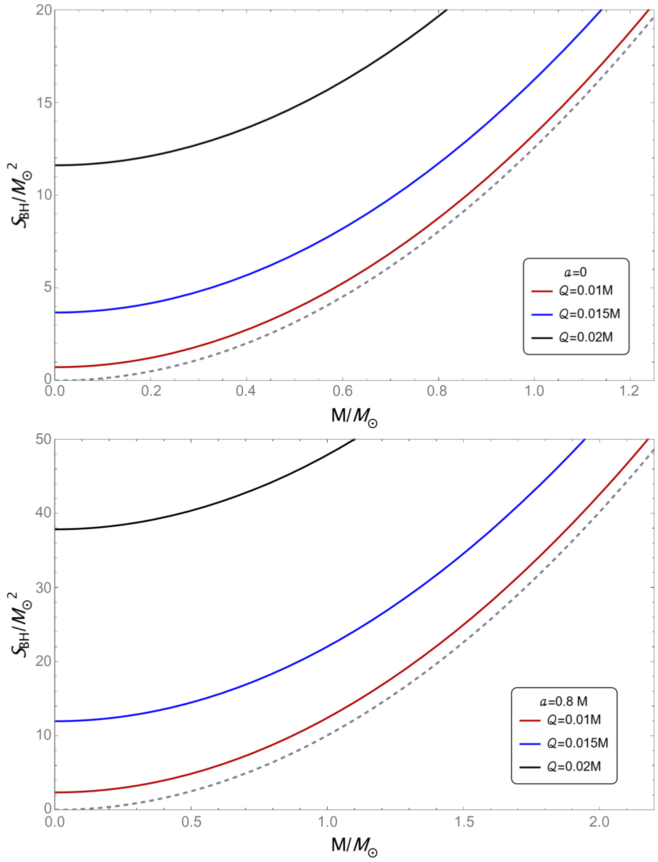

FIG. 7. Entropy of the EEH black hole as a function of the mass of the black hole, and for different values of the black hole parameters. The dashed line corresponds to the KN case and the continuous line to the EEH one.

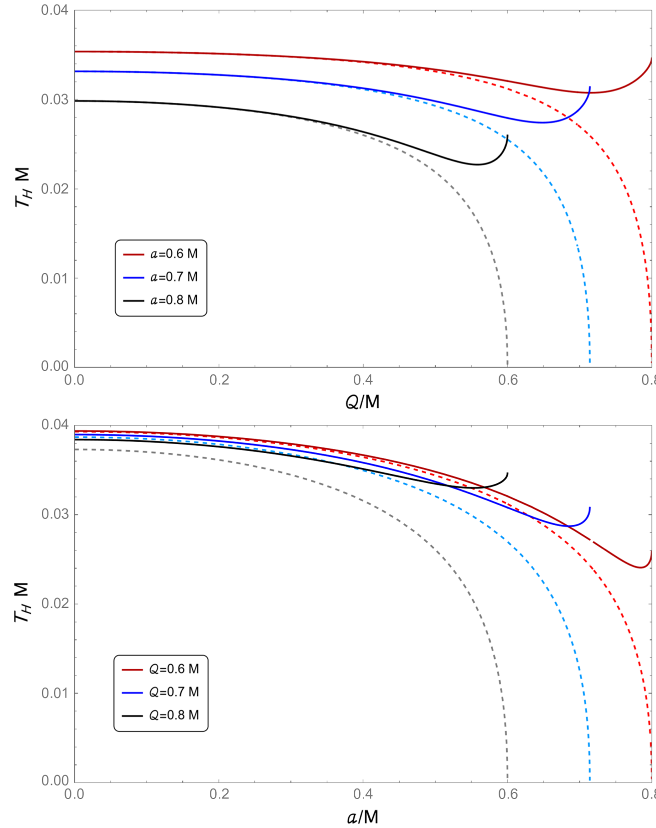

FIG. 8. Temperature of the EEH black hole as a function of the black hole parameters and for a fixed mass, $M = 1 \times 10^4 M_\odot$. The dashed line corresponds to the KN case and the continuous line to the EEH one.

symmetric with the $t - \phi$ orthogonality property, the surface gravity κ_+ must be constant over its event horizon, and this outcome is totally geometric since it does not involve field equations. The second one, by Bardeen, Carter, and Hawking [29], establishes that if Einstein's equation holds with the matter stress-energy tensor satisfying the dominant energy condition, then κ_+ must be constant on any Killing horizon. Here the $t - \phi$ orthogonality property is released.

Therefore, the zeroth law of black hole mechanics states that the surface gravity (9) of a stationary black hole is constant over the event horizon, which is essentially the requirement of transitivity of the equilibrium state. The zeroth law of black hole mechanics is an assertion of the constancy of the surface gravity on a stationary Killing horizon.

The Hawking temperature of the black hole event horizon is proportional to the surface gravity, i.e.,

$$T_{\rm H} = \frac{1}{2\pi}\kappa_+ = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, = \frac{\sqrt{M^2 - a^2 - \tilde{Q}^2}}{2\pi\left(2M^2 - \tilde{Q}^2 + 2M\sqrt{M^2 - a^2 - \tilde{Q}^2}\right)}, \tag{49}$$

which up to first order in α is given by

$$T_{\rm H} = T_{\rm H}^{\rm KN} + \frac{\alpha}{180\pi^2}\frac{(2a^2 + Q^2)}{D_c^2(R_{\rm H}^2 + a^2)^2}\frac{Q^4}{R_{\rm H}^4\sqrt{M^2 - a^2 - Q^2}}, \tag{50}$$

with $T_{\rm H}^{\rm KN} = \sqrt{M^2 - a^2 - Q^2}/[2\pi(R_{\rm H}^2 + a^2)]$. From (46), the temperature can be written in terms of the entropy as $T_{\rm H} = (r_+ - r_-)/(4S_{\rm BH})$. The Hawking temperature is presented in Fig. 8 as a function of the charge and angular momentum, which satisfy the event horizon condition. The temperature of the KN black hole decays as Q or a increase. Only for the EEH case does the temperature start to increase at some point as the pair production intensifies. Figure 9 shows the temperature as a function of the mass M of the black hole. In this case, the temperature also tends to infinity as the black hole evaporates, which is called the explosion of the black hole. The latter occurs earlier than in the KN case, i.e., for a bigger value of the mass, since it can be verified that the EH term in (50) is proportional to M^{-3}, while the KN one is proportional to M^{-1}.

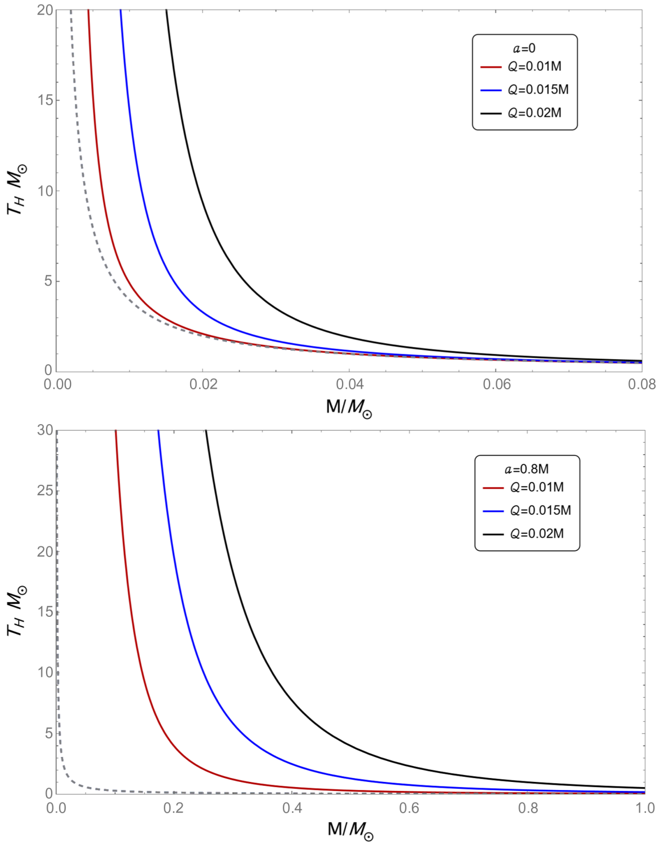

FIG. 9. Temperature of the EEH black hole as a function of the mass of the black hole and for different values of the black hole parameters. The dashed line corresponds to the KN case and the continuous line to the EEH one.

