

Modulus stabilization in the multiple-modulus framework

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In a class of modular-invariant models with multiple moduli fields, the viable lepton flavor mixing pattern can be realized if the values of moduli are selected to be at the fixed points. In this paper, we investigate a modulus stabilization mechanism in the multiple-modulus framework which is capable of providing de Sitter (dS) minima precisely at the fixed points $\tau = i$ and ω , by taking into consideration nonperturbative effects on the superpotential and the dilaton Kähler potential. Due to the existence of additional Kähler moduli, more possible vacua can occur, and the dS vacua could be the deepest under certain conditions. We classify different choices of vacua and discuss their phenomenological implications for lepton masses and flavor mixing.

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I. INTRODUCTION

The flavor problem—that of the origin of the three quark and lepton families and their pattern of masses and mixings—is an unresolved puzzle within the Standard Model (SM) of particle physics. The discovery of very small neutrino masses with large mixing enriches the flavor problem still further, requiring a further seven parameters (more or less) for its phenomenological description and demanding new physics beyond the SM. The unexpected phenomenon of large lepton mixing has caused a schism in the community between those who think that this is a hint of a family symmetry at work—in particular, non-Abelian and discrete—and those who think that it is just a random or anarchic choice of parameters. If one follows the symmetry approach, one is immediately confronted by the problem of how to break the symmetry, without which there would be massless fermions with no mixing, and this leads to the introduction of rather arbitrary flavon fields and driving fields which determine their vacuum alignments, which play a crucial role in determining the masses and mixings (for a review, see, e.g., Ref. [1]).

In an attempt to make the non-Abelian discrete family symmetries—and in particular, the accompanying flavon fields—less arbitrary, it has been suggested that a more satisfactory framework for addressing the flavor problem,

at least in the lepton sector, might be modular symmetry broken by a single complex modulus field τ [2]. Using ideas borrowed from string theory [3,4], modular symmetry on the worldsheet represents a reparametrization symmetry of the extradimensional coordinates, whose toroidal compactification is controlled by one or more moduli fields, the simplest example being a single complex modulus field τ describing the two-compact-dimensional lattice of a six-dimensional theory, modulus field τ , where its vacuum expectation value (VEV) fixes the geometry of the torus [5–7].

The resulting infinite modular symmetry in the upper half of the complex plane, $\text{PSL}(2, \mathbb{Z})$, has particularly nice features which rely on holomorphicity, the lack of complex conjugation symmetry, which seems to call for supersymmetry. The infinite modular group has a series of infinite normal subgroups called the principle congruence subgroups $\Gamma(N)$ of level N , whose elements are equal to the 2×2 unit matrix mod N (where typically N is an integer called the level of the group). For a given choice of level $N > 2$, the quotient group $\Gamma_N = \text{PSL}(2, \mathbb{Z})/\Gamma(N)$ is finite and may be identified with the groups $\Gamma_N = A_4$ [2,8–26], S_4 [27–30], or A_5 [31–33] for levels $N = 3, 4, 5$, which may subsequently be used as a family symmetry [2].

The only flavon present in such theories is the single modulus field τ , whose VEV fixes the value of Yukawa couplings which form representations of Γ_N and are modular forms. Remarkably, the resulting Yukawa couplings involved in the terms in the superpotential containing superfields whose modular weights do not sum to zero, but take even values, can exist as modular forms with a precise functional dependence on τ [2], leading to very predictive theories independent of flavons [2]. However, for general values of the modulus field τ , the resulting Yukawa couplings are not very hierarchical, so fermion

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mass hierarchies do not emerge naturally. There are also more general formulations involving the double cover of the finite groups, where modular forms may have integer values, or—more general still—fractional values, called metaplectic groups [34–48].

In all such theories, the modular symmetry acts on the modulus field τ in a nonlinear way, and also, the finite modular symmetry is necessarily broken. τ is restricted to a fundamental domain in the upper-half complex plane which does not include zero. However, it is well known that there are three fixed points where a discrete subgroup of the modular symmetry is preserved [28,49,50]: namely, $\tau = i$, which preserves Z_2^S ; $\tau = \omega = e^{2\pi i/3}$, which preserves Z_3^{ST} ; and $\tau = i\infty$, which preserves Z_N^S for level N , where S, T are the generators of the modular symmetry [2]. At these fixed points, the Yukawa couplings may have some zero components, which may correspond to massless charged leptons, with the charged-lepton mass hierarchy possibly resulting from small deviations from the fixed points [51–62]. Alternatively, the charged-lepton mass hierarchy could result from the use of so-called weighton fields [63], which are singlet fields with nonzero modular weights which develop VEVs and provide a natural suppression mechanism for Yukawa couplings.

Since string theories are usually formulated in ten dimensions, the simplest factorizable compactifications require three tori, which motivates bottom-up models based on three moduli fields τ_i [64], and several realistic models have been constructed along these lines [65–71]. In particular, the finite fixed points $\tau = i$ and $\tau = \omega$ seem to play a special role in modular symmetry, since they emerge from 10D supersymmetric orbifold examples [72]. Realistic orbifold models with three S_4 modular symmetries have been constructed based on these fixed points, with two of the moduli $\tau = i$ and $\tau = i + 2$ controlling the neutrino sector, and the third modulus $\tau = \omega$ being responsible for (diagonal) charge lepton Yukawa matrices [68]. For the chosen orbifold $(T^2)^3/(Z_2 \times Z_2)$, two of the moduli are constrained to lie at $\tau = i$, or equivalently $\tau = i$ and $\tau = i + 2$, while the third modulus is not fixed by the orbifold, but was chosen to be at $\tau = \omega$ for phenomenological reasons, although it was observed that this choice enhanced the remnant symmetry of the orbifold [68]. It would be interesting to see if such choices of moduli fields are stabilized at these points.

Interestingly, the minima of the effective supergravity potentials which are used to stabilize the moduli also seem to be situated close to the fixed points $\tau = i$ and $\tau = \omega$. Indeed, the most important physical implication of string theory might be the existence of extra dimensions, and the moduli are the most important particle species arising in the compactifications of extra dimensions [73]. In this regard, modulus stabilization is crucial for giving moduli nonzero masses and arriving at phenomenologically variable models. One important question is whether the minima of the

potential are precisely at the fixed points $\tau = i$ and $\tau = \omega$, or are close to these fixed points but not precisely at them. In the former case, fermion mass hierarchies could arise from the weighton fields [63], while in the latter case, they could arise from the deviations from the fixed points [51], as discussed above.

One approach to modulus stabilization is the use of flux compactifications, which is widely discussed in Type IIB string theory [74–78]. In the context of modular flavor symmetry, the authors in Ref. [7] consider the 3-form flux in the Type IIB model. They systematically analyze the stabilization of complex structure moduli in possible configurations of flux compactifications on a $(T^2)^3/(Z_2 \times Z_2)$ orbifold. The number of stabilized moduli depends on an integer $N_{\text{flux}}^{\text{max}}$ associated with the fluxes. The values of moduli are found to be clustered at the fixed point $\tau = \omega$ in the fundamental domain.

Another origin of the nontrivial scalar potential is the nonperturbative effects. In Refs. [79,80], the authors realize the modulus stabilization by constructing a simple nonperturbative superpotential induced by the hidden dynamics within the framework of supergravity. In heterotic strings, there is an important nonperturbative effect called gaugino condensation [81–83]. Although the potential is flat in terms of the dilaton, Kähler, and complex structure moduli at tree level, it is indeed shown that threshold corrections [84–87] or worldsheet instantons can uplift the potential and lead to nontrivial vacua [88]. In the presence of modular symmetries, the authors in Refs. [89,90] consider the stabilization of Kähler moduli. They enumerate all possible nonperturbative contributions and derive the scalar potential. Minimizing the scalar potential, they find that the anti-de Sitter (AdS) vacua can generally appear at the imaginary axis and the lower boundary of the fundamental domain. They comment that no de Sitter (dS) vacuum is found in their numerical calculations. They also discuss the case where the dilaton comes into the superpotential, and argue that their results will not change if the superpotential relies on the dilaton as a sum of exponentials. The authors of Ref. [91] adopt the same framework. However, they find that in a special case, the VEV of τ can actually be in the interior of the fundamental domain, which is very close to the fixed point $\tau = \omega$. Still, no dS vacuum is found.

Cosmological observations imply that our Universe is in a dS phase with a positive cosmological constant. If we believe the string theory is the correct ultraviolet-complete theory of particle physics and gravity, the string compactifications should yield the 4D dS cosmology. It is then interesting to investigate how to uplift the AdS vacua obtained in the simple gaugino condensation to the dS vacua. The authors of Ref. [92] find that nonperturbative effects and uplifting terms can lead to dS vacua around fixed points in the Type IIB theory. In Ref. [93], the authors show that the AdS vacua can be uplifted by the matter superpotential [94,95]. They introduce a heavy meson field,

which couples with the moduli in the Kähler potential and superpotential. Due to the existence of the meson field, the vacua can be uplifted to dS vacua, and the VEVs of τ could slightly deviate from the fixed points.

There are, however, still some possibilities to realize the dS vacua without introducing the matter superpotential. In Ref. [96], the authors investigate the modulus stabilization within the framework of one Kähler modulus plus one dilaton. They first prove three no-go theorems that forbid dS vacua, which verify previous conjectures in Refs. [89–91]. In order to evade the dS no-go theorems, they further include Shenker-like effects [97] as nonperturbative corrections to the dilaton Kähler potential. As a result, they obtain metastable dS vacua at the fixed points $\tau = i$ and ω .

In this paper, we shall consider a modulus stabilization mechanism which is capable of providing dS minima precisely at the fixed points $\tau = i$ and $\tau = \omega$, in the absence of matter fields, but taking into account the effect of the dilaton field, with nonperturbative corrections to the dilaton Kähler potential, along the lines of Ref. [96], but extended to the three-modulus case. We find that the finite fixed points can serve as dS vacua. Due to the existence of additional Kähler moduli, the vacuum structure becomes more diverse, and we thereby classify the different possible vacua. Conditions for these vacua to be dS vacua are distinct from those in the single-modulus case. Moreover, the dS vacua obtained at the fixed points can be the deepest under certain conditions, which is also different from Ref. [96]. In addition, we discuss the relation between the modulus stabilization mechanism studied in this paper and neutrino mass models with multiple modular symmetries.

The layout of the remainder of the paper is as follows: In Sec. II, we review the basic knowledge about modular symmetries and nonperturbative effects in the string theory, and we construct the scalar potential relevant for modulus stabilization. We study the modulus stabilization and investigate its phenomenological implications for lepton masses and flavor mixing in Sec. III. We summarize our main conclusion in Sec. IV.

II. THE MODULAR-INVARIANT SCALAR POTENTIAL

A. Modular symmetry

To start with, we briefly review some basic knowledge about modular symmetries. The modular group $\bar{\Gamma}$ is isomorphic to $\text{PSL}(2, \mathbb{Z})$, defined as [2]

$$\bar{\Gamma} \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm \mathbf{I}) \middle| a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}, \quad (2.1)$$

where $\mathbf{I}$ is a two-dimensional unitary matrix. Under the modular group, the modulus τ and chiral supermultiplets $\chi^{(I)}$ transform as

$$\gamma: \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \chi^{(I)} \rightarrow (c\tau + d)^{-k_I} \rho_I(\gamma) \chi^{(I)}, \quad (2.2)$$

with γ being an element of $\bar{\Gamma}$, k_I denoting the weight of the chiral supermultiplet, and $\rho_I(\gamma)$ representing the unitary representation matrix of γ . There are two generators S and T in $\bar{\Gamma}$ satisfying $S^2 = (ST)^3 = \mathbf{I}$, the matrix representations of which can be written as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.3)$$

If we enact all the elements $\gamma \in \bar{\Gamma}$ on a given point τ in the upper-half complex plane $\mathbb{C}_+ = \{\tau \in \mathbb{C} : \text{Im}\tau > 0\}$, we will obtain an orbit of τ . Then, one can always find a minimal connected set $\mathcal{G}$, where all the orbits intersect the interior of $\mathcal{G}$ at one and only one point. The set $\mathcal{G}$ is called the fundamental domain of $\bar{\Gamma}$, defined as

$$\mathcal{G} = \left\{ \tau \in \mathbb{C}_+ : -\frac{1}{2} \leq \text{Re}\tau < \frac{1}{2}, |\tau| > 1 \right\} \cup \left\{ \tau \in \mathbb{C}_+ : -\frac{1}{2} \leq \text{Re}\tau \leq 0, |\tau| = 1 \right\}. \quad (2.4)$$

Enacting $\gamma \in \bar{\Gamma}$ on $\mathcal{G}$ will generate another fundamental domain, as shown in Fig. 1.

The modular form $f(\tau)$ is a holomorphic function of τ transforming under the modular group as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma(N), \quad (2.5)$$

where the level N and weight k are, respectively, positive and even integers, and $\Gamma(N)$ denotes the principle congruence subgroups of $\bar{\Gamma}$. For a given N , the modular forms can always be decomposed into several multiplets $Y_{\mathbf{r}}^{(k)} = (f_1(\tau), f_2(\tau), \dots)^T$ that transform as irreducible unitary representations of the quotient subgroups $\Gamma_N = \bar{\Gamma}/\Gamma(N)$: namely,

$$Y_{\mathbf{r}}^{(k)}(\gamma\tau) = (c\tau + d)^k \rho_{\mathbf{r}}(\gamma) Y_{\mathbf{r}}^{(k)}(\tau), \quad \gamma \in \Gamma_N, \quad (2.6)$$

where $\rho_{\mathbf{r}}(\gamma)$ denotes the representation matrix of Γ_N . Γ_N are the finite modular groups, isomorphic to non-Abelian discrete groups—e.g., $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$.

Now, we consider the modular-invariant supersymmetric theories. The invariance of the action $\mathcal{S}$ under the modular transformations requires that the Kähler potential $K(\tau, \bar{\tau}, \chi, \bar{\chi})$ remain unchanged up to a Kähler transformation $\mathcal{K}(\tau, \bar{\tau}, \chi, \bar{\chi}) \rightarrow \mathcal{K}(\tau, \bar{\tau}, \chi, \bar{\chi}) + u(\tau, \chi) + u(\bar{\tau}, \bar{\chi})$ [where $u(\tau, \chi)$ itself is invariant under the modular transformation], and the superpotential $\mathcal{W}(\tau, \chi)$ should exactly keep invariant. For the Kähler potential, the minimal form subject to the Kähler transformation is

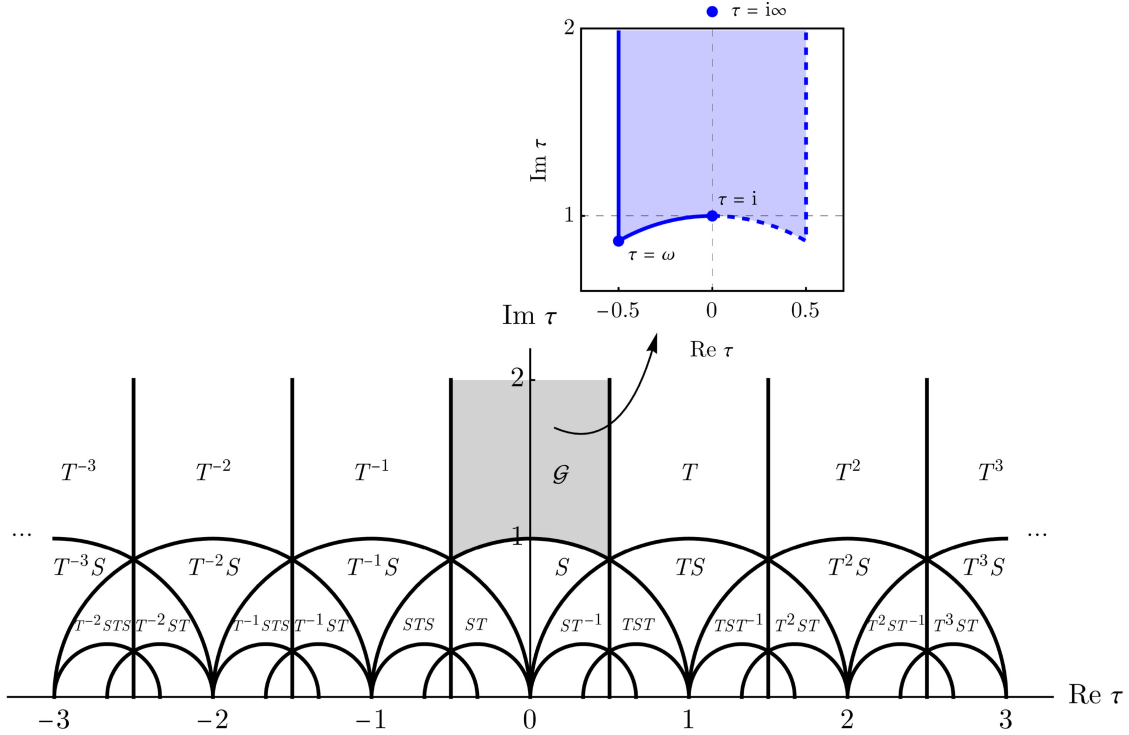


FIG. 1. Fundamental domain $\mathcal{G}$ of $\bar{\Gamma}$. Enacting $\bar{\Gamma}$ on $\mathcal{G}$ generates the entire upper-half complex plane with $\text{Im}\tau > 0$. Three fixed points— $\tau = i$, ω and $i\infty$ in $\mathcal{G}$ —are labeled by blue dots.

$$\mathcal{K}(\tau, \bar{\tau}, \chi, \bar{\chi}) = -h \log(-i\tau + i\bar{\tau}) + \sum_I \frac{|\chi^{(I)}|^2}{(-i\tau + i\bar{\tau})^{k_I}},$$

where h is a positive constant. The superpotential $\mathcal{W}(\tau, \chi)$ can be generally written as

$$\mathcal{W}(\tau, \chi) = \sum_p \sum_{\{I_1, \dots, I_p\}} (Y_{I_1 \dots I_p}(\tau) \chi^{(I_1)} \dots \chi^{(I_p)})_1. \quad (2.7)$$

In order for $\mathcal{W}(\tau, \chi)$ to be invariant under the modular transformation, the Yukawa couplings $Y_{I_1 \dots I_p}$ should take the modular forms

$$Y_{I_1 \dots I_p}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 \dots I_p}(\tau), \quad \gamma \in \Gamma_N, \quad (2.8)$$

where ρ_Y denotes the representation matrix and k_Y is the weight of $Y_{I_1 \dots I_p}(\tau)$. Note that $k_Y = k_{I_1} + \dots + k_{I_p}$, and $\rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_p} \ni \mathbf{1}$ should be satisfied.

The modular symmetry can be extended to the framework of multiple moduli [64]. Supposing that there are a series of modular groups $\bar{\Gamma}^1, \bar{\Gamma}^2, \dots, \bar{\Gamma}^M$ associated with different moduli $\tau_1, \tau_2, \dots, \tau_M$, the modular transformation of each modulus field would be

$$\gamma_i: \tau_i \rightarrow \gamma_i \tau_i = \frac{a_i \tau_i + b_i}{c_i \tau_i + d_i}. \quad (2.9)$$

Similarly to the single-modulus case, we can obtain a set of finite modular groups $\Gamma_{N_i}^i = \bar{\Gamma}^i / \Gamma^i(N_i)$. The chiral superfield $\chi^{(I)}$ then transforms under the modular group $\Gamma_{N_1}^1 \times \Gamma_{N_2}^2 \times \dots \times \Gamma_{N_M}^M$ as

$$\begin{aligned} \chi^{(I)}(\tau_1, \dots, \tau_M) &\rightarrow \chi^{(I)}(\gamma_1 \tau_1, \dots, \gamma_M \tau_M) \\ &= \prod_{i=1, \dots, M} (c_i \tau_i + d_i)^{-k_{I,i}} \\ &\quad \otimes \rho_{I,i}(\gamma_i) \chi^{(I)}(\tau_1, \tau_2, \dots, \tau_M), \end{aligned} \quad (2.10)$$

where we label the elements in $\Gamma_{N_i}^i$ as γ_i . In addition, $k_{I,i}$ and $\rho_{I,i}$ are the weights of $\chi^{(I)}$ and the corresponding representation matrices in $\Gamma_{N_i}^i$, respectively, and $\otimes$ represents the outer product of the representation matrices $\rho_{I,1}, \rho_{I,2}, \dots, \rho_{I,M}$.

Correspondingly, the Kähler potential can be rewritten as

$$\begin{aligned} \mathcal{K}(\tau_1, \dots, \tau_M, \bar{\tau}_1, \dots, \bar{\tau}_M, \chi, \bar{\chi}) &= - \sum_{i=1, \dots, M} h_i \log(-i\tau_i + i\bar{\tau}_i) \\ &\quad + \sum_I \frac{|\chi^{(I)}|^2}{\prod_{i=1, \dots, M} (-i\tau_i + i\bar{\tau}_i)^{k_{I,i}}}. \end{aligned} \quad (2.11)$$

The superpotential $\mathcal{W}(\tau_1, \dots, \tau_M, \chi)$ becomes a modular-invariant function of all the moduli fields as well as the superfields, which takes the form

$$\mathcal{W}(\tau_1, \dots, \tau_M, \chi) = \sum_p \sum_{\{I_1, \dots, I_p\}} (Y_{I_1, \dots, I_p} \chi^{(I_1)} \dots \chi^{(I_p)})_1. \quad (2.12)$$

Under the modular group, the modular forms $Y_{I_1, \dots, I_p}$ transform as

$$\begin{aligned} Y_{I_1, \dots, I_p}(\gamma_1 \tau_1, \dots, \gamma_M \tau_M) \\ = \prod_{i=1, \dots, M} (c_i \tau_i + d_i)^{k_{Y,i}} \otimes_{i=1, \dots, M} \rho_{Y,i}(\gamma_i) Y_{I_1, \dots, I_p}(\tau_1, \dots, \tau_M). \end{aligned} \quad (2.13)$$

Since the modular symmetries associated with different moduli are independent of each other, one modulus field obtaining its VEV will not affect the others. Once all the moduli acquire their individual VEVs, the entire modular symmetry will be spontaneously broken down. However, there are some fixed points of τ , where the modular symmetry is only partially broken and we are left with residual symmetries [28]. There are three different fixed points in the fundamental domain $\mathcal{G}$ (cf. Fig. 1): namely,

- (1) $\tau_C = i$, which is invariant under S and preserves a Z_2^S symmetry.
- (2) $\tau_L = -1/2 + i\sqrt{3}/2$, which is invariant under ST and preserves a Z_3^{ST} symmetry.
- (3) $\tau_T = i\infty$, which is invariant under T and preserves a Z_2^T symmetry.

It is very interesting to investigate whether these special points which are fixed by residual symmetries also have dynamical origins. This is exactly the main motivation for our work.

B. $\mathcal{N} = 1$ supergravity theory

We consider the $\mathcal{N} = 1$ supergravity theory in the Abelian heterotic orbifolds, which should generally include the dilaton, the Kähler moduli, the complex structure moduli, gauge fields, and twisted and untwisted matter fields. Here we focus on a simple scenario, where only the Kähler moduli τ_i and the dilaton field S are relevant for the scalar potential.

Let us first consider the case of one Kähler modulus τ plus one dilaton field S . In the framework of supergravity theory, supersymmetry should be regarded as a local symmetry. In this case, the Kähler potential and the superpotential are dependent on each other via the following modular-invariant Kähler function:

$$G(\tau, \bar{\tau}, S, \bar{S}) = \kappa^2 \mathcal{K}(\tau, \bar{\tau}, S, \bar{S}) + \log |\kappa^3 \mathcal{W}(\tau, S)|^2, \quad (2.14)$$

with $\kappa^2 = 8\pi/M_P^2$ (M_P is the Planck mass). Assuming the Kähler potential of τ to be the minimal form, $\mathcal{K}(\tau, \bar{\tau}, S, \bar{S})$ can be essentially expressed as

$$\mathcal{K}(\tau, \bar{\tau}, S, \bar{S}) = \Lambda_K^2 [K(S, \bar{S}) - 3 \log(2\text{Im}\tau)], \quad (2.15)$$

where Λ_K is a mass scale and $K(S, \bar{S})$ represents the Kähler potential for the dilaton.¹ At tree level, we have a simple relation $K(S, \bar{S}) \propto -\log(S + \bar{S})$, which is related to the 4D universal gauge coupling via $g_a^2/2 = 1/\langle S + \bar{S} \rangle$ once the dilaton gets its VEV. However, if nonperturbative effects such as the Shenker-like effects are included [97], additional corrections $\delta K(S, \bar{S})$ could be added into $K(S, \bar{S})$. We will see later that such effects play a crucial role in generating dS vacua. On the other hand, it is straightforward to check that $\text{Im}\tau \rightarrow |c\tau + d|^{-2} \text{Im}\tau$ under the modular transformation, hence $e^{\kappa^2 \mathcal{K}}$ should possess a weight of 6. The modular invariance of $G(\tau, \bar{\tau})$ implies that the transformation of $\mathcal{K}(\tau, \bar{\tau}, S, \bar{S})$ under the modular group is compensated by that of $\mathcal{W}(\tau)$. As a result, under the modular transformation, $\mathcal{W}(\tau)$ should transform as

$$\mathcal{W}(\tau) \rightarrow (c\tau + d)^{-3} \mathcal{W}(\tau), \quad (2.16)$$

indicating that the superpotential possesses a weight of -3 . In the next subsection, we will show that the superpotential satisfying Eq. (2.16) can be induced by a nonperturbative effect—gaugino condensation.

Once the Kähler potential and superpotential are known, we can construct the scalar potential V as [98]

$$V = e^{\kappa^2 \mathcal{K}} (\mathcal{K}^{i\bar{j}} D_i \mathcal{W} D_{\bar{j}} \mathcal{W}^* - 3\kappa^2 |\mathcal{W}|^2), \quad (2.17)$$

where $D_i = \partial_i + (\partial_i \mathcal{K})$, with ∂_i being the first derivative with respect to the Kähler moduli (which is simply $\partial/\partial\tau$ in the single-modulus case) together with the dilaton, and $\mathcal{K}^{i\bar{j}}$ is the inverse of the Kähler metric $\mathcal{K}_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}$. The scalar potential given in Eq. (2.17) is modular invariant, which is proved in Appendix A.