B. First law

The fundamental thermodynamic relation, which contains all of the information about the thermodynamic state of the black hole matter, was given by Smarr in 1973 [44],

$$M^2 = \frac{1}{4}\frac{A}{4\pi} + \frac{4\pi}{A}\left[J^2 + \frac{1}{4}\tilde{Q}^4\right] + \frac{1}{2}\tilde{Q}^2, \tag{51}$$

which is equivalent to (38). The parameters A, J, $\tilde{Q}$ are a complete set of quantities for the black hole system, $M = M(A, J, \tilde{Q})$. In ordinary thermodynamics, extensive parameters may be added together for composite systems. Nevertheless, (51) is not homogeneous of first order in A, J, and $\tilde{Q}$. Although the areas of two black holes are additive, the sum of the areas of two black holes of masses M_1 and M_2 is always less than the area of a black hole obtained by merging these two together, i.e., $(M_1^2 + M_2^2) \leq (M_1 + M_2)^2$. Moreover, it is possible to identify the event horizon area A with the entropy $S_{\rm BH}$:

$$S_{\rm BH} = \frac{1}{4} k_B A, \tag{52}$$

where k_B is the Boltzmann constant. Then, (51) in units in which $k_B = 1/8\pi$ can be rewritten as

$$M^2 = \left[2S_{\rm BH} + \frac{1}{8S_{\rm BH}}\left(J^2 + \frac{1}{4}\tilde{Q}^4\right) + \frac{1}{2}\tilde{Q}^2\right], \tag{53}$$

which is often used in its inverted form:

$$S_{\rm BH} = \frac{1}{4}M^2 - \frac{1}{8}\tilde{Q}^2 + \frac{1}{4}M^2\left(1 - \frac{J^2}{M^4} - \frac{\tilde{Q}^2}{M^2}\right)^{1/2}. \tag{54}$$

Equation (53) represents the fundamental thermodynamic relation $M = M(S_{\rm BH}, J, \tilde{Q})$ in terms of the parameters $S_{\rm BH}$, J, and $\tilde{Q}$.

The first law of thermodynamics states that the total energy in any thermodynamic system is conserved,

$$dE = T_{\rm H} dS_{\rm BH} + dW = T_{\rm H} dS_{\rm BH} + \Omega_{r_+} dJ + \Phi_{r_+} d\tilde{Q}, \tag{55}$$

where dE is the change of energy, corresponding to the change in the mass of the black hole, and dW is the work done on the system by exterior agents. When the system is one rotating with angular velocity Ω_{r_+} and charged up to electric potential Φ_{r_+}, the changes in its angular momentum J and charge $\tilde{Q}$ contribute to the work. If M changes infinitesimally by dM, then

$$dM = T_{\rm H} dS_{\rm BH} + \Omega_{r_+} dJ + \Phi_{r_+} d\tilde{Q}, \tag{56}$$

where $T_{\rm H}$, Ω_{r_+}, and Φ_{r_+} are given by

$$T_{\rm H} = \frac{\partial M}{\partial S_{\rm BH}} = M^{-1}\left[1 - \frac{J^2 + \frac{1}{4}\tilde{Q}^4}{16 S_{\rm BH}^2}\right], \tag{57}$$

$$\Omega_{r_+} = \frac{\partial M}{\partial J} = \frac{J}{8 M S_{\rm BH}}, \tag{58}$$

$$\Phi_{r_+} = \frac{\partial M}{\partial \tilde{Q}} = \frac{\tilde{Q}(\tilde{Q}^2 + 8S_{\rm BH})}{16 M S_{\rm BH}}. \tag{59}$$

These are the corresponding intensive parameters, constant everywhere on the event horizon. Ω_{r_+} is the angular velocity of the event horizon linked to the angular momentum J, Φ_{r_+} is the electric potential connected with the screened electric charge $\tilde{Q}$, and $T_{\rm H}$ is the temperature related to the entropy $S_{\rm BH}$.

If one considers $M = M(S_{\rm BH}, J, \tilde{Q}^2)$, then (53) is a homogeneous function of degree $1/2$ in these parameters. The Euler theorem requires

$$\frac{1}{2}M = T_{\rm H} S_{\rm BH} + \Omega_{r_+} J + \Theta \tilde{Q}^2 = T_{\rm H} S_{\rm BH} + \Omega_{r_+} J + \frac{1}{2}\Phi_{r_+}\tilde{Q}, \tag{60}$$

with the electric potential per unit charge

$$\Theta = \frac{\Phi_{r_+}}{2\tilde{Q}} = \frac{\tilde{Q}^2 + 8S_{\rm BH}}{32MS_{\rm BH}}. \qquad (61)$$

This relation was first introduced by Smarr [44] and constitutes the black hole version of the Gibbs-Duhem relation of ordinary thermodynamics. Differentiating (60) and using the first law (56), one obtains

$$-\frac{1}{2}dM = S_{\rm BH}dT_{\rm H} + Jd\Omega_{r_+} + \tilde{Q}^2 d\Theta, \qquad (62)$$

wherefrom

$$\left(\frac{\partial M}{\partial T_{\rm H}}\right)_{\Omega_{r_+},\Theta} = -2S_{\rm BH}, \qquad (63)$$

$$\left(\frac{\partial M}{\partial \Omega_{r_+}}\right)_{T_{\rm H},\Theta} = -2J, \qquad (64)$$

$$\left(\frac{\partial M}{\partial \Theta}\right)_{T_{\rm H},\Omega_{r_+}} = -2\tilde{Q}^2. \qquad (65)$$

C. Second law

In ordinary thermodynamics, the second law requires that the entropy of a closed system always increases. While this law may be satisfied by a system including a black hole, it is not quite instructive in its original form. For instance, if ordinary matter falls into a black hole, the ordinary entropy S_0 becomes invisible to an exterior observer, hence telling that ordinary entropy increases do not provide any insight. Therefore, one must include the black hole entropy in the total entropy in order to give a more useful law, the generalized second law of thermodynamics: the sum of the ordinary entropy outside black holes S_0 [45] and the total black hole entropy always increases [46],

$$\begin{aligned} dS_0 + dS_{\rm BH} = dS_0 + &\frac{2\pi}{\sqrt{M^2 - a^2 - \tilde{Q}^2}} \\ &\times \Big[\Big(\tilde{Q}^2 - 2M\sqrt{M^2 - a^2 - \tilde{Q}^2} - 2M^2\Big)dM \\ &+ \Big(M + \sqrt{M^2 - a^2 - \tilde{Q}^2}\Big)\tilde{Q}d\tilde{Q} + adJ\Big] \geq 0. \end{aligned} \qquad (66)$$

When matter entropy flows into a black hole, the generalized second law demands that the increase in black hole entropy shall do more than compensate for the disappearance of ordinary entropy from being observed.

D. Third law

To lead a system to absolute zero temperature requires an infinite number of steps. From (49), the black hole temperature vanishes when $M^2 - a^2 - \tilde{Q}^2 = 0$. KN-like black holes satisfying this condition are called extreme. For $T_{\rm H} = 0$, the black hole entropy is not only nonvanishing, but also depends on the angular momentum per unit mass a, an analogue of a thermodynamic intensive parameter directly related to the black hole angular velocity Ω_{r_+}. Hence, it is impossible to achieve $\kappa_+ = 0$ in a finite series of physical processes. Furthermore, it is impossible the existence of the case $M^2 = a^2 + \tilde{Q}^2$ equivalent to an extreme black hole, it has lost its two horizons, and the surface gravity κ_+, and therefore the expected Hawking temperature vanishes.

Surface gravity establishes a limit for the black hole, consistent with the third law of thermodynamics. From this, an expression is established for the Hawking temperature of a KN-like black hole as a function of its mass M, angular moment J, and screened charge $\tilde{Q}$. As the black hole loses mass, its temperature increases in an inversely proportional way. The temperature of the black hole decreases when the mass increases, and therefore it has a specific negative heat.

E. Heat capacity and phase transitions

One can rewrite (16) and (17) in terms of the Hawking temperature as follows:

$$\Omega_{r_+} = a\left(\frac{r_+ - r_-}{4\pi T_{\rm H}}\right), \qquad (67)$$

$$\Phi_{r_+} = \tilde{Q}r_+\left(\frac{r_+ - r_-}{4\pi T_{\rm H}}\right). \qquad (68)$$

This is a relation between increments in mechanical and geometrical properties. It turns out that Ω_{r_+} is precisely the angular velocity of the black hole event horizon, at which any test particle approaching it circumnavigates, and Φ_{r_+} is the black hole electric potential in the sense that it equals the line integral of the hole's electric field from infinity to any location on the horizon.

From the first law (56), the heat capacity at constant Q and M is given by

$$C_{\tilde{Q}} = T_{\rm H}\left(\frac{\partial S_{\rm BH}}{\partial T_{\rm H}}\right)_{\tilde{Q},M} = -\Omega_{r_+}\frac{\partial J}{\partial T_{\rm H}}, \qquad (69)$$

while the heat capacity at constant J and M is

$$C_J = T_{\rm H}\left(\frac{\partial S_{\rm BH}}{\partial T_{\rm H}}\right)_{J,M} = -\Phi_{r_+}\frac{\partial \tilde{Q}}{\partial T_{\rm H}}. \qquad (70)$$

For J or $\tilde{Q}$ held constant, it can be rewritten in terms of M, J, $\tilde{Q}$, T, and $S_{\rm BH}$ as

$$C_{J,\tilde{Q}} = T_{\rm H}\left(\frac{\partial S_{\rm BH}}{\partial T_{\rm H}}\right)_{J,\tilde{Q}} = \frac{8MS_{\rm BH}^3 T_{\rm H}}{J^2 + \tilde{Q}^4/4 - 8T_{\rm H}^2 S_{\rm BH}^3}. \quad (71)$$

The specific heat diverges when its denominator becomes zero. This corresponds to a phase transition of second order, which separates two black holes with different mass/horizon radii; there is a critical mass M_0 separating the two phases. Analyzing where the transition occurs, we numerically (see Fig. 10) derive that in the EEH black hole the phase transition occurs for bigger charges $\tilde{Q}$ than those for the KN black hole, and fixed parameters (M, a). Moreover, in the EEH black hole the phase transition occurs for bigger values of the angular momentum a and fixed parameters $(M, \tilde{Q})$. The determination of the critical mass M_0 that separates the two phases can be done from the expression of the entropy as a function of the temperature. One of the methods is to search for a turning point in the $(T_{\rm H}, S_{\rm BH})$ curve [47].