C. Gaugino condensation

In the heterotic string constructions, gaugino condensation is a simple example that can lead to the spontaneous breakdown of supersymmetry. A gauge group G_a undergoing gaugino condensation will give rise to a nonperturbative superpotential of the form [81–83]

$$\mathcal{W} \sim e^{-f_a/b_a}, \quad (2.18)$$

where f_a is the gauge kinetic function and b_a is the beta function of the group G_a . The gaugino condensation

¹In fact, the Kähler potential for the dilaton could also depend on τ . Here, we neglect the τ dependence for simplicity.

typically occurs at an energy scale $\Lambda_W \sim 10^{16}$ GeV [82]. At the tree level, the gauge kinetic function simply takes the form $f_a = k_a S$, with k_a being the level of the Kac-Moody algebra of G_a , which is apparently modulus independent. However, if the orbifolds of our interest arise in $\mathcal{N} = 2$ subsectors, threshold corrections to the gauge kinetic functions induced by integrating out heavy string states should be taken into consideration [74–78]. In the single-modulus case, the modified f_a can be written as

$$f_a = k_a S + b_a \log \eta^6(\tau) + \dots, \quad (2.19)$$

where $\eta(\tau)$ is the Dedekind η function (See Appendix B for the definition). The modulus-dependent term $b_a \log \eta^6(\tau)$ indicates that $\mathcal{W}$ indeed transforms under the modular group with a weight of -3 , and the dots denote additional contributions to threshold corrections which are also modulus dependent but keep invariant under the modular transformation. Apart from the threshold corrections, one-loop anomaly cancellation could also lead to significant modifications to f_a [99–102], which however can be absorbed into S by redefining the dilaton field [103]. Substituting Eq. (2.19) into Eq. (2.18), we arrive at the following parametrized form of $\mathcal{W}$:

$$\mathcal{W}(\tau, S) = \Lambda_W^3 \frac{\Omega(S)H(\tau)}{\eta^6(\tau)}, \quad (2.20)$$

where $\Omega(S)$ denotes a function of the dilaton field S , and $H(\tau)$ is a dimensionless modular-invariant function.² We can further require $H(\tau)$ to be a rational function to avoid any singularity in the fundamental domain; thus, the most general form of $H(\tau)$ should be [104]

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)), \quad (2.21)$$

with $j(\tau)$ being the modular-invariant Klein j function invariant under the modular transformation defined in Appendix B, m and n being non-negative integers, and $\mathcal{P}$ denoting a polynomial with respect to $j(\tau)$. In the following, we take $\mathcal{P} = 1$ for simplicity. It is interesting to mention that $j(\omega) = 0$ and $j(i) = 12^3 = 1728$ are satisfied at the two fixed points $\tau = \omega$ and i , respectively. Hence, $H(\tau)$ would be vanishing at $\tau = i$ (or ω) if $m \neq 0$ (or $n \neq 0$).

Once we substitute the Kähler potential in Eq. (2.15) and the superpotential in Eq. (2.20) into Eq. (2.17), the single-modulus scalar potential can be immediately expressed as [96]

²For the single gaugino condensation, a generic form of $\Omega(S)$ should be $\Omega(S) = v + e^{-S/b_a}$, with v being a constant.

$$V(\tau, \bar{\tau}, S, \bar{S}) = \Lambda_V^4 \mathcal{C}(\tau, \bar{\tau}, S, \bar{S}) [\mathcal{M}(\tau, \bar{\tau}) + (\mathcal{A}(S, \bar{S}) - 3)|H(\tau)|^2], \quad (2.22)$$

with

$$\begin{aligned} \mathcal{C}(\tau, \bar{\tau}, S, \bar{S}) &= \frac{e^{K(S, \bar{S})} |\Omega(S)|^2}{(2\text{Im}\tau)^3 |\eta(\tau)|^{12}}, \\ \mathcal{M}(\tau, \bar{\tau}) &= \frac{(2\text{Im}\tau)^2}{3} \left| iH'(\tau) + \frac{3H(\tau)}{2\pi} \hat{G}_2(\tau, \bar{\tau}) \right|^2, \\ \mathcal{A}(S, \bar{S}) &= \frac{|\Omega_S + K_S \Omega|^2}{K_{S\bar{S}} |\Omega|^2}, \end{aligned} \quad (2.23)$$

where we have $\Lambda_V = (\kappa^2 \Lambda_W^6)^{1/4}$ and $K^{S\bar{S}} = (K_{S\bar{S}})^{-1}$, and the subscripts S and $\bar{S}$ represent the first derivatives with respect to S and $\bar{S}$, respectively. Moreover, $\hat{G}_2$ is the nonholomorphic Eisenstein function of weight 2 defined as

$$\hat{G}_2(\tau, \bar{\tau}) = G_2(\tau) - \frac{\pi}{\text{Im}\tau}, \quad (2.24)$$

where the Eisenstein series G_2 is a holomorphic counterpart of $\hat{G}_2(\tau, \bar{\tau})$, and it can be related to the Dedekind η function via

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi} G_2(\tau). \quad (2.25)$$

III. MODULUS STABILIZATION

Before going into the details of minimizing the scalar potential, we can first gain some general insights without specifying the form of the scalar potential. One salient feature of the scalar potential is that it diverges in the limit $\text{Im}\tau \rightarrow \infty$. Hence, the fixed point $\tau \rightarrow i\infty$ seems not to be the vacuum. The finite fixed points, however, are able to be the minima of the scalar potential. In fact, since V is a zero-weight modular form, $\partial V/\partial\tau$ must be a modular form with weight 2 (see Appendix A for proof). Then, if we consider the modular transformation of $\partial V/\partial\tau$ under the generator S at $\tau = i$, we will arrive at

$$(\partial V/\partial\tau)|_{\tau=i} = (-i)^2 (\partial V/\partial\tau)|_{\tau=i} = -(\partial V/\partial\tau)|_{\tau=i}. \quad (3.1)$$

Similarly, if we consider the modular transformation of $\partial V/\partial\tau$ under the generator ST at $\tau = \omega$, we will obtain

$$(\partial V/\partial\tau)|_{\tau=\omega} = (-\omega - 1)^2 (\partial V/\partial\tau)|_{\tau=\omega} = \omega (\partial V/\partial\tau)|_{\tau=\omega}. \quad (3.2)$$

Equations (3.1) and (3.2) tell us that $\partial V/\partial\tau$ has to be zero at $\tau = i$ and ω . Therefore, the finite fixed points should be the extrema of the scalar potential in the Kähler modulus space.

However, identifying whether they are exactly the minima requires an in-depth analysis of certain scalar potentials.

Another important aspect is that the dilaton sector also plays a crucial role in modulus stabilization. The effects of the dilaton are mainly twofold. On the one hand, a positive $\mathcal{A}(S, \bar{S})$ term in the superpotential could uplift the minima to dS vacua. On the other hand, stringy corrections to the Kähler potential $K(S, \bar{S})$ are required to stabilize the potential in the dilaton sector. In fact, the scalar potential should satisfy $\partial V / \partial S = 0$ at the minima. Hence, we arrive at

$$\frac{\partial V}{\partial S} = \frac{\Lambda_V^4 e^K}{(2\text{Im}\tau)^3 |\eta(\tau)|^{12}} (\Omega_S + K_S \Omega) \mathcal{Q} = 0, \quad (3.3)$$

where

$$\mathcal{Q} = e^{-2i\sigma} |H|^2 \left[(\Omega_S + K_S \Omega) \left(\frac{K_S}{K_{S\bar{S}}} - \frac{K_{S\bar{S}\bar{S}}}{K_{S\bar{S}}^2} \right) + \frac{\Omega_{S\bar{S}}}{K_{S\bar{S}}} + \frac{\Omega K_{S\bar{S}}}{K_{S\bar{S}}} + \frac{\Omega_S K_S}{K_{S\bar{S}}} \right] + \bar{\Omega} (\mathcal{M} - 2|H|^2), \quad (3.4)$$

with σ being the phase angle of $\Omega_S + K_S \Omega$. Then, we can immediately gain the following two possibilities of the necessary conditions for S to be stabilized:

$$\begin{aligned} \text{Condition A: } & \Omega_S + K_S \Omega = 0; \\ \text{Condition B: } & \Omega_S + K_S \Omega \neq 0, \quad \mathcal{Q} = 0. \end{aligned} \quad (3.5)$$

Indeed, *Condition A* corresponds to the case where $\mathcal{A}(S, \bar{S})$ is vanishing—i.e., the scalar potential can be written as a factorized form of the dilaton and Kähler moduli sector. In Ref. [96], the authors have proved three no-go theorems regarding the dS vacua under *Condition A*, indicating that *Condition A* can never lead to dS vacua—i.e., the dependence of the scalar potential on the dilaton and Kähler moduli should not be factorized. Hence, one must switch to *Condition B* in order to obtain dS vacua.

Nevertheless, even if it is possible for the extrema that satisfy *Condition B* to be the dS vacua, such vacua may still be unstable in the dilaton sector. Indeed, one can prove that if only the tree-level Kähler potential for the dilaton, $K(S, \bar{S}) \propto -\log(S + \bar{S})$, is included, $\tau = i$ and ω could never be the dS vacua, no matter which form $\Omega(S)$ takes, since the extrema are unstable in the dilaton sector [96]. In order to evade the dS no-go theorems, one should go beyond the minimal Kähler potential of S . It is found in Ref. [96] that nonperturbative Shenker-like effects can result in nontrivial corrections to $K(S, \bar{S})$, rendering the dilaton sector metastable at the fixed points $\tau = i$ and ω . Differently from the gaugino condensation, which has a generic strength $\delta\mathcal{L} = e^{-1/g_s^2}$ with g_s being the string coupling constant, Shenker-like effects are inherently stringy effects which lead to modifications of $\mathcal{O}(e^{-1/g_s})$.

In the rest of this paper, we first explore how $\mathcal{A}(S, \bar{S})$ can modify the modulus stabilization, assuming the dilaton has been stabilized, and then we show concrete examples of the Shenker-like effects that can generate nontrivial $\mathcal{A}(S, \bar{S})$ in Appendix D.

A. Minimizing the single-modulus scalar potential

For the convenience of the readers, we first briefly review the minimization of the scalar potential with a single modulus, which has been widely studied under both *Condition A* and *Condition B* in the previous literature [89–91, 104]. In Refs. [90, 104], assuming $\mathcal{A}(S, \bar{S}) = 0$, the authors analyze different scalar potentials by varying the indices m and n in Eq. (2.21). They have numerically searched the minima of the scalar potentials and concluded that the vacua should appear either on the lower boundary of the fundamental domain or on the imaginary axis of τ . The authors in Ref. [91] find a special case where $m \neq 0$ and $n = 0$ can lead to global minima very close to but not precisely at the fixed point $\tau = \omega$. In summary, there are four different types of AdS vacua depending on the choices of m and n if *Condition A* is satisfied:

- (1) $m = 0$ and $n = 0$: The vacuum $\langle \tau \rangle = 1.235i$.
- (2) $m = 0$ and $n \neq 0$: The vacuum $\langle \tau \rangle = i$.
- (3) $m \neq 0$ and $n = 0$: The vacua are close to, but not precisely at, $\tau = \omega$.
- (4) $m \neq 0$ and $n \neq 0$: The vacua are located at the lower boundary of the fundamental domain.

A detailed analysis of minimizing the one-modulus scalar potential under *Condition B* can be found in Ref. [96]. In this case, the finite fixed points can be dS or Minkowski vacua. The types of these vacua dramatically depend on the values of $\mathcal{A}(S, \bar{S})$, which can be analyzed by calculating the Hessian matrices (See Appendix C). The main conclusions are collected as follows:

- (1) $m = 0$ and $n = 0$: $\tau = \omega$ is always the dS vacuum, while $\tau = i$ can be the dS vacuum if $3 < \mathcal{A}(S, \bar{S}) < 3.5964$ is satisfied;
- (2) $m > 1$, $n = 0$: $\tau = \omega$ is always a dS vacuum if $\mathcal{A}(S, \bar{S}) > 3$, and $\tau = i$ is a Minkowski vacuum.
- (3) $m = 0$, $n > 1$: $\tau = i$ is a dS vacuum within a window of $\mathcal{A}(S, \bar{S})$ which increases with n , and $\tau = \omega$ is a Minkowski vacuum.
- (4) $m = 1$ or $n = 1$: $\tau = i$ or $\tau = \omega$ could be the minimum in terms of the Kähler modulus, but it is actually unstable in the dilaton sector.
- (5) $m > 1$, $n > 1$: Both $\tau = i$ and $\tau = \omega$ are Minkowski vacua when $\mathcal{A}(S, \bar{S}) > 3$.

As can be seen above, the inclusion of dilaton effects will not only uplift the vacua to dS/Minkowski vacua, but also shift the VEVs of τ toward the fixed points. For illustration, we consider a specific case with $m = 2$ and $n = 0$. In Fig. 2, we exhibit the distribution of $\log_{10}(\Delta V / |V_{\min}|)$, with ΔV defined as the difference between V and its

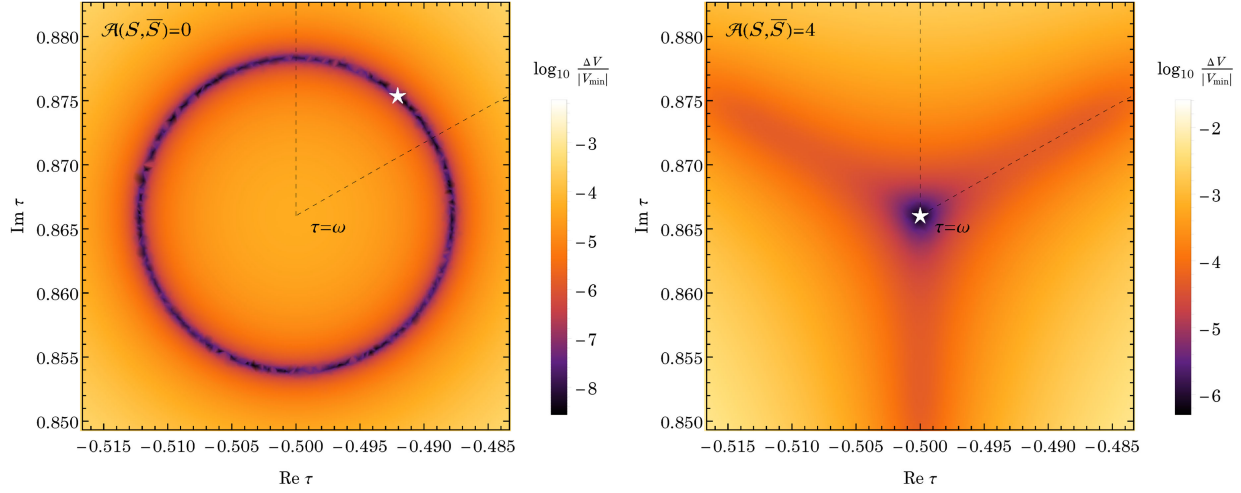


FIG. 2. Density plots of the distribution of $\log_{10}(\Delta V/|V_{\min}|)$, with $\Delta V = V - V_{\min}$ in the vicinity of $\tau = \omega$. We take $m = 2$ and $n = 0$, for instance. In the left panel, we choose $\mathcal{A}(S, \bar{S}) = 0$, while $\mathcal{A}(S, \bar{S})$ is fixed to be 4 in the right panel. We use white stars to label the vacua. The dashed lines correspond to the boundaries of the fundamental domain.

minimal value $V_{\min}$ in the vicinity of $\tau = \omega$ under the assumptions $\mathcal{A}(S, \bar{S}) = 0$ and $\mathcal{A}(S, \bar{S}) = 4$. In the case where $\mathcal{A}(S, \bar{S}) = 0$, the fixed point $\tau = \omega$ turns out to be a local maximum, and the global AdS vacuum appears at $\tau = -0.492 + 0.875i$ with $V_{\min} = -2.48 \times 10^7 \Lambda_V^4$, which is consistent with the result in Ref. [91]. However, if $\mathcal{A}(S, \bar{S}) > 3$ [e.g., $\mathcal{A}(S, \bar{S}) = 4$], we obtain a vacuum precisely at $\tau = \omega$, where the value of V is found to be $V_{\min} = 8.29 \times 10^6 \Lambda_V^4$, indicating that $\tau = \omega$ is indeed a dS vacuum.

Moreover, the presence of the dilaton term in the superpotential may also transition the fixed points into global minima. This can be understood by considering the expression of the scalar potential in Eq. (2.22). Since both $\mathcal{M}(\tau, \bar{\tau})$ and $|H(\tau)|^2$ are non-negative, the scalar potential would be semipositive definite if $\mathcal{A}(S, \bar{S}) \geq 3$. $\mathcal{M}(\tau, \bar{\tau})$ depends on $H(\tau)$ and $H'(\tau)$, both of which could be vanishing at $\tau = i$ or ω when $m > 1$ or $n > 1$, hence there would be at least one finite fixed point corresponding to the global minimum of the scalar potential.

In order to verify that finite fixed points can truly be the global vacua, we also implement a numerical approach. In specific, we initiate our analysis by randomly generating starting points in the fundamental domain. Subsequently, we employ the gradient descent technique to meticulously search for the local minima. The results are shown in Fig. 3, where the blue dots and orange diamonds represent the complete sets of local minima obtained with vanishing $\mathcal{A}(S, \bar{S})$ and nonvanishing $\mathcal{A}(S, \bar{S})$ [$\mathcal{A}(S, \bar{S}) = 3.3$], respectively. By including the dilaton effects, both $\tau = i$ and ω can be the vacua for certain ranges of $\mathcal{A}(S, \bar{S})$. It is interesting to point out that there are additional vacua inside the fundamental domain even if $\mathcal{A}(S, \bar{S}) > 3$. For example, when $m = 2$ and $n = 0$, we have another dS

vacuum at $\tau = -0.489 + 0.872i$, which is very close to $\tau = \omega$. However, this vacuum is not the global one, as $V|_{\tau=-0.489+0.872i} = 8.29 \times 10^6 \Lambda_V^4$, while $V|_{\tau=i} = 0$. In addition, when $m = 2$ and $n = 3$, we can find an additional vacuum on the lower boundary of the fundamental domain with $\tau = -0.211 + 0.978i$, which is again not the deepest, as both $\tau = i$ and ω turn out to be Minkowski vacua.

B. Modulus stabilization in the three-modulus framework

Since the compactification of 10D heterotic string theory will generally lead to three moduli, associated with three 2D tori,³ we should extend the single-modulus stabilization into this more complete scenario and explore how the nonperturbative effects can give a dynamical explanation of the VEVs of moduli with multiple modular symmetries.

In the three-modulus case, the modular-invariant function $H(\tau)$ in the superpotential should be replaced by a more general form,

$$\mathcal{H}(\tau_1, \tau_2, \tau_3) = \sum_{\substack{m_1, m_2, m_3 \\ n_1, n_2, n_3}} H^{(m_1, n_1)}(\tau_1) H^{(m_2, n_2)}(\tau_2) H^{(m_3, n_3)}(\tau_3), \quad (3.6)$$

³It should be mentioned that here, we focus on the scenario where the extra 6D space can be factorized into three T^2 tori. However, nonfactorizable toroidal manifolds [105–108] can have different geometries from factorizable ones, since the number of fixed tori could be less. Consequently, the moduli are not separable and may be incorporated into some larger symmetry groups—e.g., the Siegel modular group [5, 109–112]. The scenario for modulus stabilization in the nonseparable case could be different, which is beyond the scope of the present paper.

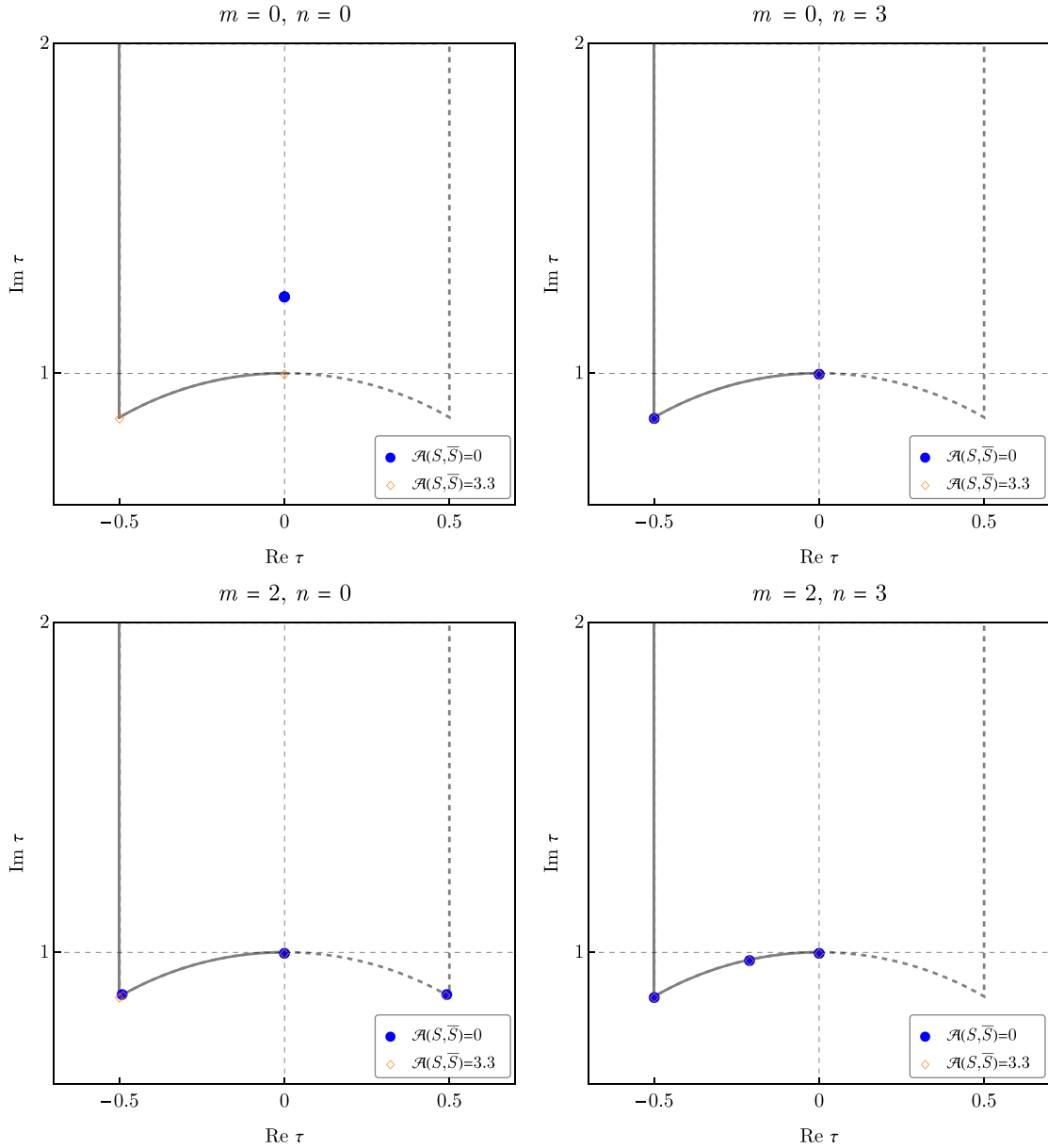


FIG. 3. Complete sets of the minima of the single-modulus scalar potential in the fundamental domain, where the blue dots and orange diamonds represent the vacua obtained with vanishing $\mathcal{A}(S, \bar{S})$ and nonvanishing $\mathcal{A}(S, \bar{S})$ [$\mathcal{A}(S, \bar{S}) = 3.3$], respectively. Different values of m and n are taken into consideration.

where $H^{(m_i, n_i)} = (j(\tau) - 1728)^{m_i/2} j(\tau)^{n_i/3}$ for $i = 1, 2, 3$. Given the infinite number of modular-invariant $\mathcal{H}(\tau_1, \tau_2, \tau_3)$, it is difficult to investigate the modulus stabilization for all $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ in a systematic way. Instead, we try to find the minimal superpotential that can lead to global dS vacua at the fixed points. One may notice that the simplest $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ should be a factorized form⁴ $H^{(m_1, n_1)}(\tau_1)H^{(m_2, n_2)}(\tau_2)H^{(m_3, n_3)}(\tau_3)$, which, however,

⁴The factorized form of the superpotential was considered in Ref. [104], where $H^{(m_i, n_i)}(\tau_i)$ for all the moduli take the same form.

would become zero as long as one $H^{(m_i, n_i)}(\tau_i)$ is vanishing at the fixed points. Consequently, this scenario will essentially lead to Minkowski vacua at the fixed points. Instead, we consider $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ as the summation of three different $H^{(m_i, n_i)}(\tau_i)$ —namely,⁵

⁵It seems more natural to expect a factorized form for $\mathcal{H}(\tau_1, \tau_2, \tau_3)$, since the loop-level corrections from each torus contribute to the superpotential as exponential forms, as can be seen in Eq. (2.18). However, $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ in Eq. (3.7) may still be realized by, e.g., introducing multiple dilatons, each of which is associated with one torus.

$$\mathcal{H}(\tau_1, \tau_2, \tau_3) = H^{(m_1, n_1)}(\tau_1) + H^{(m_2, n_2)}(\tau_2) + H^{(m_3, n_3)}(\tau_3). \quad (3.7)$$

Then, $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ would be nonzero as long as at least one of $H^{(m_i, n_i)}(\tau_i)$ is nonvanishing, making the realization of dS vacua more likely. As a result, the Kähler potential and superpotential can be rewritten as

$$\mathcal{K}(\tau_i, \bar{\tau}_i, S, \bar{S}) = \Lambda_K^2 \{K(S, \bar{S}) - \log[(2\text{Im}\tau_1)(2\text{Im}\tau_2)(2\text{Im}\tau_3)]\}, \quad (3.8)$$

$$\mathcal{W}(\tau_i, S) = \frac{\Lambda_W^3 \Omega(S) [H^{(m_1, n_1)}(\tau_1) + H^{(m_2, n_2)}(\tau_2) + H^{(m_3, n_3)}(\tau_3)]}{\eta^2(\tau_1)\eta^2(\tau_2)\eta^2(\tau_3)}, \quad (3.9)$$

respectively, where the variables τ_i go through $\{\tau_1, \tau_2, \tau_3\}$. The scalar potential in this scenario turns out to be

$$V(\tau_i, \bar{\tau}_i, S, \bar{S}) = \Lambda_V^4 \tilde{\mathcal{C}}(\tau_i, \bar{\tau}_i, S, \bar{S}) \{ \tilde{\mathcal{M}}(\tau_i, \bar{\tau}_i) + [\mathcal{A}(S, \bar{S}) - 3] |\mathcal{H}(\tau_i)|^2 \}, \quad (3.10)$$

with

$$\begin{aligned} \tilde{\mathcal{C}}(\tau_i, \bar{\tau}_i, S, \bar{S}) &= \prod_{i=1}^3 \frac{e^{K(S, \bar{S})} |\Omega(S)|^2}{(2\text{Im}\tau_i) |\eta(\tau_i)|^4}, \\ \tilde{\mathcal{M}}(\tau_i, \bar{\tau}_i) &= \sum_{i=1}^3 (2\text{Im}\tau_i)^2 \left| i \frac{\partial H^{(m_i, n_i)}(\tau_i)}{\partial \tau_i} + \sum_{j=1}^3 \frac{H^{(m_j, n_j)}(\tau_j)}{2\pi} \hat{G}_2(\tau_i, \bar{\tau}_i) \right|^2. \end{aligned} \quad (3.11)$$

One can observe that apart from $\tau_{1,2,3}$ and $\mathcal{A}(S, \bar{S})$, there are six additional parameters that can affect the minima of the scalar potential—namely, $m_{1,2,3}$ and $n_{1,2,3}$. In the following, we discuss the minimization of the scalar potential given in Eq. (3.10), mainly focusing attention on the finite fixed points i and ω . In order to identify whether they are indeed the minima of the potential, we again calculate the Hessian matrices at the fixed points and make them positive definite.