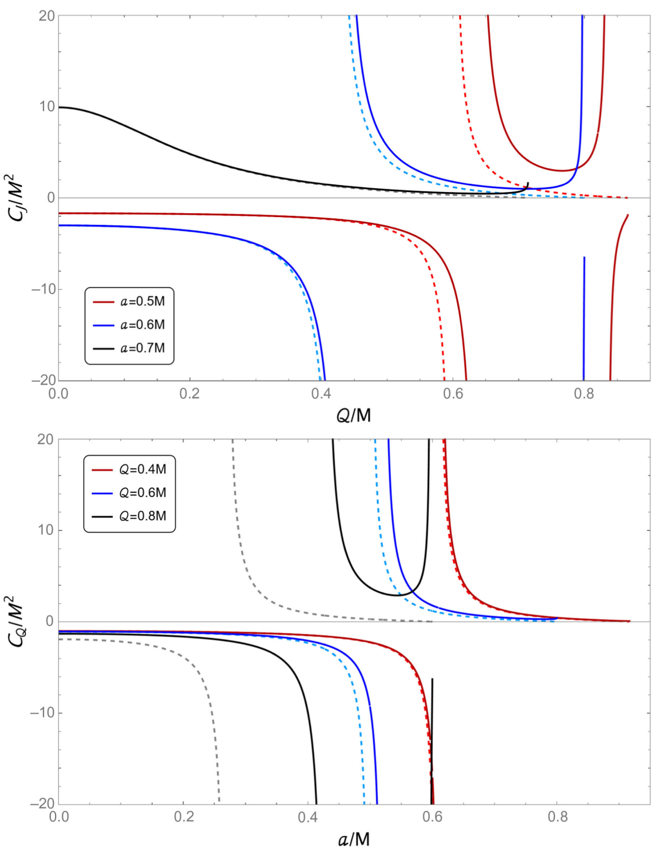

FIG. 10. Heat capacity (70) at constant $J = aM$ (top) and Q (bottom), for fixed $M = 1\times 10^4 M_\odot$. The dashed lines correspond to the KN case, where there is only one phase transition. The continuous lines correspond to the EEH case, in which an additional phase can occur for near-extreme values of the black hole parameters (see figure on the top).

The temperature at which the phase transition occurs can also be calculated by solving for $T_{\rm H}$ the denominator of the specific heat equaled to zero, obtaining

$$T_0 = \frac{M_0}{\pi(r_+^2 + a^2)}\left\{\sqrt{1 + \frac{2\pi(r_+^2 + a^2)}{M_0^2}} - 1\right\}, \quad (72)$$

which is lower than the one corresponding to the KN case. The critical mass in terms of T_0 and S_0 is given by

$$M_0 = \frac{1}{T_0} - \frac{1}{2}T_0 S_0. \quad (73)$$

For the EEH rotating BH, associated with the divergence of the heat capacities, there are cases of one or two phase transitions (see Fig. 10), and considering that a negative heat capacity is associated with instability and a positive heat capacity points to stability of the system, from the heat capacity at constant charge, the BH goes from instability to stability for charges in the interval $Q \leq 0.6$, while in the case of charges $Q \approx 0.8$ the system again becomes unstable. Contrasting conclusions come from the heat capacity at constant J, where for $a \geq 0.7$ the system is stable, $C_J > 0$, but for rotation in the range $0 \leq a \leq 0.6$ the system transits from instability to stability and then again to instability after its second phase transition.

A detailed study of the thermodynamics of the EEH BH through the behavior of the Gibbs free energy would require the introduction of the cosmological constant, which can be related to the thermodynamic pressure in a standard definition of the pressure as $p = -\Lambda/8\pi$; then, along with its conjugate variable, the volume v, a state equation could be derived and then it would be possible to carry out the analysis of the critical points, similarly to those performed for the static EEH cases in [15,16,22], where the metric functions included a cosmological constant as in [15,16], or with an "effective" cosmological constant as in [22].

V. SUPERMASSIVE, STELLAR, AND PRIMORDIAL BLACK HOLES

The charge of a black hole may be different from zero during the accretion of charged matter like plasma and during the gravitational collapse of a star to a KN black hole. In the latter conditions, processes of charge splitting and consequent pair production by vacuum polarization occur, when the gravitational energy of the collapsing core is transformed into electromagnetic energy and electron-positron pairs created by vacuum polarization [34]. Moreover, due to the conservation of angular momentum, it is expected that such rotating black holes will have big spin values. Black holes formed by collapsing stars are stellar black holes, with masses that are a few solar masses above the critical mass of neutron stars, $M > 3.2M_{\odot}$.

As mentioned above, the size of the EH effects depends on the mass of the black hole. The screened charge (5) includes the coefficient

$$\left(\frac{\alpha}{90\pi D_c^2 M^2}\right)\frac{(Q/M)^2}{(R_{\mathrm{H}}/M)^4}\approx 9.24\times 10^7\left(\frac{M_{\odot}}{M}\right)^2\frac{(Q/M)^2}{(R_{\mathrm{H}}/M)^4}, \tag{74}$$

with the bounded quantities $1 \leq R_{\mathrm{H}}/M \leq 2$ and $0 \leq |Q|/M \leq 1$. Then, the term $1/M^2$ is responsible for the size of the EH correction. In the previous examples, a mass of $M = 1\times 10^4 M_{\odot}$ was used, which results in the value $\sim 0.924(Q/M)^2/(R_{\mathrm{H}}/M)^4$. The more massive the black hole is, the bigger the charge needed to visualize the EH effects. Moreover, similar terms emerge in the EH modifications of the black hole properties, like the entropy (48) and temperature (50).

A. Supermassive black holes

The EH corrections are inversely proportional to the mass of the black hole. The biggest astrophysical black holes are supermassive black holes, with masses millions to billions times the mass of the Sun. For instance, Sagittarius A* has a mass of $M = 4.0\times 10^6 M_{\odot}$, while the mass of the black hole at the center of the galaxy M87 (we will call it M87*) is $M = 6.5\times 10^9 M_{\odot}$ [48]. Equation (74) for Sag A* is $\sim 5.774\times 10^{-6}(Q/M)^2/(R_{\mathrm{H}}/M)^4$. Choosing the biggest charge value $Q = M$, which corresponds to a static extreme black hole and results in $R_{\mathrm{H}} = M$, the EH effect on the charge would be of the order of $\sim 5.774\times 10^{-6}$. Analogously, for M87* it would be $\sim 2.187\times 10^{-12}$. This means that for a supermassive black hole, with an extreme charge and therefore zero rotation, the EH effects would barely be visible.

Astrophysical black holes are not expected to carry big charges, and the Event Horizon Telescope measurements of the shadows of Sag A* and M87* suggest that both rotate [48]. Even in the case in which supermassive black holes are described by extreme EEH static solutions, the tiny variations induced in the shadows are in general not observable unless one has extremely accurate mass and distance measurements of the black hole, which is usually not the case. A Kerr black hole with a slightly altered mass could produce the exact same shadow.

B. Stellar black holes

For stellar black holes, the EH effects become relevant for smaller charges. For a stellar black hole with a mass of ten solar masses, Eq. (74) is $\sim 9.24\times 10^5(Q/M)^2/(R_{\mathrm{H}}/M)^4$. One can then consider charges of the order of $Q\sim M/1000$ and still have an impact from the EH effects. Such small charges allow big values of the angular momentum, which is expected from its conservation during the process of gravitational collapse. On the other hand, small charges are also required in order for the EH

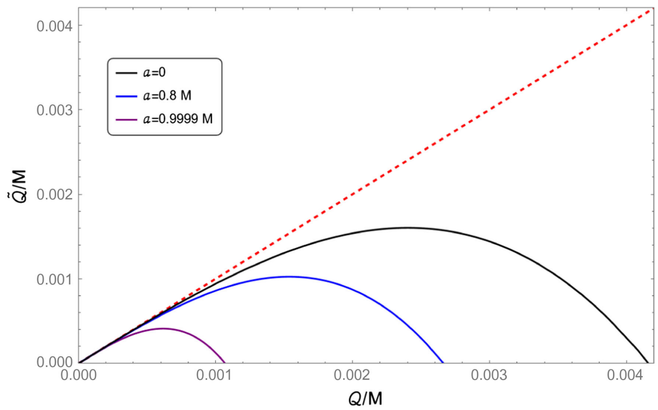

FIG. 11. Screened charge $\tilde{Q}$ (5) as a function of the real charge Q for different values of the angular momentum a and a fixed stellar mass, $M = 10M_{\odot}$. The dashed line corresponds to the KN case.