We also thoroughly search the minima of V in the entire fundamental domain for different $m_{1,2,3}$, $n_{1,2,3}$, and $\mathcal{A}(S, \bar{S})$ using the gradient descent approach, which could help us identify whether the fixed points can be the global minima of the scalar potential.

The second derivatives of V in terms of Kähler moduli are expressed as

$$\begin{aligned} \frac{\partial^2 V}{\Lambda_V^4 \partial \tau_i^2} &= \frac{\partial^2 \tilde{\mathcal{C}}}{\partial \tau_i^2} [\tilde{\mathcal{M}} + (\mathcal{A} - 3) |\mathcal{H}|^2] + \tilde{\mathcal{C}} \left[\frac{\partial^2 \tilde{\mathcal{M}}}{\partial \tau_i^2} + (\mathcal{A} - 3) \mathcal{H}^* \frac{\partial^2 H^{(m_i, n_i)}}{\partial \tau_i^2} \right], \\ \frac{\partial^2 V}{\Lambda_V^4 \partial \tau_i \partial \bar{\tau}_j} &= \frac{\partial^2 \tilde{\mathcal{C}}}{\partial \tau_i \partial \bar{\tau}_j} [\tilde{\mathcal{M}} + (\mathcal{A} - 3) |\mathcal{H}|^2] + \tilde{\mathcal{C}} \left[\frac{\partial^2 \tilde{\mathcal{M}}}{\partial \tau_i \partial \bar{\tau}_j} + (\mathcal{A} - 3) \left| \frac{\partial H^{(m_i, n_i)}}{\partial \tau_i} \right|^2 \right], \end{aligned} \quad (3.12)$$

with

$$\begin{aligned} \frac{\partial^2 \tilde{\mathcal{C}}}{\partial \tau_i \partial \tau_j} &= -i \delta_{ij} \tilde{\mathcal{C}} \frac{\partial}{\partial \tau_i} \frac{\hat{G}_2(\tau_i, \bar{\tau}_i)}{2\pi}, \\ \frac{\partial^2 \tilde{\mathcal{C}}}{\partial \tau_i \partial \bar{\tau}_j} &= i \delta_{ij} \tilde{\mathcal{C}} \frac{\partial}{\partial \tau_i} \frac{[\hat{G}_2(\tau_j, \bar{\tau}_j)]^*}{2\pi}, \\ \frac{\partial^2 \tilde{\mathcal{M}}}{\partial \tau_i \partial \tau_j} &= \delta_{ij} (2\text{Im}\tau_i)^2 \left[i \frac{\partial^2 H^{(m_i, n_i)}(\tau_i)}{\partial \tau_i^2} + \frac{\mathcal{H}(\tau_i)}{2\pi} \frac{\partial \hat{G}_2(\tau_i, \bar{\tau}_i)}{\partial \tau_i} \right] \frac{\mathcal{H}^*(\tau_i)}{\pi} \frac{\partial \hat{G}_2^*(\tau_i, \bar{\tau}_i)}{\partial \tau_i}, \\ \frac{\partial^2 \tilde{\mathcal{M}}}{\partial \tau_i \partial \bar{\tau}_j} &= \delta_{ij} (2\text{Im}\tau_i)^2 \left[\left| i \frac{\partial^2 H^{(m_i, n_i)}(\tau_i)}{\partial \tau_i^2} + \frac{\mathcal{H}(\tau_i)}{2\pi} \frac{\partial \hat{G}_2(\tau_i, \bar{\tau}_i)}{\partial \tau_i} \right|^2 + \left| \frac{\mathcal{H}(\tau_i)}{2\pi} \frac{\partial \hat{G}_2(\tau_i, \bar{\tau}_i)}{\partial \bar{\tau}_i} \right|^2 \right], \end{aligned} \quad (3.13)$$

where all the derivatives above are calculated at $\tau = i$ or ω .

According to the choices of (m_i, n_i) , we have the following three distinct classes:

- (1) *Class A*: $(m_1, n_1) = (m_2, n_2) = (m_3, n_3)$. This is an exactly symmetric class, where three modulus parameters can be exchanged. It is natural to expect that the global minima should appear at $\tau_1 = \tau_2 = \tau_3$. Then, one can find that $\tilde{\mathcal{C}}(\tau_i, \bar{\tau}_i, S, \bar{S})$ and $\tilde{\mathcal{M}}(\tau_i, \bar{\tau}_i)$ in Eq. (3.11) reduce to the single-modulus case, and thus the results for the minima are the same as those obtained in the single-modulus case.
- (2) *Class B*: $(m_1, n_1) = (m_2, n_2) \neq (m_3, n_3)$. In this case, we can freely exchange τ_1 and τ_2 without affecting the value of the scalar potential, hence it is effectively a two-modular case, where only two Kähler moduli, τ_1 (or τ_2) and τ_3 , are independent.
- (3) *Class C*: $(m_1, n_1) \neq (m_2, n_2) \neq (m_3, n_3)$. This class becomes more complicated, since there is no symmetry among the three moduli parameters. In this class, we should consider all three moduli as free parameters.

We first focus on *Class B*. In this class, the number of independent real variables is reduced to four, indicating that the Hessian matrices should be four dimensional. On the other hand, Eq. (3.13) tells us that all the mixed second derivatives in terms of different moduli are vanishing. Moreover, the imaginary parts of $\partial^2 V / (\partial \tau_i \partial \tau_j)$ are also zero at the finite fixed points. Hence, we arrive at the following diagonal Hessian matrices:

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 V}{\partial s_1^2} & 0 & 0 & 0 \\ 0 & \frac{\partial^2 V}{\partial t_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 V}{\partial s_3^2} & 0 \\ 0 & 0 & 0 & \frac{\partial^2 V}{\partial t_3^2} \end{pmatrix}. \quad (3.14)$$

Therefore, in order for finite fixed points to be the minima of the scalar potential, we should require each element in Eq. (3.14) to be positive.

Since effectively we have two independent modulus parameters τ_1 and τ_3 , there are 12 kinds of arrangements of the indices (m_i, n_i) depending on whether they are zero or not, including

- (1) $(m_1, n_1) = (0, 0)$, $(m_3, n_3) = (0, \underline{n}_3)$,
- (2) $(m_1, n_1) = (0, 0)$, $(m_3, n_3) = (\underline{m}_3, 0)$,
- (3) $(m_1, n_1) = (0, 0)$, $(m_3, n_3) = (\underline{m}_3, \underline{n}_3)$,
- (4) $(m_1, n_1) = (0, \underline{n}_1)$, $(m_3, n_3) = (\underline{m}_3, 0)$,
- (5) $(m_1, n_1) = (0, \underline{n}_1)$, $(m_3, n_3) = (\underline{m}_3, \underline{n}_3)$,
- (6) $(m_1, n_1) = (\underline{m}_1, 0)$, $(m_3, n_3) = (\underline{m}_3, \underline{n}_3)$,

together with their counterparts formed by exchanging the subscripts 1 and 3. Note that we use $\underline{m}_i$ and $\underline{n}_i$ to underline nonvanishing m_i and n_i . In the following, we choose $\underline{m}_i = 2$ and $\underline{n}_i = 3$ for illustration. Then, the powers of $j(\tau) - 1728$ and $j(\tau)$ in $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ become integers, which

simplifies the calculation. Such a parameter choice also allows us to avoid the problem that the scalar potential cannot be stabilized in the dilaton sector [96].

We can take $(m_1, n_1) = (0, 0)$ and $(m_3, n_3) = (0, 3)$ as an example. In order for the Hessian matrix in Eq. (3.14) to be positive definite, one should require

$$\left(\frac{\partial^2 V}{\partial s_1^2}, \frac{\partial^2 V}{\partial t_1^2} \right) \Big|_{\tau_1 = i \text{ or } \omega} > 0, \quad \left(\frac{\partial^2 V}{\partial s_3^2}, \frac{\partial^2 V}{\partial t_3^2} \right) \Big|_{\tau_3 = i \text{ or } \omega} > 0. \quad (3.15)$$

Substituting the values of m_i and n_i into the above inequalities, we arrive at the following conditions:

$$\begin{aligned} \tau_1 = i: \quad & 3.596 - \mathcal{A} > 0, \quad \mathcal{A} - 0.4036 > 0. \\ \tau_1 = \omega: \quad & \mathcal{A} - 2 > 0. \\ \tau_3 = i: \quad & 117.2 - \mathcal{A} > 0, \quad \mathcal{A} + 113.2 > 0. \\ \tau_3 = \omega: \quad & \mathcal{A} - 2 > 0. \end{aligned} \quad (3.16)$$

Then, we can immediately find that $\tau_1 = i$ and ω can be the vacua of the scalar potential only if $0.4036 < \mathcal{A} < 3.596$ and $\mathcal{A} > 2$, respectively. These conditions are exactly consistent with those in the one-modulus case with $(m, n) = (0, 0)$, which can be understood as follows. From Eqs. (C3) and (3.13), one could realize that the main difference between the second derivatives in the three-modulus case and those in the single-modulus case is that we replace $H(\tau)$ with $\mathcal{H}(\tau_i) = H^{(m_1, n_1)}(\tau_1) + H^{(m_2, n_2)}(\tau_2) + H^{(m_3, n_3)}(\tau_3)$. Given that $\partial H^{(0,0)} / \partial \tau_i = 0$ and $\partial^2 H^{(0,0)} / \partial \tau_i^2 = 0$, $|\mathcal{H}|^2$ can actually be extracted out as an overall factor in Eq. (3.12). As a consequence, we obtain the same conditions for $\tau = i, \omega$ to be the vacua as those in the single-modulus case. Meanwhile, the conditions for $\tau_3 = i$ and ω to be the vacua turn out to be $-113.2 < \mathcal{A} < 117.2$ and $\mathcal{A} > 2$, respectively, the former of these being different from that in the single-modulus case with $(m, n) = (0, 3)$ obtained in Ref. [96]. This is because $\partial^2 H^{(0,3)} / \partial \tau_i^2 \neq 0$ at the fixed points, and therefore one cannot extract an overall $|\mathcal{H}|^2$ in Eq. (3.12). As a summary, we arrive at the following conditions for different fixed points to be the dS vacua in the case where $(m_1, n_1) = (m_2, n_2) = (0, 0)$ and $(m_3, n_3) = (0, 3)$:

- (1) $\tau_1 = \tau_2 = \tau_3 = i$: $3 < \mathcal{A} < 3.596$.
- (2) $\tau_1 = \tau_2 = i$, $\tau_3 = \omega$: $3 < \mathcal{A} < 3.596$.
- (3) $\tau_1 = \tau_2 = \omega$, $\tau_3 = i$: $3 < \mathcal{A} < 117.2$.
- (4) $\tau_1 = \tau_2 = \tau_3 = \omega$: $\mathcal{A} > 3$.

We have also numerically searched the minima of V by scanning the parameter space of τ_1 , τ_2 , τ_3 , and $\mathcal{A}$. The results support the above conclusions. The numerical calculation also reveals where the deepest vacuum is. Assuming $\mathcal{A} = 3.3$, we arrive at $V|_{(\omega, \omega, \omega)} = 3.331 \Lambda_V^4$, $V|_{(i, i, \omega)} = 3.475 \Lambda_V^4$, $V|_{(i, i, i)} = 2.656 \times 10^6 \Lambda_V^4$, and

TABLE I. Possible vacua of the scalar potential at fixed points for different choices of m_i and n_i in *Class B*, where we set $(m_1, n_1) = (m_2, n_2) \neq (m_3, n_3)$, together with the corresponding constraints on $\mathcal{A}(S, \bar{S})$ for the vacua not to be AdS vacua. Note that $\tau_1 = \tau_2$ should be satisfied in *Class B*. The global minima in each case are indicated in bold font.

$(m_{1,2}, n_{1,2})$	(m_3, n_3)	$\tau_{1,2}$	τ_3	$\mathcal{A}(S, \bar{S})$	$(m_{1,2}, n_{1,2})$	(m_3, n_3)	$\tau_{1,2}$	τ_3	$\mathcal{A}(S, \bar{S})$
(0,0)	(0,3)	i	i	(3,3.596)	(0,3)	(0,0)	i	i	(3,3.596)
		i	ω	(3,3.596)			i	ω	(3,60.43)
		ω	i	(3,117.2)			ω	i	(3,3.596)
		ω	ω	(3, +∞)			ω	ω	(3, +∞)
(0,0)	(2,0)	i	i	(3,3.596)	(2,0)	(0,0)	i	i	(3,3.596)
		i	ω	(3,3.596)			i	ω	(3,198624)
		ω	i	(3,99314)			ω	i	(3,3.596)
		ω	ω	(3, + ∞)			ω	ω	(3, + ∞)
(0,0)	(2,3)	i	i	(3,3.596)	(2,3)	(0,0)	i	i	(3,3.596)
		i	ω	(3,3.596)			i	ω	(3, 3.43 $\times 10^8$)
		ω	i	(3, 1.72 $\times 10^8$)			ω	i	(3,3.596)
		ω	ω	(3, +∞)			ω	ω	(3, +∞)
(2,0)	(0,3)	i	i	(3,117.3)	(0,3)	(2,0)	i	i	(3,60.45)
		i	ω	[3, +∞)			i	ω	[3, 117.3)
		ω	i	(3,114.1)			ω	i	[3, +∞)
		ω	ω	(3, + ∞)			ω	ω	(3, + ∞)
(2,0)	(2,3)	i	i	[3, +∞)	(2,3)	(2,0)	i	i	[3, +∞)
		i	ω	[3, +∞)			i	ω	(3, + ∞)
		ω	i	(3, + ∞)			ω	i	[3, + ∞)
		ω	ω	(3, + ∞)			ω	ω	(3, + ∞)
(0,3)	(2,3)	i	i	(3,60.45)	(2,3)	(0,3)	i	i	(3,117.3)
		i	ω	(3,60.45)			i	ω	[3, +∞)
		ω	i	[3, +∞)			ω	i	(3,117.3)
		ω	ω	[3, +∞)			ω	ω	[3, +∞)

$V|_{(\omega, \omega, i)} = 2.546 \times 10^6 \Lambda_V^4$. It is then apparent that $\tau_1 = \tau_2 = \tau_3 = \omega$ corresponds to the global minimum.

Following the same procedure, we can also calculate the vacua for other arrangements of (m_i, n_i) . The results are summarized in Table I, where we show possible vacua situated at the fixed points, together with the corresponding constraints on $\mathcal{A}(S, \bar{S})$. Some remarks are as follows:

- (1) As mentioned before, since $(m_1, n_1) = (m_2, n_2)$ is assumed, we only consider the vacua with $\tau_1 = \tau_2$ that preserve the symmetry between τ_1 and τ_2 . Although the vacuum may also exist when τ_1 and τ_2 take different values, the symmetric vacua should be in general deeper.
- (2) $(\tau_1, \tau_2) = (\omega, \omega)$ is always the vacuum, while other fixed points could be the vacua for certain ranges of $\mathcal{A}(S, \bar{S})$. If there is at least one pair of (m_i, n_i) equal to (0,0), the global minimum of the scalar potential would be the dS vacuum. If none of (m_i, n_i) equals zero, the Minkowski vacuum could exist. Similarly to the single-modulus case, numerically we could find dS vacua close to the fixed points $\tau = \omega$ in some cases, but they are not the deepest vacua of the scalar potential.

- (3) If we exchange the values of (m_1, n_1) and (m_3, n_3) , we will arrive at a mirrored case, in which similar vacua could also be easily obtained by reversing the values of τ_1 and τ_3 . The allowed ranges of $\mathcal{A}(S, \bar{S})$ for dS vacua may change by roughly a factor of 2, since there are actually two moduli, τ_1 and τ_2 , associated with (m_1, n_1) .
- (4) In the single-modulus case, it is shown that $\tau = i$ will always be the minimum as long as $m > 1$ [95], since the Hessian matrix is positive definite and does not depend on $\mathcal{A}(S, \bar{S})$. However, this is not the case in the three-modulus extension. Taking $(m_1, n_1) = (0, 0)$ and $(m_3, n_3) = (2, 0)$, for instance, nonvanishing $H^{(0,0)}(\omega)$ recruits the dependence on $\mathcal{A}(S, \bar{S})$ in the Hessian matrix, setting an upper bound on $\mathcal{A}(S, \bar{S})$ for $\tau = i$ to be the minimum, which is of the order of $[\partial^2 H^{(2,0)}(\tau_i)/\partial \tau_i^2]|_{\tau_i=i}$.
- (5) Among all the dS vacua, two of them are phenomenologically interesting. These two vacua appear at $\tau_1 = \tau_2 = i$ and $\tau_3 = \omega$ when $(m_1, n_1) = (2, 0)$ and $(m_3, n_3) = (0, 0)$, and at $\tau_1 = \tau_2 = \omega$ and $\tau_3 = i$ when $(m_1, n_1) = (0, 0)$ and $(m_3, n_3) = (2, 0)$. In

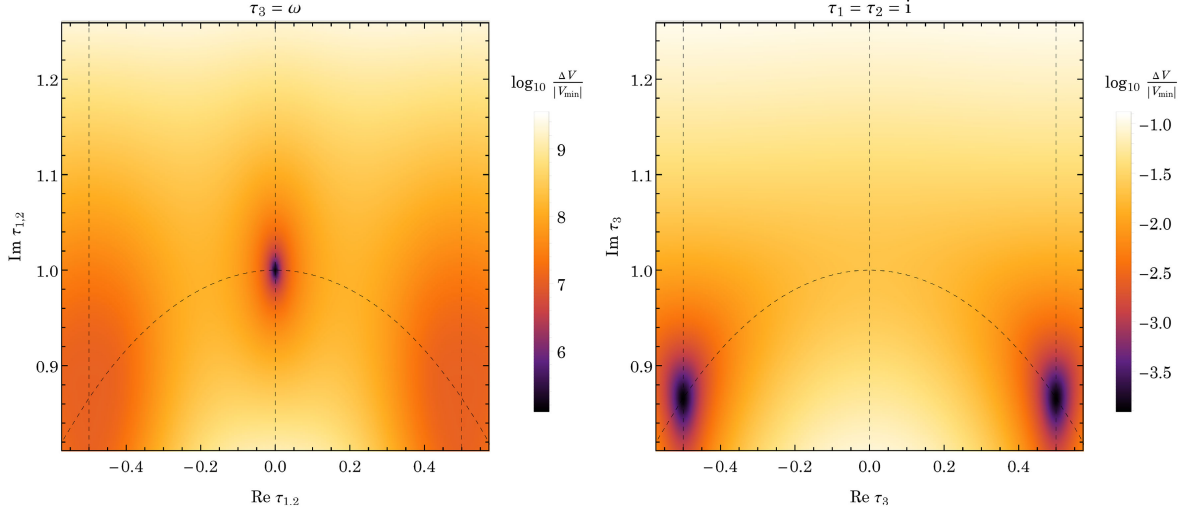


FIG. 4. Illustration for the vacua of the scalar potential in the case where $(m_1, n_1) = (m_2, n_2) = (2, 0)$ and $(m_3, n_3) = (0, 0)$. We focus on the global vacuum $(\tau_1, \tau_2, \tau_3) = (i, i, \omega)$. For each plot, we fix τ_1 (τ_2) or τ_3 and exhibit the projection of $\log_{10}(\Delta V/|V_{\min}|)$ in terms of the other modulus parameter. Left: $\tau_3 = \omega$ is fixed. Right: $\tau_1 = \tau_2 = i$ is fixed.

Fig. 4, we show the projections of $\log_{10}(\Delta V/|V_{\min}|)$ with the choice $(m_1, n_1) = (2, 0)$ and $(m_3, n_3) = (0, 0)$ by fixing $\tau_3 = \omega$ (left panel) and $\tau_1 = \tau_2 = i$ (right panel), where one can indeed find that the global minimum appears when $\tau_1 = \tau_2 = i$ and $\tau_3 = \omega$ in this case. In the next subsection, we will demonstrate that they can lead to viable models which can account for neutrino masses and flavor mixing.

- (6) In the above analysis of searching for the minima of the scalar potential, we regard $\mathcal{A}$ as a free parameter. As we mentioned at the beginning of this section, obtaining the required values of $\mathcal{A}$ is nontrivial, but it depends on specific Shenker-like corrections in the dilaton Kähler potential. As preliminary examples, in Appendix D, we construct concrete Shenker-like terms that can stabilize the dilaton sector and generate feasible $\mathcal{A}(S, \bar{S})$ as shown in Table I.

At the end of this subsection, let us discuss *Class C*. As each pair of (m_i, n_i) should be different from the others, there are in total four different choices of (m_i, n_i) :

- (1) $(m_1, n_1) = (0, 0), (m_2, n_2) = (0, \underline{n}_2), (m_3, n_3) = (\underline{m}_3, 0).$
- (2) $(m_1, n_1) = (0, 0), (m_2, n_2) = (0, \underline{n}_2), (m_3, n_3) = (\underline{m}_3, \underline{n}_3).$
- (3) $(m_1, n_1) = (0, 0), (m_2, n_2) = (\underline{m}_2, 0), (m_3, n_3) = (\underline{m}_3, \underline{n}_3).$
- (4) $(m_1, n_1) = (0, \underline{n}_1), (m_2, n_2) = (\underline{m}_2, 0), (m_3, n_3) = (\underline{m}_3, \underline{n}_3).$

Although this entire nonsymmetric class would be much more complicated since all the moduli should be regarded as free variables, one can still follow the similar method adopted with *Class B* to determine the vacua. It is straightforward to obtain the following conditions for the finite fixed points to be the minima:

$$\left. \left(\frac{\partial^2 V}{\partial s_1^2}, \frac{\partial^2 V}{\partial s_2^2}, \frac{\partial^2 V}{\partial s_3^2} \right) \right|_{\tau_i = i \text{ or } \omega} > 0, \quad \left. \left(\frac{\partial^2 V}{\partial t_1^2}, \frac{\partial^2 V}{\partial t_2^2}, \frac{\partial^2 V}{\partial t_3^2} \right) \right|_{\tau_i = i \text{ or } \omega} > 0. \quad (3.17)$$

Taking $(m_1, n_1) = (0, 0)$, $(m_2, n_2) = (0, 3)$, and $(m_3, n_3) = (2, 0)$, for instance, according to Eq. (3.17), $\tau = i$ or ω being the dS minimum requires

$$\begin{aligned} \tau_1 = \tau_2 = \tau_3 = i: & \quad 3 < \mathcal{A} < 3.596. \\ \tau_1 = \tau_2 = i, \quad \tau_3 = \omega: & \quad 3 < \mathcal{A} < 3.596. \\ \tau_1 = \tau_3 = i, \quad \tau_2 = \omega: & \quad 3 < \mathcal{A} < 3.596. \\ \tau_2 = \tau_3 = i, \quad \tau_1 = \omega: & \quad 3 < \mathcal{A} < 118.5. \\ \tau_1 = \tau_2 = \omega, \quad \tau_3 = i: & \quad 3 < \mathcal{A} < 198624. \\ \tau_1 = \tau_3 = \omega, \quad \tau_2 = i: & \quad 3 < \mathcal{A} < 200907. \\ \tau_2 = \tau_3 = \omega, \quad \tau_1 = i: & \quad 3 < \mathcal{A} < 3.596. \\ \tau_1 = \tau_2 = \tau_3 = \omega: & \quad \mathcal{A} > 3. \end{aligned} \quad (3.18)$$

Hence, either i or ω can be the dS vacuum for certain ranges of $\mathcal{A}$. Differently from *Class B*, we have two degenerate global minima in the fundamental domain. To be specific, the numerical calculation shows that $(\tau_1, \tau_2, \tau_3) = (\omega, \omega, i)$

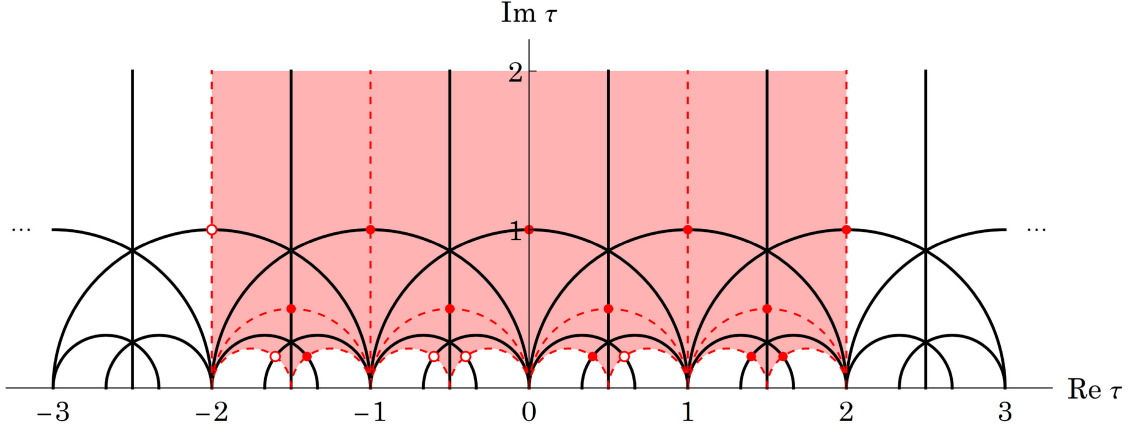


FIG. 5. The fundamental domain $\mathcal{G}(4)$ of $\Gamma(4)$ is shaded red. Red dots label the values of τ which can be converted to $\tau = i$ via the modular transformation $\gamma \in \Gamma_4$, where the hollow dots are removed due to redundancies.

and $(\tau_1, \tau_2, \tau_3) = (\omega, i, \omega)$ correspond to the minimal value of the potential, $V_{\min} = 0.851\Lambda_V^4$ [$\mathcal{A}(S, \tilde{S}) = 3.3$ has been assumed].

The other three cases can also be analyzed in a similar way. If $(m_1, n_1) = (0, 0)$, $(m_2, n_2) = (0, 3)$, and $(m_3, n_3) = (2, 3)$, the global minima appear at $(\tau_1, \tau_2, \tau_3) = (\omega, \omega, \omega)$ with $V_{\min} = 0.833\Lambda_V^4$. If $(m_1, n_1) = (0, 0)$, $(m_2, n_2) = (2, 0)$, and $(m_3, n_3) = (2, 3)$, the global minimum would be at $(\tau_1, \tau_2, \tau_3) = (\omega, i, \omega)$ with $V_{\min} = 0.851\Lambda_V^4$. In addition, if no pair of (m_i, n_i) is selected to be $(0, 0)$, the global minima would become Minkowski vacua.