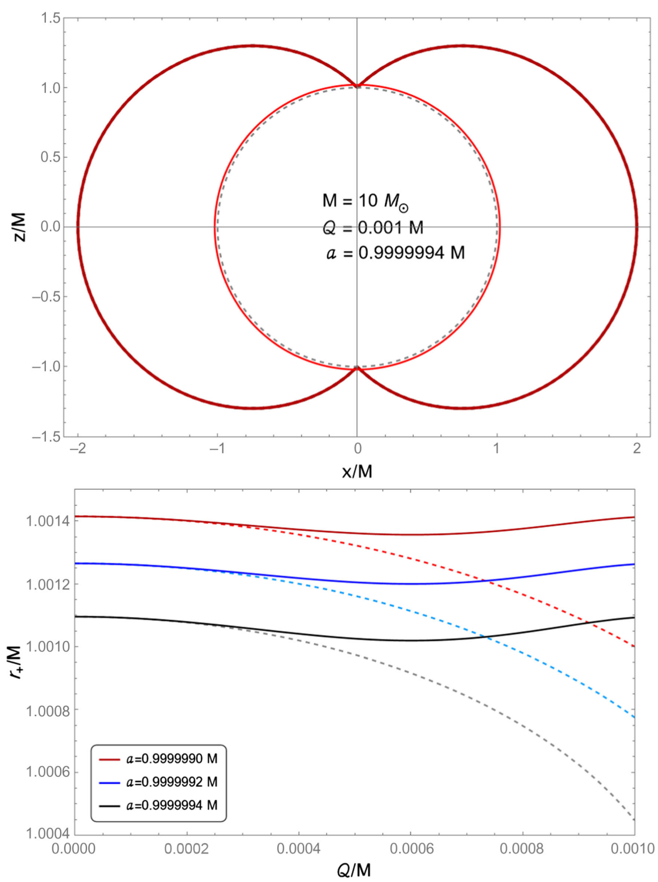

FIG. 12. EH effects on the event horizon and ergoregion (top) and the radius of the event horizon as a function of Q (bottom), for different values of a, and for a fixed stellar mass $M = 10M_{\odot}$. The continuous lines correspond to the EEH case, while the dashed ones correspond to the KN case.

modifications to be considered first-order corrections to the case of a KN stellar black hole.

The screened charge of a stellar black hole with $M = 10M_\odot$ is displayed in Fig. 11. If this black hole is static, the maximum allowed charge is of the order of $Q \sim 0.004M$. Nevertheless, for $a = 0.8$, the charge $Q = 0.004M$ is not allowed since the EH term would no longer be a correction. Then, one would have to choose values of the parameters that are below the curve $\tilde{Q}$ and $\tilde{Q} > 0$. In the following figures, in order to compare the EH effects for different values of the parameters at the same scale, we use near-extreme values close to $Q \sim 0.001M$, and $a \sim 0.9999995M$.

Examples of modifications of the event horizon and ergoregion are shown in Fig. 12. The ergoregion is barely affected. The EH effects on the event horizon are visible, but small.

The size of the EH effects on the thermodynamic properties are also different for stellar black holes. For bigger values of the angular momentum, both the entropy and temperature grow (see Fig. 13), while in the KN case, both keep decreasing. The black hole begins to heat up when reaching a value of a when the pair production intensifies.

Something similar happens when increasing the charge of the black hole, although in this case, the EH effects on both the entropy and temperature become visible with very small increments of Q (see Fig. 14). In the KN case, the EH effect of such small values of Q is not visible, and then the thermodynamic properties are more similar to those of a Kerr black hole. Hence, stellar black holes are much more sensitive to changes in their charge since the vacuum polarization produced by the corresponding electric field also modifies the geometric and thermodynamic properties.

C. Primordial black holes

The existence of galaxies today implies inhomogeneity in the early stages of the Universe. One would expect some regions to become sufficiently compressed for gravitational attraction to overcome pressure and produce black holes with masses from $10^{-5}g$ upwards [49]. These are called

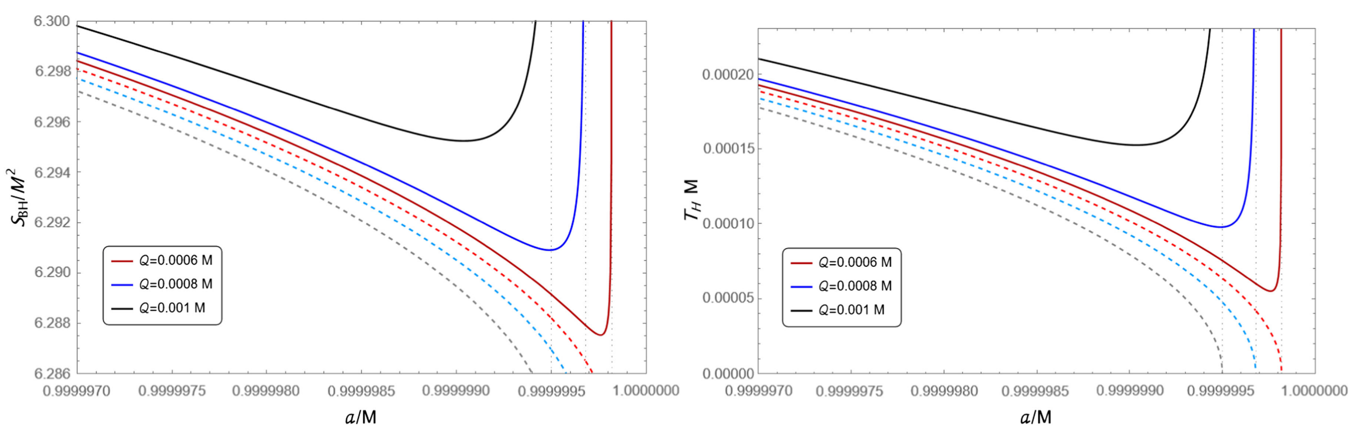

FIG. 13. Entropy (left) and temperature (right) of the EEH black hole (continuous lines) as a function of the angular momentum a, compared to the KN ones (dashed lines), for a stellar black hole with mass $M = 10M_\odot$.

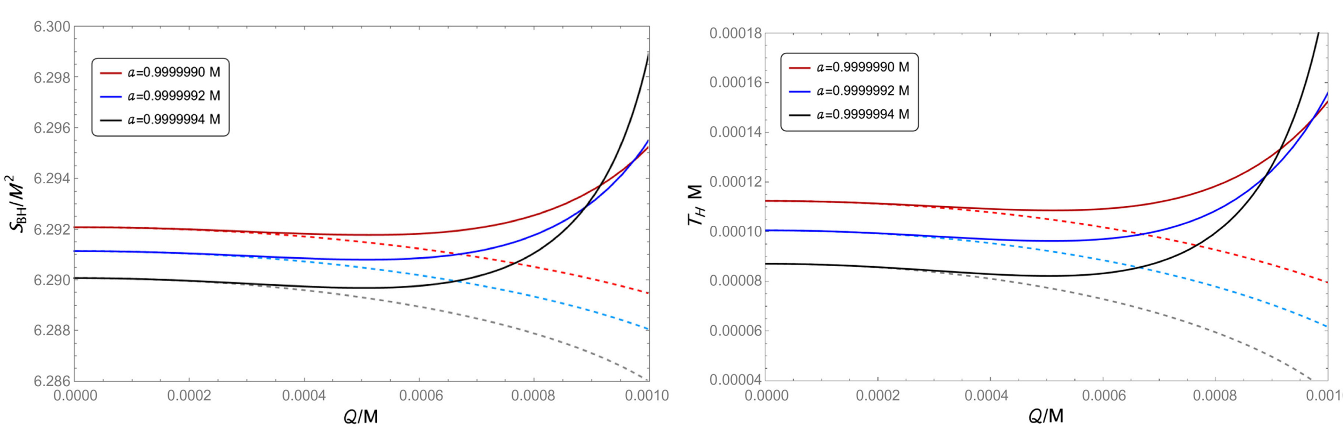

FIG. 14. Entropy (left) and the temperature (right) of the EEH black hole (continuous lines) as a function of the black hole charge Q, compared to the KN ones (dashed lines), for a stellar black hole with mass $M = 10M_\odot$.

primordial black holes, and some would have grown by accreting nearby matter.

For a primordial black hole with the mass of the Earth, $M \sim 5.97 \times 10^{24}$ kg $\sim 3 \times 10^{-6} M_{\odot}$, Eq. (74) is $\sim 3.08 \times 10^{13} (Q/M)^2/(R_{\rm H}/M)^4$. Charges of the order of $Q \sim 10^{-7} M$ can produce EH effects.

From the time of evaporation by Hawking radiation (A11), using the age of the Universe $\sim 13.78 \times 10^9$ years, the mass of an isolated Schwarzschild black hole, which would be evaporated by now, is of the order of $M \sim 10^{12}$ kg. For a primordial KN black hole with this mass, Eq. (74) is $\sim 1.84 \times 10^{26} (Q/M)^2/(R_{\rm H}/M)^4$, and the EH effects would become relevant for charges of the order of $Q \sim 10^{-14} M$.

VI. SUMMARY AND CONCLUSIONS

QED vacuum corrections to the Maxwell-Lorentz theory can be accounted for by the effective QED theory resulting after one loop of nonperturbative quantization, i.e., the EH NLED [1]. The vacuum is treated as a specific type of medium, the polarizability and magnetizability properties of which are determined by clouds of virtual charges surrounding the real charges and currents. This fact can be interpreted as a kind of dielectric constant of the vacuum. The EH effective Lagrangian is only valid for constant fields. The EEH generalization of the KN black hole solution was recently performed by Bretón *et al.* [9]. They considered the QED interpretation of the EH NLED and generated a rotating, electrically charged black hole solution by assuming that the nonlinearity only influences the electric charge by screening it. The geometry is only affected through the screened values of the real charges and the induced magnetic dipole moment. The black hole solution is then reduced to a screened KN one, as it happens for the static solution, which is considered as screened RN [6,7].