C. Phenomenological implications for lepton masses and flavor mixing

The simplest factorizable compactifications with more than one torus motivate several bottom-up models based on multiple moduli fields, which can account for lepton masses, flavor mixing, and CP violation [64–68]. The main idea is to introduce multiple modular symmetries, each of which is related to one modulus field. The transformation of each modulus under the corresponding modular group is independent of the transformations of all other moduli, as shown in Eq. (2.9). Similarly to the single-modulus case, the chiral supermultiplets and Yukawa couplings are arranged as irreducible representations under different finite modular groups $\Gamma_{N_i}^i$. It should be mentioned that we also need to introduce a couple of extra flavon fields, which transform as bimultiplets under the $\Gamma_{N_i}^i$ groups. Once these flavons obtain their individual VEVs, multiple modular symmetries are spontaneously broken to a unified finite modular symmetry, and then the flavor structures in the charged-lepton and neutrino sectors are governed by the VEVs of different moduli fields. The VEVs of bimultiplets can be determined by introducing driving fields [64], which are assumed to be irrelevant for the modulus stabilization.

The modulus stabilization we have discussed in the previous sections is based on the infinite modular group $\tilde{\Gamma}$.

As has been shown in Fig. 5, the modulus parameter inside the fundamental domain $\mathcal{G}$ can be mapped into other domains via modular transformations, hence we should have an infinite number of degenerate vacua of τ in the upper-half complex plane. If we consider a specific finite modular group Γ_N , enacting Γ_N on $\mathcal{G}$ will give rise to the fundamental domain of $\Gamma(N)$ —namely, $\mathcal{G}(N) = \Gamma_N \mathcal{G}$. Any transformation $\gamma \in \Gamma_N$ acting on $\mathcal{G}(N)$ will leave $\mathcal{G}(N)$ invariant, indicating that $\mathcal{G}(N)$ is actually a target space of Γ_N [50]. In Fig. 5, we exhibit the fundamental domain $\mathcal{G}(4)$ of $\Gamma(4)$. In order to illustrate the degeneracy of vacua inside $\mathcal{G}(4)$, we take the fixed point $\tau = i$, for instance. The red dots in Fig. 5 denote the values of τ which can be converted to $\tau = i$ via the modular transformation $\gamma \in \Gamma_4$. Given that $S^2 = (ST)^3 = T^4 = \mathbf{I}$ should be satisfied, we have the following equalities:

$$\begin{aligned} 2+i &= -2+i, & \frac{2}{5}+\frac{i}{5} &= -\frac{2}{5}+\frac{i}{5}, & \frac{7}{5}+\frac{i}{5} &= \frac{3}{5}+\frac{i}{5}, \\ -\frac{7}{5}+\frac{i}{5} &= -\frac{3}{5}+\frac{i}{5}, & \frac{8}{5}+\frac{i}{5} &= -\frac{8}{5}+\frac{i}{5}, \end{aligned} \quad (3.19)$$

indicating that there are redundant points on the boundary of $\mathcal{G}(4)$, which are represented by the hollow dots in Fig. 5. Then, one can easily observe that if $\tau = i$ turns out to be the vacuum, there will be 11 additional degenerate vacua in the target space of the S_4 group. In the single-modulus case, if two moduli can be related to each other via a modular transformation, the resulting physical observables would be the same, since the modular transformation in the neutrino sector compensates for that in the charged-lepton sector, and the final physical quantities will be modular invariant. Nevertheless, if multiple modulus parameters come into the superpotential, we are unable to arbitrarily vary the values of moduli via modular transformations without changing the results of physical observables, due to the relative phases among the moduli. For example, (i, i, i) and

TABLE II. Models with multiple modular symmetries investigated in the previous literature. The values of modulus parameters in the charged-lepton and neutrino sectors that can generate viable lepton masses and flavor mixing, together with the corresponding flavor mixing patterns, are summarized. In addition, we explicitly present benchmark values of m_i and n_i that can realize such kinds of vacua.

Modular group	Charged-lepton sector	Neutrino sector	Flavor pattern	References
$S_4^A \times S_4^B \times S_4^C$	$\tau_C = \omega$ $(m_C, n_C) = (0, 0)$	$\tau_A = -\omega^2$ $\tau_B = 1/2 + i/2$ $(m_A, n_A) = (0, 0)$ $(m_B, n_B) = (2, 0)$	TM ₁	Ref. [64]
$S_4^I \times S_4^{\nu}$	$\tau_I = \omega$ $(m_I, n_I) = (0, 0)$	$\tau_{\nu} = -1/2 + i/2$ $(m_{\nu}, n_{\nu}) = (2, 0)$	TM ₁	Ref. [65]
$S_4^F \times S_4^{Na}$	$\tau_F = \omega$ $(m_F, n_F) = (0, 0)$	$\tau_N = -1/2 + i/2$ $(m_N, n_N) = (2, 0)$	CM + TM ₁	Ref. [66]
$S_4^A \times S_4^B \times S_4^C$	$\tau_C = \omega$ $(m_C, n_C) = (0, 0)$	$\tau_A = 1/2 + i/2$ $\tau_B = 3/2 + i/2$ $(m_A, n_A) = (2, 0)$ $(m_B, n_B) = (2, 0)$	Littlest modular seesaw	Refs. [67,71]
$S_4^A \times S_4^B \times S_4^C$	$\tau_C = \omega$ $(m_C, n_C) = (0, 0)$	$\tau_A = i + 2$ $\tau_B = i$ $(m_A, n_A) = (2, 0)$ $(m_B, n_B) = (2, 0)$	Littlest modular seesaw	Ref. [68]
$A_4^I \times A_4^{\nu}$	$\tau_I = 3/2 + i/(2\sqrt{3})$ $(m_I, n_I) = (0, 0)$	$\tau_{\nu} = i$ $(m_{\nu}, n_{\nu}) = (2, 0)$	TM ₂	Ref. [69]
$A_5^I \times A_5^{\nu}$	$\tau_I = i\infty$	$\tau_{\nu} = i$	GR ₂	Ref. [70]
...				

^aIn Ref. [66], the authors work in a SU(5) grand unified extension of flavor models involving two modular S_4 groups. S_4^F acts on quarks and left-handed lepton doublets, while S_4^{Na} acts on the right-handed neutrino sector. An approximate TM₁ lepton flavor mixing and a Cabbibo mixing (CM) in the quark sector are realized in their model.

($i, i, i + 2$) would in principle result in different physical consequences.

In Table II, we summarize various lepton flavor models with multiple modular symmetries, where the values of τ are taken to be precisely at the fixed points. Moreover, we also show some benchmark values of m_i and n_i that can lead to such kinds of vacua. Except for the modular $A_5^I \times A_5^{\nu}$ model [70] where the value of τ_I is fixed to be $i\infty$, the VEVs of moduli required in all the other models can indeed be realized in our formalism. In particular, in the modular $S_4^A \times S_4^B \times S_4^C$ model discussed in Ref. [65], the TM₁ mixing pattern requires $\tau_A = -\omega^2$, $\tau_B = 1/2 + i/2$ and $\tau_C = \omega$, which can be fulfilled by choosing $(m_A, n_A) = (m_C, n_C) = (0, 0)$ and $(m_B, n_B) = (2, 0)$. The littlest seesaw models can also be realized in the framework of the $S_4^A \times S_4^B \times S_4^C$ symmetry [67,68,71], where the required VEVs of moduli can be generated by choosing $(m_A, n_A) = (m_B, n_B) = (2, 0)$ and $(m_C, n_C) = (0, 0)$. Although our primary focus in this paper is on the vacua of the three-modulus scalar potential, the case with two moduli fields is analogous to *Class B* discussed previously, which involves two sets of identical (m_i, n_i) . Consequently, it is straightforward to derive the conditions for the vacua to be located at the fixed points in the two-modulus scenario. Therefore, we

indeed find a dynamical origin of the VEVs of moduli fields in the modular-invariant models with multiple moduli.

IV. SUMMARY

The modular symmetry provides us with a satisfactory and appealing framework for addressing the flavor problem. The only flavons present in such a framework are one or more moduli fields τ . It seems that the fixed points $\tau = i$ and $\tau = \omega$ play a special role in both the phenomenological model building and the 10D supersymmetric orbifold examples. However, revealing the origin of the VEVs of moduli is still an intricate challenge.

In this paper, we study the modulus stabilization within the multiple-modulus framework. In line with Ref. [96], we consider the Kähler moduli and dilaton but neglect their coupling with matter fields. The influence of the dilaton sector is twofold. On the one hand, the tree-level dilaton Kähler potential will be modified by additional nonperturbative stringy effects—e.g., Shenker-like effects—which are vital for us to evade several dS no-go theorems. On the other hand, the dilaton will enter the superpotential as a functional form $\Omega(S)$. The parametrized form of the superpotential turns out to be Eq. (3.9), where the modular-invariant function $H(\tau)$ in the single-modulus case is

replaced by $\mathcal{H}(\tau_1, \tau_2, \tau_3) = H^{(m_1, n_1)}(\tau_1) + H^{(m_2, n_2)}(\tau_2) + H^{(m_3, n_3)}(\tau_3)$ in the three-modulus case. The scalar potential in the three-modulus scenario is then given by Eq. (3.10), where the contribution from the dilaton sector is parametrized by $\mathcal{A}(S, \bar{S})$.

We numerically search the minima of the scalar potential in the entire parameter space of τ_i and $\mathcal{A}(S, \bar{S})$, and we calculate the Hessian matrices at the fixed points $\tau = i$ and ω . Due to the existence of additional Kähler moduli, the vacua look rather different from those in the single-modulus case. In fact, both the finite fixed points $\tau = i$ and $\tau = \omega$ could be the dS vacua of the scalar potential if specific conditions on $\mathcal{A}(S, \bar{S})$ are satisfied. We classify different choices of vacua by varying the indices (m_i, n_i) , and we summarize conditions for the vacua to be dS minima in Table I, which are also distinct from the single-modulus case. In addition, dS vacua can be found in the interior of the fundamental domain (even close to the fixed points), which are, however, not the global minima of the potential.

Modulus stabilization as discussed in this paper has significant phenomenological implications for fermion masses and flavor mixing, once the finite modular groups are specified. In particular, we find that the vacua $(\tau_1, \tau_2, \tau_3) = (\omega, \omega, i)$ [obtained by setting $(m_1, n_1) =$

$(m_2, n_2) = (0, 0)$ and $(m_3, n_3) = (2, 0)$] and $(\tau_1, \tau_2, \tau_3) = (i, i, \omega)$ [obtained by setting $(m_1, n_1) = (m_2, n_2) = (2, 0)$ and $(m_3, n_3) = (0, 0)$] can lead to the TM_1 mixing and littlest modular seesaw model, respectively. It should be mentioned that there are several degenerate vacua inside the fundamental domain $\mathcal{G}(N)$ of $\Gamma(N)$. Therefore, it would be interesting to explore whether the domain wall problem could exist and how to break this degeneracy, which we leave for future work.

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APPENDIX A: WHY IS THE SCALAR POTENTIAL MODULAR INVARIANT?

Before going further, it is useful to find out how the derivative of modular forms changes under the modular transformation. Supposing $f(\tau)$ is a modular form, we have

$$f'(\tau) \equiv \frac{d}{d\tau} f(\tau) \xrightarrow{\gamma} \frac{d}{d(\gamma\tau)} f(\gamma\tau) = \frac{d}{d\tau} [(c\tau + d)^k f(\tau)] \cdot \frac{d\tau}{d(\gamma\tau)} = ck(c\tau + d)^{k+1} f(\tau) + (c\tau + d)^{k+2} f'(\tau), \quad (A1)$$

where we have used the relations

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad d(\gamma\tau) = \frac{d\tau}{(c\tau + d)^2}. \quad (A2)$$

From Eq. (A1), we can easily find that $f'(\tau)$ becomes a modular form with weight 2 only if $f(\tau)$ is a zero-weight modular form. In this regard, we introduce the derivative D_i , which is covariant under the modular transformation. Keeping Eq. (2.14) in mind, we find that

$$D_i \mathcal{W} = \partial_i \mathcal{W} + (\partial_i \mathcal{K}) \mathcal{W} = \partial_i \mathcal{W} + (\partial_i G - \partial_i \log |\mathcal{W}|^2) \mathcal{W} = (\partial_i G) \mathcal{W}. \quad (A3)$$

Since G is a modular-invariant function—i.e., a modular form with weight 0— $D_i \mathcal{W}$ then turns out to be a modular form with $k = -1$.

Now, we can write down the transformation properties of all the components in the scalar potential:

$$\begin{aligned} e^{\mathcal{K}} &\rightarrow (c\tau + d)^6 e^{\mathcal{K}}, \\ \mathcal{K}^{i\bar{j}} &\rightarrow |c\tau + d|^{-4} \mathcal{K}^{i\bar{j}}, \\ D_i \mathcal{W} &\rightarrow (c\tau + d)^{-1} D_i \mathcal{W}. \end{aligned} \quad (A4)$$

Taking the above transformation rules into consideration, we can conclude that the scalar potential V is indeed invariant under the modular transformation.

APPENDIX B: THE DEDEKIND η FUNCTION AND KLEIN j FUNCTION

In this appendix, we present the definitions of several important modular forms. The Dedekind η function is a modular form with a weight of $-1/2$, defined as

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (B1)$$

where $q = e^{2\pi i \tau}$. One can express $\eta(\tau)$ as the following q expansions:

$$\eta = q^{1/24}(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22})). \quad (\text{B2})$$

The Eisenstein series $G_{2k}(\tau)$ is another kind of modular form with a weight of $2k$, the definition of which is

$$G_{2k}(\tau) = \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ (n_1, n_2) \neq (0,0)}} (n_1 + n_2\tau)^{-2k}, \quad (\text{B3})$$

which converges to the holomorphic function in the upper-half complex plane for the integer $k \geq 2$. The series does not converge when $k = 1$, but one can still define $G_2(\tau)$ via a specific prescription on the order of summation. With the help of $\eta(\tau)$ and $G_4(\tau)$, one can define a modular-invariant function, which is called the Klein j function, as

$$j(\tau) = \frac{3^6 5^3}{\pi^{12}} \frac{G_4(\tau)^3}{\eta(\tau)^{24}}, \quad (\text{B4})$$

which is also a modular form with weight 0.