Energy extraction from a black hole by the Penrose process [24–26] allowed an analogy between the four laws of black hole mechanics [29] and the four laws of thermodynamics. The area of the event horizon is related to the entropy of the black hole, and the surface gravity to its temperature. We considered the rotating electrically charged EEH black hole solution as a thermodynamic system affected by captured matter on a plane perpendicular to the rotational axis, and analyzed the EH effects on the black hole properties. Both the event horizon and the static limit surface grow compared to those of the KN black hole, but the ergoregion diminishes. Consequently, less energy can be extracted from an EEH black hole, which was verified after analyzing the Penrose process. The irreducible mass of the EEH black hole is bigger than the KN one, and less stored rotational and electromagnetic energy is available to be extracted from an EEH black hole, which can be interpreted as part of a KN black hole's stored energy already being used for the pair production. In this sense, the process of particle creation carries away both the charge and part of the angular momentum from a KN black hole, as concluded in [38].

We then presented the four laws of black hole thermodynamics and studied the EH corrections to the entropy and temperature of the black hole. The KN entropy is smaller than the EEH one, which implies that the latter is a more probable state. In a charging process, the EEH entropy first decreases, but then increases for bigger charges, when the pair production intensifies. If such big charge values were previously reached, some work would first have to be done in order to counteract the pair production (the entropy decreases). Afterwards, the black hole discharges and the entropy increases. A similar behavior occurs for changes in the angular momentum. Additionally, as the mass is reduced, the entropy decreases, but in the EEH case the limiting entropy value is different from zero.

The temperature of the EEH black hole is also higher than the KN one, and also increases after reaching big values of charge and angular momentum. The produced pairs in the equatorial plane heat up the black hole. As a consequence, the more virtual particles arise, the higher the temperature. On the other hand, in the KN case the temperature tends to zero for near-extreme values of the parameters. Furthermore, the temperature also tends to infinity as the mass tends to zero, which is known as the explosion of the black hole in the evaporation process. But in the EEH case, such an explosion occurs earlier, i.e., for bigger masses.

We also studied the heat capacity, which defines the phase transitions. We derived that in an EEH black hole the phase transition occurs at bigger values of charge or angular momentum than in a KN black hole, as well as at a lower temperature. Moreover, a new phase transition occurs in the EEH case for near-extreme values of the parameters, which does not occur in the KN case. Additionally, following [46], we presented a proportional relation between the EEH entropy and the KN one under Hawking radiation and found that the EEH entropy grows in time at a higher rate than the KN entropy.

For a primordial black hole with the mass of the Earth, charges of the order of $Q \sim 10^{-7} M$ can produce EH effects. For a supermassive black hole, with an extreme charge and therefore zero rotation, the EH effects would barely be visible. For stellar black holes, the EH effects are visible for small values of the charge of the black hole. We concluded that a small increment of the charge would very rapidly increase the entropy and temperature of a stellar black hole due to pair production. Both also increase for near-extreme values of the angular momentum. During the gravitational collapse of stars to stellar black holes, processes of charge separation and consequent pair production by vacuum polarization occur [34], which can drastically modify their electromagnetic structure. Hence, EH effects on both the geometric and thermodynamic properties of such black holes should be considered.

We highlight the contributions of our paper, as compared with previous studies of EEH static black holes (see, for instance, [15,16,22]). First of all, the EEH rotating BH presents several characteristics associated with the BH angular momentum. The horizons are different, and there is an ergoregion from which chargeless test particles can extract energy (Sec. V); also, the available extractable energy is less than that for the KN BH. Moreover, since in [16,22] the BH charge was magnetic and the magnetic field did not contribute to the pair production rate, another novelty of our paper is the analysis of the effects of electron-positron production in thermodynamic quantities like the entropy, temperature, and surface gravity, which we presented in the form of the KN term plus the nonlinear EH contribution, a form that facilitates distinguishing the EH contribution. In spite of the fact that the rotation or angular momentum introduces the distortion of the observed BH shadows, nowadays there are no available BH observations with the needed precision to distinguish the nonlinear EH effects.

Note added. During the revision process of this paper, one of the coauthors, Alfredo Macías passed away.

ACKNOWLEDGMENTS

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APPENDIX A: HAWKING RADIATION AND BLACK HOLE EVAPORATION

In order to write the generalized Stefan-Boltzmann law for the screened KN black hole, we change to Eddington-Finkelstein coordinates [26,50] ν and ξ,

$$d\nu = dt + \frac{r^2+a^2}{\Delta}dr, \qquad d\xi = d\phi + \frac{a}{\Delta}dr. \tag{A1}$$

Therefore, the screened KN metric takes the form

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}d\nu^2 + 2d\nu dr \\ &-\frac{2a(r^2+a^2-\Delta)\sin^2\theta}{\Sigma}d\nu d\xi + \Sigma^2 d\theta^2 \\ &-2a\sin^2\theta dr d\xi + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}d\xi^2. \end{aligned} \tag{A2}$$

During the process of Hawking radiation, the black hole's area, and therefore its entropy, decreases, since the black hole mass decreases, in violation of the second law of thermodynamics (45). This fact reflects a failure of the energy condition as a result of the quantum fluctuations that produce the radiation. The generalized second law of thermodynamics (66) predicts that the emergent Hawking radiation entropy contributions more than compensate for the decrease of the black hole entropy.

The evaporation process of a black hole can be regarded as blackbody radiation. Under this basic assumption, the temperature of the black hole measured by a stationary observer at infinity is the Hawking temperature, and the radiation process needs to satisfy a Stefan-Boltzmann law. The generalized Stefan-Boltzmann law, which can be used to investigate Hawking radiation with energy, charge, and angular momentum, can be written as [46]

$$\begin{aligned} -\sigma T_{\rm H}^4 A &= \frac{dM}{d\nu} - \frac{a}{r_+^2+a^2}\frac{dJ}{d\nu} - \frac{\tilde{Q}r_+}{r_+^2+a^2}\frac{d\tilde{Q}}{d\nu} \\ &= \frac{dM}{d\nu} - \Omega_{r_+}\frac{dJ}{d\nu} - \Phi_{r_+}\frac{d\tilde{Q}}{d\nu}, \end{aligned} \tag{A3}$$

where σ is a positive constant related to the number of quantized matter fields coupled to gravity, A is the area of the event horizon, and r_+ is the radius of the event horizon. Identifying the coefficients as the event horizon angular velocity (16) and electric potential (17), the correction terms in the modified laws are, respectively, the dJ and $d\tilde{Q}$ terms in the first law of black hole thermodynamics (56). So the generalized Stefan-Boltzmann law can be seemingly regarded as consistent with the first law. Thus, the Bekenstein-Hawking entropy satisfies the relation $dS_{\rm BH} = -\sigma T_{\rm H}^3 A d\nu$, and the evolution equation reads

$$\frac{dS_{\rm BH}}{d\nu} = -\frac{\sigma}{16\pi^2}\frac{(r_+ - r_-)^3}{(r_+^2+a^2)^2}. \tag{A4}$$

In the process of Hawking radiation, the Bekenstein-Hawking entropy decreases with time. Then, in order to satisfy the generalized second law (66), the ordinary entropy outside the black hole should increase, $dS_0/d\nu > 0$.

On the other hand, we can rewrite the entropy of the EEH rotating black hole as the entropy of the KN one plus an EH correction, i.e., (48). One can also rewrite the evolution equation as

$$\begin{aligned} \frac{dS_{\rm BH}}{d\nu} = \frac{dS_{\rm BH}^{\rm KN}}{d\nu} - \frac{\alpha\sigma}{180\pi^3}\frac{Q^4}{D_c^2 R_{\rm H}^4}\frac{(R_{\rm H}^2+3a^2)}{(R_{\rm H}^2+a^2)^3} \\ \times\sqrt{M^2-a^2-Q^2}, \end{aligned} \tag{A5}$$

with

$$\frac{dS_{\rm BH}^{\rm KN}}{d\nu} = -\frac{\sigma}{2\pi^2}\frac{(M^2-a^2-Q^2)^{3/2}}{(R_{\rm H}^2+a^2)^2}, \tag{A6}$$

or rewrite (A5) as

$$\frac{dS_{\rm BH}}{d\nu} = \left[1 + \frac{\alpha}{90\pi}\frac{Q^4}{D_c^2 R_{\rm H}^4}\frac{(R_{\rm H}^2 + 3a^2)}{(R_{\rm H}^2 + a^2)} \times \left(\frac{1}{M^2 - a^2 - Q^2}\right)\right]\frac{dS_{\rm BH}^{\rm KN}}{d\nu}, \tag{A7}$$

which is the proportional relation between both entropies, with a positive coefficient that grows as one approaches the extreme case. This means that the EEH entropy changes at a higher rate than the KN one under Hawking radiation. Hence, the ordinary entropy outside the black hole S_0 should also change at a higher rate.

As pointed out by Damour *et al.* [38], the process of particle creation in the EH framework differs from the process of Hawking radiation.

First of all, the EH process does not require a time-varying background geometry, and is necessarily caused by the presence of an electric or magnetic field. The rate of pair production around a KN black hole is obtained from locally applying the Schwinger formula [3] to the case of a curved KN geometry, based on the equivalence principle. Then, pair production occurs when the electric field of the black hole reaches the critical value D_c.

Second, pair production in the EH process uses part of the electromagnetic and rotational energy of the KN black hole [51] and increases its irreducible mass, as verified from (35). The entropy and irreducible mass are related via (42) and (46) as

$$S = 4\pi M_{\rm ir}^2. \tag{A8}$$

Consequently, when EH pair production takes place the entropy increases, as analyzed, for instance, in Eq. (48) and Fig. 6, satisfying the second law of black hole thermodynamics. On the contrary, in the Hawking radiation process the entropy decreases, in violation of the second law. The irreducible mass is radiated away and decreases with time, which is called the evaporation of the black hole. A rough procedure to analyze the latter is the following. Let us consider a Schwarzschild black hole, whose mass is the irreducible mass, i.e., $M = M_{\rm ir}$ from (38) with $\tilde{Q} = J = 0$. The Stefan-Boltzmann law (A3) for the Schwarzschild black hole is given by

$$\frac{dM}{dt} = -\sigma A T_{\rm H}^4 \propto -M^2\left(\frac{1}{M}\right)^4 = -\frac{1}{M^2}. \tag{A9}$$

After integrating from an initial black hole mass M_0 decreasing up to a mass close to the Plank mass $m_{\rm p}$, with the assumption $M_0 \gg m_{\rm p}$, one obtains [52]

$$t(M_0) \propto (M_0^3 - m_{\rm p}^3) \propto \left(\frac{M_0}{m_{\rm p}}\right)^3. \tag{A10}$$

Putting back the units by using the Plank time $t_{\rm p}$, such that $t(M_0) = t_{\rm p}(M_0/m_{\rm p})^3$, the time required for a black hole of mass M_0 to evaporate is

$$t(M_0) \approx 10^{65}\left(\frac{M_0}{M_\odot}\right)^3 \text{ years}. \tag{A11}$$

Finally, the EH effects may be relevant even for astrophysical black holes, in particular for stellar black holes with masses larger than the critical mass of neutron stars [34], $M \geq 3.2M_\odot$, as analyzed in the previous section. On the other hand, from (A11) the required time for the evaporation of a stellar Schwarzschild black hole of $4M_\odot$ by Hawking radiation is around 10^{66} years, which is more than 10^{56} times the age of the Universe $\sim 10^{10}$ years.

Summarizing, an astrophysical black hole surrounded by matter would follow the laws of black hole thermodynamics. It stores electromagnetic and rotational energy, and its irreducible mass, area, and entropy always increase. Only a completely isolated black hole could begin to evaporate by vacuum polarization processes as follows. A charged rotating black hole with an electric field bigger than the critical field D_c, would create particles by the EH process, which carries away both the charge and part of the angular momentum from the black hole. When the electric field of the black hole is smaller than D_c, the charged black hole would continue to lose its charge by spontaneous particle emission [53]. For the rotational energy, something similar happens. As shown by Page [54], a rapidly rotating black hole would lose angular momentum through the Hawking process several times faster than its mass. The area and entropy of a rotating black hole would initially increase with time as heat flows into the hole from particle pairs created in the ergoregion. As the rotation decreases, the thermal emission becomes dominant, drawing heat out of the hole and decreasing its area. In this way, any initially rotating black hole spins down to a nearly nonrotating state before most of its mass has been given up. When the charge and rotation of the black hole are lost, i.e., $M = M_{\rm ir}$ from (38), the irreducible mass is radiated away by the Hawking process, which corresponds to the evaporation of a Schwarzschild black hole.

APPENDIX B: HYPOTHETICAL θ DEPENDENCY

The EEH rotating black hole solution (1) was derived following the QED interpretation of the EH NLED [9], which states that the effects of the vacuum polarization are nearly constant and affect the geometry only through the screened values of the real charges [34]. When the electromagnetic field of a KN black hole reaches the critical field D_c, pair production occurs and the clouds of virtual charges affect the real charge. Such modifications are described by the definition of the charge $\tilde{Q}$ [Eq. (2)], which has to be evaluated at constant r and θ, depending on the point at which one analyzes the EH effects. In the previous sections, we considered the EEH black hole as a thermodynamic system affected only by captured matter on the plane perpendicular to the rotational axis. Consequently, $\tilde{Q}$ was fixed for $\theta = \pi/2$ and $r = R_{\mathrm{H}}$, i.e., we used the charge (5).

Furthermore, although the QED EH effective Lagrangian is only valid for constant fields, in order to gain some physical insight into the EH modifications we consider other locations, choosing different values of θ in (2). If the EH corrections were dependent on θ, the charge of the black hole would be screened at locations close to the equatorial plane and magnified near the poles, as can be seen in Fig. 15. The effect on $\tilde{Q}$ can strongly change for some values of θ. For instance, for a particle captured at the poles of the black hole, where $\theta = 0, \pi$, one obtains

$$\tilde{Q}^2 = Q^2\left\{1 - \frac{5\alpha}{225\pi}\frac{Q^2}{(R_{\mathrm{H}}^2 + a^2)^2 D_c^2}\left[1 - \frac{4a^2}{(R_{\mathrm{H}}^2 + a^2)}\right.\right. \times\left(1 - \frac{a^2}{(R_{\mathrm{H}}^2 + a^2)}\right)\left(7 - \frac{12a^2}{(R_{\mathrm{H}}^2 + a^2)}\right. \left.\left.\left. + \frac{12a^4}{(R_{\mathrm{H}}^2 + a^2)^2}\right)\right]\right\}, \tag{B1}$$

and the charge is magnified for some values of the angular momentum a, i.e., $\tilde{Q} > Q$, contrary to our QED screened Maxwell assumption. This fact is presented in Fig. 16, where the modified charge of the black hole can appear either screened or magnified for some values of the parameters and different fixed values of θ. In particular, the charge can be magnified after previously being screened, due to the increment of the angular momentum. This could be interpreted as most of the produced virtual charges being dragged by the rotation of the black hole, modifying the charge distribution. In the rotating case, the dragged virtual charges near the equatorial plane would screen the charge in that location, while at the poles the induced magnetic moment would magnify it. Moreover, the event horizon is shrunk near the poles and is expanded near the equatorial plane, as presented in Fig. 17. In this case, while the event horizon of the KN black hole is a sphere of radius R_{H}, the event horizon of the EEH one would no longer be a sphere, but would be distorted, as shown in Fig. 18.

We now briefly analyze the effects on the thermodynamic quantities. We again consider a particle captured by the black hole at different locations of θ.

The minimum energy required for energy extraction by the Penrose process (31) does not depend explicitly on θ, and is defined by the condition $\dot{r} = 0$. Nevertheless, the condition $\dot{\theta} = 0$ should also be satisfied, as concluded from the analysis of (30). Consequently, one can still use the definition of the irreducible mass (34) and the related Smarr formula (38). Hence, near the poles of the black hole, where $\tilde{Q} > Q$ and $M_{\mathrm{ir}} < M_{\mathrm{ir}}^{\mathrm{KN}}$, there is more stored energy available for extraction, since both the electromagnetic energy $\frac{\tilde{Q}^2}{4M_{\mathrm{ir}}} > \frac{Q^2}{4M_{\mathrm{ir}}^{\mathrm{KN}}}$ and the rotational one $\frac{J^2}{4M_{\mathrm{ir}}^2} > \frac{J^2}{4(M_{\mathrm{ir}}^{\mathrm{KN}})^2}$ increase.

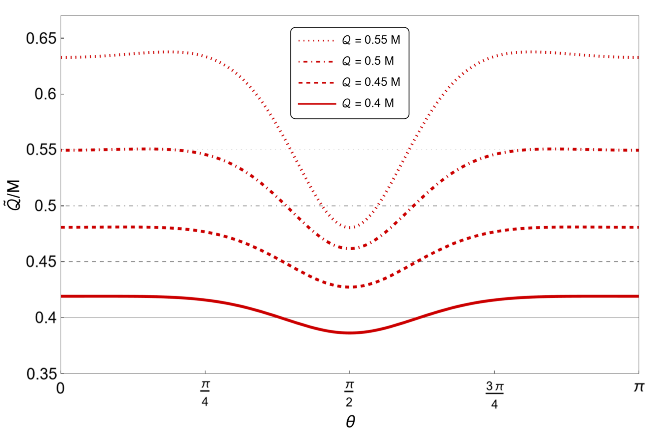

FIG. 15. Hypothetical charge distribution at the event horizon R_{H} as a function of θ and for different values of Q. The black hole parameters are $M = 1 \times 10^4 M_{\odot}$ and $a = 0.8M$. The minimum value of $\tilde{Q}$ occurs at the equatorial plane $\theta = \pi/2$, where the induced magnetic field vanishes, i.e., $H_Q = 0$.

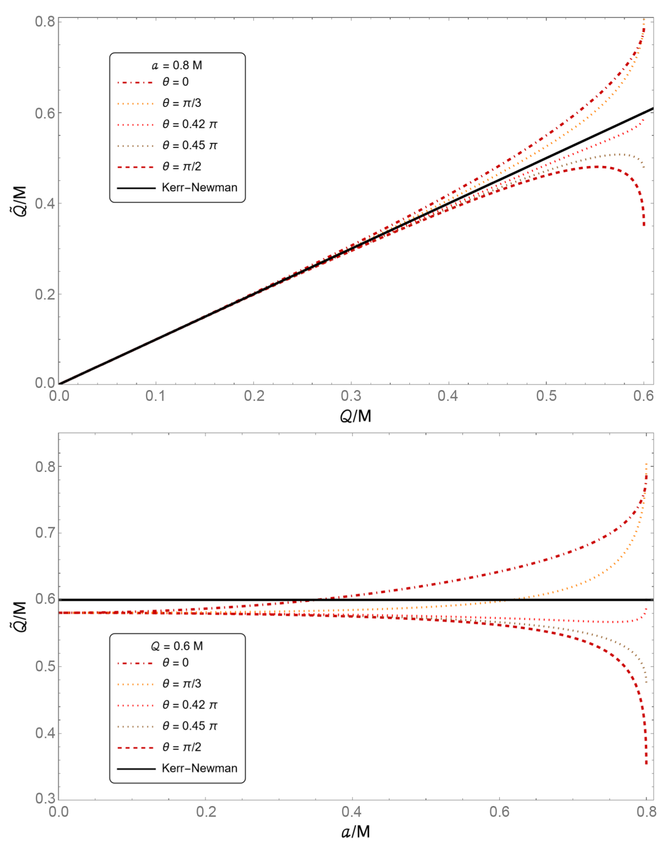

FIG. 16. $\tilde{Q}$ [Eq. (2)] as a function of Q and a, for different values of θ and fixed mass $M = 1 \times 10^4 M_{\odot}$.

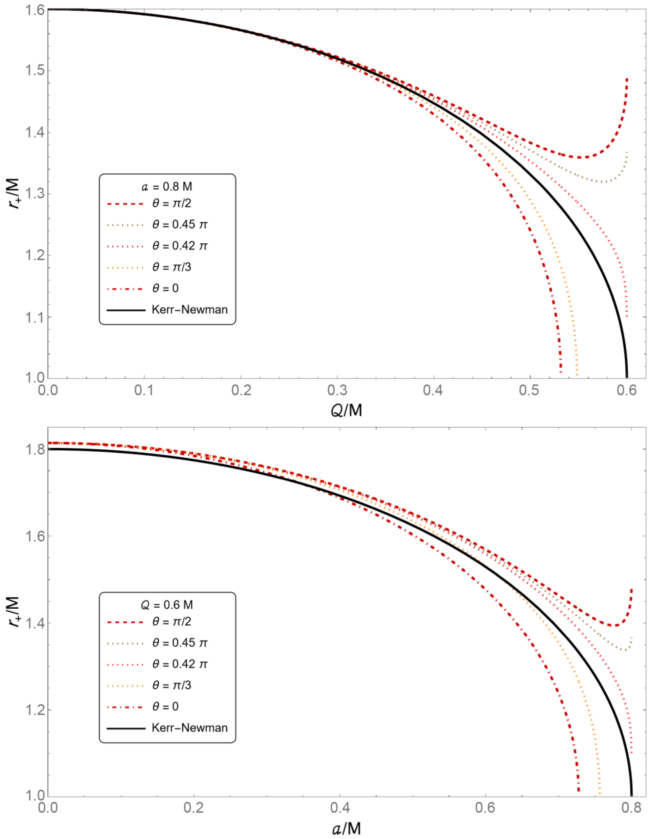

FIG. 17. Event horizon radius r_+ (6) as a function of Q and a, for different values of θ and fixed mass $M = 1\times 10^4 M_\odot$.

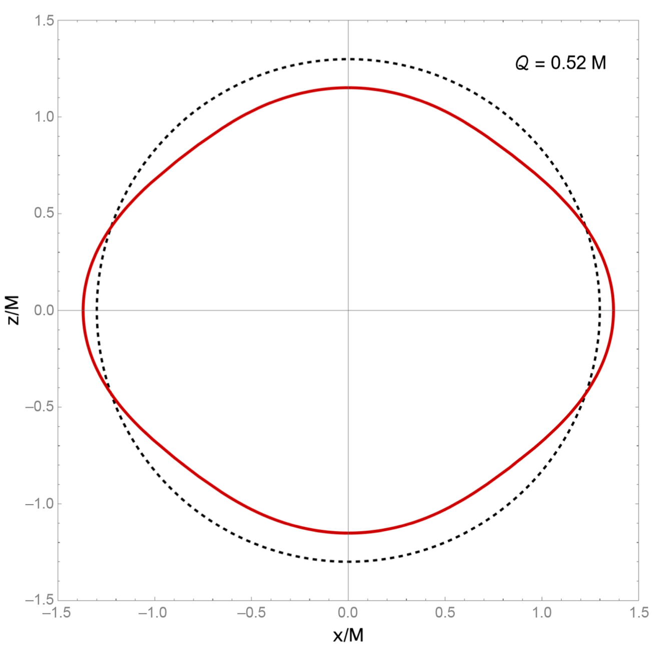

FIG. 18. Hypothetical distorted event horizon of the EEH rotating black hole (continuous line) compared to the KN one (dashed line), for fixed $M = 1\times 10^4 M_\odot$ and $a = 0.8M$.

For the entropy, which is related to the horizon surface, something similar happens. On the one hand, the equatorial plane is a location with bigger entropy, as shown in Fig. 6. In fact, the entropy takes there its highest value (see Fig. 19). On the other hand, the location with the lowest entropy is at the poles. This could be interpreted as there being more clouds of virtual particles near the equatorial plane, which would also indicate that the pair production intensifies there. If one relates the entropy with probability, it is more probable to find the clouds of virtual particles close to the equator. It is also more probable for a particle to be captured at the equatorial plane than at the poles.

The entropy after integration of θ reads

$$S_{\rm BH} = S_{\rm BH_{KN}} + \frac{\alpha}{10800 D_c^2}\frac{Q^4}{R_{\rm H}^2}\left(\frac{1}{\sqrt{M^2-a^2-Q^2}}\right) \times \left[\frac{405 R_{\rm H}^9}{(R_{\rm H}^2+a^2)^5} - \frac{165\arctan(a/R_{\rm H})}{a} - \frac{165 a^8 R_{\rm H} + 850 a^6 R_{\rm H}^3 + 368 a^4 R_{\rm H}^5 + 430 a^2 R_{\rm H}^7}{(R_{\rm H}^2+a^2)^5}\right], \tag{B2}$$

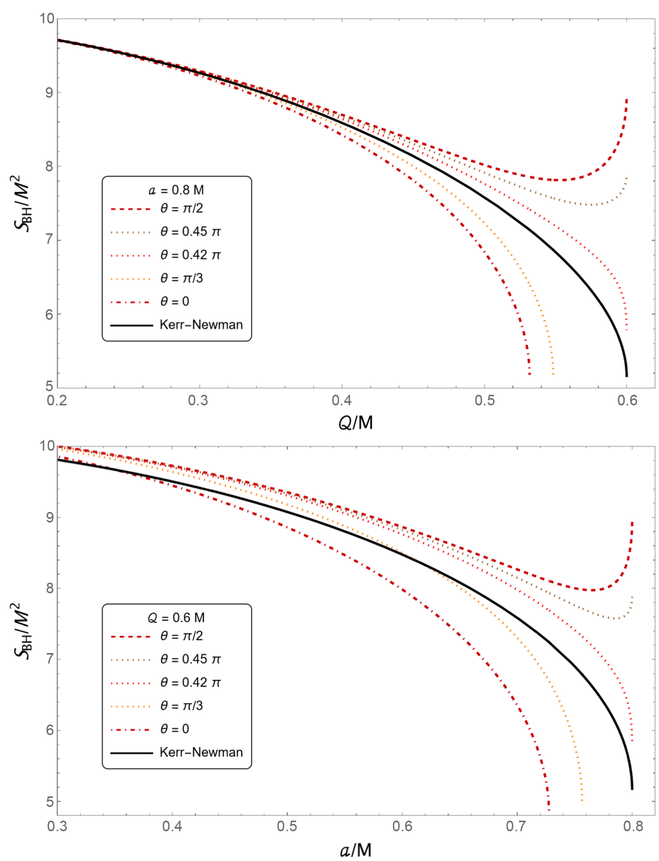

FIG. 19. Entropy $S_{\rm BH}$ [Eq. (46)] as a function of Q and a, for different values of θ and fixed mass $M = 1\times 10^4 M_\odot$.

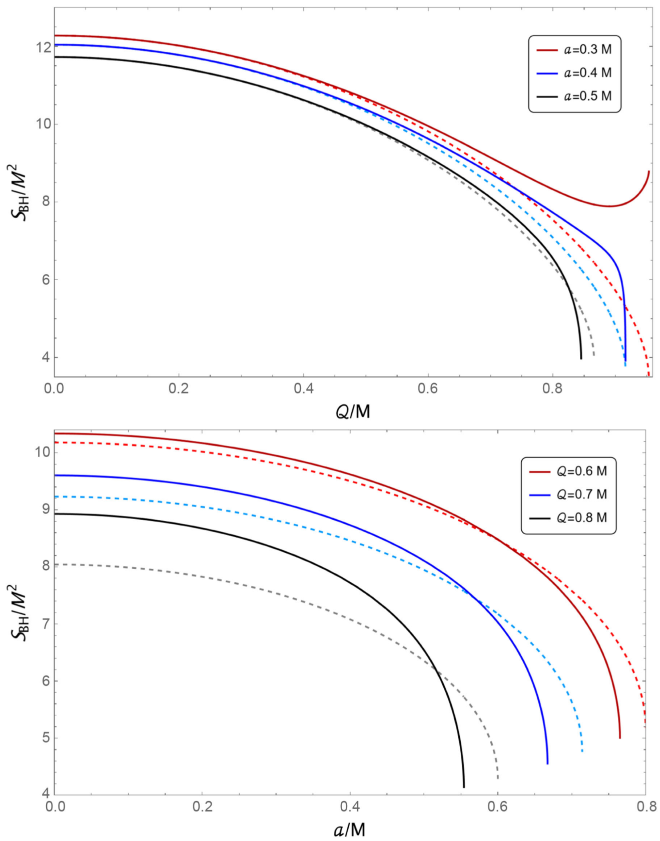

FIG. 20. Entropy as a function of Q and of a, for a fixed mass $M = 1 \times 10^4 M_\odot$. The dashed line corresponds to the usual KN case and the continuous line to that with EH corrections.

which only depends on the black hole parameters a, Q, and M. These can be restricted to values at which the entropy is always bigger than the entropy of the KN black hole, i.e., $S_{\mathrm{BH}} > S_{\mathrm{BH_{KN}}}$. Figure 20 displays the entropy (B2) as a function of its charge and angular momentum. If the angular momentum increases, the entropy might be smaller than in the usual KN case. For smaller values of a, the entropy is bigger. In the static limit, $a \to 0$, the entropy is always bigger.

The temperature would be given by

$$T_{\mathrm{H}} = T_{\mathrm{H}}^{\mathrm{KN}} + \frac{\alpha}{180\pi^2 D_c^2}\left(\frac{Q^4}{\sqrt{M^2 - a^2 - Q^2}}\right) \times \left[\frac{4a^4 \sin^2\theta \cos^2\theta}{(R_{\mathrm{H}}^2 + a^2)}\frac{\Theta_0}{\Sigma_{\mathrm{KN}}^8} + \frac{(2a^2 + Q^2)}{(R_{\mathrm{H}}^2 + a^2)^2}\frac{\Theta_1}{\Sigma_{\mathrm{KN}}^2}\right], \qquad \text{(B3)}$$

with $\Sigma_{\mathrm{KN}} = R_{\mathrm{H}}^2 + a^2\cos^2\theta$,

$$\Theta_0 = 15R_{\mathrm{H}}^8 - 58R_{\mathrm{H}}^6 a^2 \cos^2\theta + 32R_{\mathrm{H}}^4 a^4 \cos^4\theta - 38R_{\mathrm{H}}^2 a^6 \cos^6\theta + a^8 \cos^8\theta,$$

and

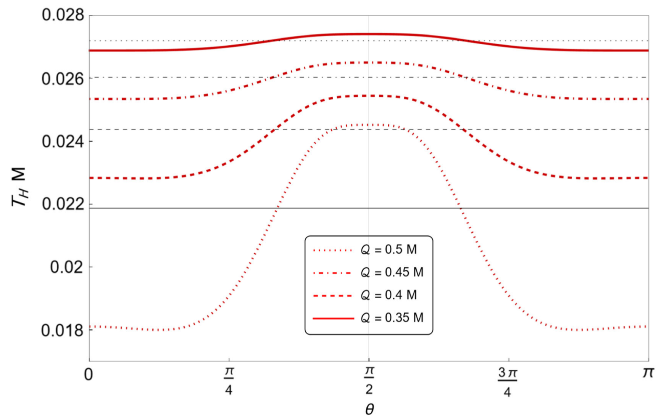

FIG. 21. Temperature as a function of θ and for different values of Q. The black hole parameters are $M = 1 \times 10^4 M_\odot$ and $a = 0.8M$. The maximum temperature occurs at the equatorial plane.

$$\Theta_1 = \left[1 - 4\frac{a^2\cos^2\theta}{\Sigma_{\mathrm{KN}}}\left(1 - \frac{a^2\cos^2\theta}{\Sigma_{\mathrm{KN}}}\right) \times \left(7 - 12\frac{a^2\cos^2\theta}{\Sigma_{\mathrm{KN}}} + 12\frac{a^4\cos^4\theta}{\Sigma_{\mathrm{KN}}^2}\right)\right].$$

Figure 21 displays the temperature as a function of θ. It is higher near the equatorial plane and lower near the poles. The temperature of the black hole is displayed in Fig. 22 as a function of Q and a and for different values of θ.

Furthermore, one could average the charge distribution (2) by integrating $\tilde{Q}(R_{\mathrm{H}}, \theta)$ around the angular coordinates as

$$\hat{Q} \equiv \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \tilde{Q}(R_{\mathrm{H}}, \theta)\sin\theta d\theta d\phi, \qquad \text{(B4)}$$

$$= Q\left\{1 - \frac{\alpha}{21600\pi D_c^2}\frac{Q^2}{R_{\mathrm{H}}^3}\left[\frac{405R_{\mathrm{H}}^9}{(R_{\mathrm{H}}^2 + a^2)^5} - \frac{165\arctan(a/R_{\mathrm{H}})}{a} - \frac{165a^8R_{\mathrm{H}} + 850a^6R_{\mathrm{H}}^3 + 368a^4R_{\mathrm{H}}^5 + 430a^2R_{\mathrm{H}}^7}{(R_{\mathrm{H}}^2 + a^2)^5}\right]\right\} \qquad \text{(B5)}$$

and study the black hole solution as the KN one with an effective average charge $\hat{Q}$, which only depends on the black hole parameters. One can restrict their values in such a way that the charge is always screened and not magnified. The latter is the same restriction as that for the entropy, which would again be given by (B2). The irreducible mass would be (37) with $\hat{Q}$.

Hence, when $\hat{Q}$ is screened the entropy is bigger than the KN one, it is a more probable state, and there would be less stored energy since part of the energy is used for the pair production. When it is magnified, the entropy is smaller, it is a less probable state, and more energy can be extracted. One can reduce the entropy of a thermodynamic system by doing work on it, and one can then obtain energy from this low-entropy system, which would now increase the entropy. As interpreted above, the clouds of virtual charges would be dragged closer to the equatorial plane and more energy could be extracted by particles located near the poles.

To sum up, the EH corrections described by the solution (1) must be considered constant since the EH NLED is only valid for constant fields. This allows one to study the thermodynamic properties of the EEH rotating black hole as corrections to the KN case at different locations, like the poles or the equatorial plane. One could consider the charge $\tilde{Q}$ [Eq. (2)] as explicitly depending on θ, and the charge distribution would be different than in the KN case, which would modify the black hole properties. In any case, this charge distribution $\tilde{Q}$ comes from the underlying symmetry of the space-time structure and the energy-momentum tensor [9].

On the other hand, if one considered the standard approach [12], where the EH Lagrangian is derived as the low-energy limit of the Born-Infeld theory, one should obtain another charge distribution. In this case it is expected a similar thermodynamic behavior of the rotating EEH solution found within the Yajima-like approach.

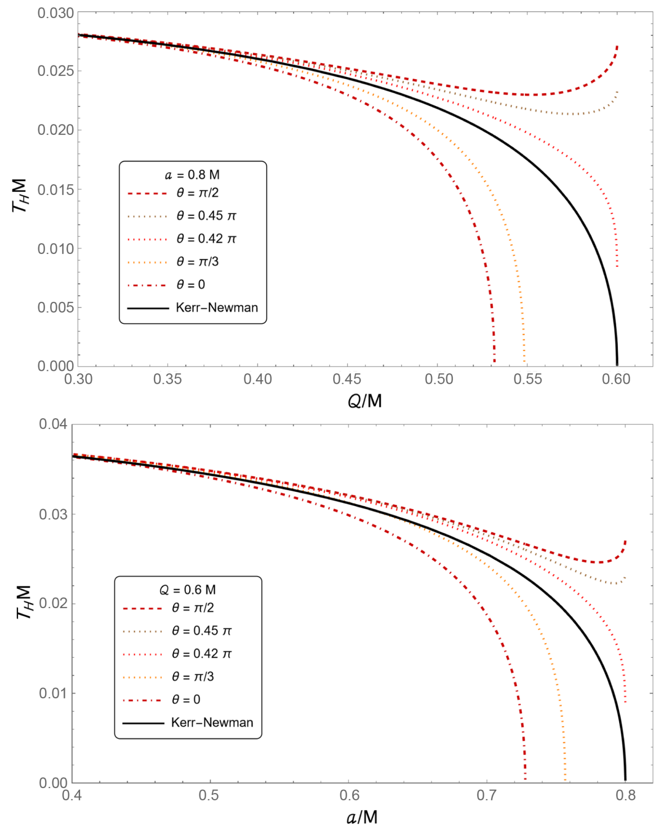

FIG. 22. Temperature T_{H} [Eq. (47)] as a function of Q and a for different values of θ and fixed mass $M = 1 \times 10^4 M_{\odot}$.

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