APPENDIX C: HESSIAN MATRIX ANALYSIS IN THE SINGLE-MODULUS CASE

Nonzero $\mathcal{A}(S, \bar{S})$ can reshape the scalar potential, and thus shift the vacua. The dependence of the vacua on the

value of $\mathcal{A}(S, \bar{S})$ can be analyzed by calculating the Hessian matrices—in particular, at the fixed points. Since we have assumed that the scalar potential is stabilized in terms of the dilaton S via Shenker-like terms *a priori*, we only need to calculate the second derivatives of V with respect to τ and $\bar{\tau}$, and convert the complex variables into real variables $\{s, t\}$ (where s and t are the real and imaginary parts of τ , respectively) using the following relations:

$$\begin{aligned} \frac{\partial^2 V}{\partial s^2} &= 2 \frac{\partial^2 V}{\partial \tau \partial \bar{\tau}} + 2 \text{Re} \left[\frac{\partial^2 V}{\partial \tau^2} \right], \\ \frac{\partial^2 V}{\partial t^2} &= 2 \frac{\partial^2 V}{\partial \tau \partial \bar{\tau}} - 2 \text{Re} \left[\frac{\partial^2 V}{\partial \tau^2} \right], \\ \frac{\partial^2 V}{\partial s \partial t} &= -2 \text{Im} \left[\frac{\partial^2 V}{\partial \tau^2} \right]. \end{aligned} \quad (\text{C1})$$

In general, the second derivatives of the scalar potential would be very complicated, given that V relies on the moduli τ in a highly nonlinear way. However, one can easily check that the first derivatives of $\mathcal{C}$ and $\mathcal{M}$ defined in Eq. (2.23) with respect to τ vanish at the fixed points $\tau = i$ and ω , rendering the calculations of the second derivatives at the fixed points much simpler. As a result, we arrive at

$$\begin{aligned} \frac{\partial^2 V}{\partial \tau^2} &= \frac{\partial^2 \mathcal{C}}{\partial \tau^2} [\mathcal{M} + (\mathcal{A} - 3)|H|^2] + \mathcal{C} \left[\frac{\partial^2 \mathcal{M}}{\partial \tau^2} + (\mathcal{A} - 3)H^* \frac{\partial^2 H}{\partial \tau^2} \right], \\ \frac{\partial^2 V}{\partial \tau \partial \bar{\tau}} &= \frac{\partial^2 \mathcal{C}}{\partial \tau \partial \bar{\tau}} [\mathcal{M} + (\mathcal{A} - 3)|H|^2] + \mathcal{C} \left[\frac{\partial^2 \mathcal{M}}{\partial \tau \partial \bar{\tau}} + (\mathcal{A} - 3) \left| \frac{\partial H}{\partial \tau} \right|^2 \right], \end{aligned} \quad (\text{C2})$$

with

$$\begin{aligned} \frac{\partial^2 \mathcal{C}}{\partial \tau^2} &= -i\mathcal{C} \frac{\partial}{\partial \tau} \frac{\hat{G}_2(\tau, \bar{\tau})}{6\pi}, \\ \frac{\partial^2 \mathcal{C}}{\partial \tau \partial \bar{\tau}} &= i\mathcal{C} \frac{\partial}{\partial \tau} \frac{[\hat{G}_2(\tau, \bar{\tau})]^*}{6\pi}, \\ \frac{\partial^2 \mathcal{M}}{\partial \tau^2} &= \frac{(2\text{Im}\tau)^2}{3} \left[i \frac{\partial^2 H(\tau)}{\partial \tau^2} + \frac{H(\tau)}{2\pi} \frac{\partial \hat{G}_2(\tau, \bar{\tau})}{\partial \tau} \right] \frac{H^*(\tau)}{\pi} \frac{\partial \hat{G}_2^*(\tau, \bar{\tau})}{\partial \tau}, \\ \frac{\partial^2 \mathcal{M}}{\partial \tau \partial \bar{\tau}} &= \frac{(2\text{Im}\tau)^2}{3} \left[\left| i \frac{\partial^2 H(\tau)}{\partial \tau^2} + \frac{H(\tau)}{2\pi} \frac{\partial \hat{G}_2(\tau, \bar{\tau})}{\partial \tau} \right|^2 + \left| \frac{H(\tau)}{2\pi} \frac{\partial \hat{G}_2(\tau, \bar{\tau})}{\partial \bar{\tau}} \right|^2 \right], \end{aligned} \quad (\text{C3})$$

where all the derivatives above are calculated at $\tau = i$ or ω . One can check that the imaginary parts of $\partial^2 V / \partial \tau^2$ are zero at the finite fixed points. Hence, we arrive at the 2×2 diagonal Hessian matrices,

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 V}{\partial s^2} & 0 \\ 0 & \frac{\partial^2 V}{\partial t^2} \end{pmatrix}. \quad (\text{C4})$$

In order for finite fixed points to be the minima of the scalar potential, we should require both $\partial^2 V / \partial s^2$ and $\partial^2 V / \partial t^2$ in Eq. (C4) to be positive at the fixed points. For example, if we set $(m, n) = (0, 0)$, the conditions for $\tau = i$ and ω to be the minima are given by

$$\begin{aligned} \tau = i: \quad & 3.596 - \mathcal{A} > 0, \quad \mathcal{A} - 0.4036 > 0. \\ \tau = \omega: \quad & \mathcal{A} - 2 > 0. \end{aligned} \quad (\text{C5})$$

Notice that $\mathcal{A} > 3$ should also be satisfied if we require dS vacua, which can be directly obtained from Eq. (2.22), given that $\mathcal{M} = 0$ and $|H| \geq 0$ at the fixed points. Hence, $\tau = i$ could be the dS vacuum if $3 < \mathcal{A}(S, \bar{S}) < 3.5964$ is satisfied, while $\tau = \omega$ can always be the dS vacuum as long as $\mathcal{A}(S, \bar{S}) > 3$.

APPENDIX D: STABILIZING THE DILATON SECTOR

In this appendix, we investigate how the Shenker-like terms in the Kähler potential can stabilize the dilaton sector. Concrete examples for the single Kähler modulus have been provided in Ref. [96]. We build on their discussion and extend it to the multiple-modulus framework.

As mentioned above, the Kähler potential of the dilaton S takes the simple form $K(S, \bar{S}) = -\log(S + \bar{S})$ at the tree level, resulting in a 4D universal gauge coupling $g_4^2/2 = 1/\langle S + \bar{S} \rangle$. Nevertheless, stringy nonperturbative effects, scaling as $\delta\mathcal{L} \sim e^{-1/g_s}$, can also exist in the heterotic models due to the dualities of heterotic theories with type-I and type-IIA string theories [97,113,114]. Such effects manifest as the Shenker-like terms in the Kähler potential. In the following, we adopt the linear multiplet superfield formalism for the dilaton, akin to the approach taken in Ref. [96], which makes it more convenient for us to parametrize the Shenker-like terms. To be more specific, the dilaton is represented by a real scalar ℓ embedded into a linear multiplet superfield L . Then, the coupling coefficient g_4 can be connected to the dilaton via [96]

$$\frac{g_4^2}{2} = \left\langle \frac{\ell}{1 + f(\ell)} \right\rangle, \quad (\text{D1})$$

where $f(\ell)$ is a function which parametrizes the stringy nonperturbative effects. Keeping in mind that in the chiral superfield formalism we have $g_4^2/2 = 1/\langle S + \bar{S} \rangle$, we can thereby establish a relation between ℓ and S —namely,

$$\frac{\ell}{1 + f(\ell)} = \frac{1}{S + \bar{S}}. \quad (\text{D2})$$

In the linear multiplet formalism, the dilaton “Kähler potential”⁶ reads

$$K(\ell) = \log(\ell) + g(\ell), \quad (\text{D3})$$

where $g(\ell)$ denotes the Shenker-like terms, satisfying the following differential equation:

$$\ell \frac{df}{d\ell} = -\ell \frac{dg}{d\ell} + f. \quad (\text{D4})$$

As a parametrized form which can manifest the structure of the 10D heterotic action, $f(\ell)$ is usually taken as [96]

$$f(\ell) = \sum_{n=0} A_n \ell^{q_n} e^{-B/\sqrt{\ell}}, \quad (\text{D5})$$

with A_n , q_n , and B being constants. Substituting the above expression of $f(\ell)$ into the differential equation (D4), and considering the initial condition $g(0) = 0$, which guarantees that the nonperturbative effects vanish as the string couplings tend to zero, we can gain the general expression of $g(\ell)$ as [96]

$$g(\ell) = \sum_{n=0} A_n B^{2q_n} \{ 2(1 - q_n) \Gamma(-2q_n, B/\sqrt{\ell}) - \Gamma(1 - 2q_n, B/\sqrt{\ell}) \}, \quad (\text{D6})$$

where $\Gamma(a, x)$ is the upper incomplete gamma function

$$\Gamma(s, x) = \int_x^\infty y^{s-1} e^{-y} dy. \quad (\text{D7})$$

Finally, the scalar potential in Eq. (2.22) can be rewritten as

$$V(\ell, \tau, \bar{\tau}) = \Lambda_V^4 \left(\prod_{i=1}^3 \frac{\ell e^{g(\ell) - (f(\ell)+1)/b_a \ell}}{(2\text{Im}\tau_i) |\eta(\tau_i)|^4} \right) \{ [A(\ell) - 3] |\mathcal{H}(\tau_i)|^2 + \tilde{\mathcal{M}}(\tau_i, \bar{\tau}_i) \}, \quad (\text{D8})$$

where the definitions of $\mathcal{H}(\tau_1, \tau_2, \tau_3)$ and $\tilde{\mathcal{M}}(\tau_1, \tau_2, \tau_3)$ can be found in Eqs. (3.7) and (3.11), respectively, and

$$A(\ell) = \frac{(1 + b_a \ell)^2 (1 + \ell g'(\ell))}{b_a^2 \ell^2}. \quad (\text{D9})$$

Let us now examine the stability of the vacua presented in Table I in the dilaton sector. For simplicity, we consider a trivial polynomial

$$f(\ell) = A_0 e^{-B/\sqrt{\ell}}, \quad (\text{D10})$$

with

⁶More accurately, it should be described as a kinetic potential.

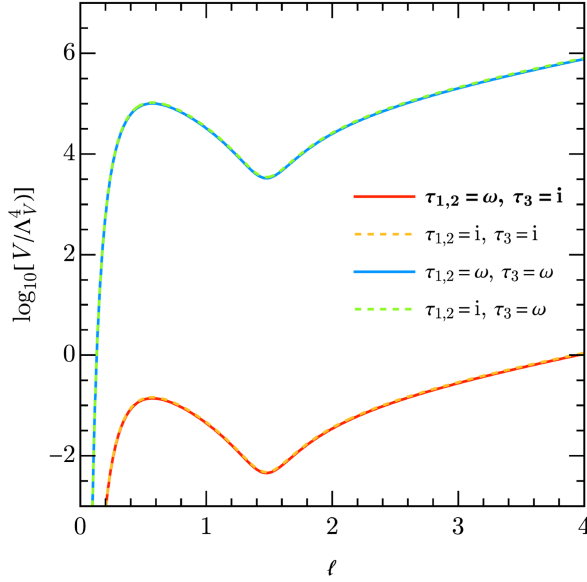


FIG. 6. The projection of the scalar potential in the ℓ direction in the case of $(m_1, n_1) = (m_2, n_2) = (0, 0)$, $(m_3, n_3) = (2, 0)$, where we choose $A_0 = 26$, $B = \pi$, and $b_a = 0.4$. In order to obtain these curves, we have fixed $\tau_{1,2}$ and τ_3 to be either i or ω . Note that the choice $\tau_{1,2} = \omega$, $\tau_3 = i$ corresponds to the deepest minimum in this case.

$$A_0 = 26, \quad B = \pi, \quad b_a = 0.4. \quad (\text{D11})$$

We still restrict ourselves to the fixed points of the Kähler moduli. In the case of $(m_1, n_1) = (m_2, n_2) = (0, 0)$, $(m_3, n_3) = (2, 0)$, by fixing τ_1 , τ_2 , and τ_3 at their respective vacua, we exhibit the projection of the scalar potential in the ℓ direction in Fig. 6. One can observe that, regardless of which fixed points the Kähler moduli take, the scalar potential always reaches a local minimum at $\langle \ell \rangle \simeq 1.47$. Furthermore, at this minimum we have

$$g_4 \simeq 0.99, \quad \langle f(\ell) \rangle \simeq 1.95, \quad \langle \mathcal{A}(\ell) \rangle \simeq 3.09. \quad (\text{D12})$$

It is easy to identify that $\mathcal{A}(\ell) \simeq 3.09$ satisfies the conditions for the fixed points of the Kähler moduli to be the metastable vacua in the case where $(m_1, n_1) = (m_2, n_2) = (0, 0)$, $(m_3, n_3) = (2, 0)$. Therefore, the Kähler moduli and the dilaton can indeed be simultaneously stabilized due to the inclusion of Shenker-like terms. We also scrutinize the remaining cases shown in Table I, and find that the simple polynomial $f(\ell)$, as defined in Eq. (D10) with the parameter choices $A_0 = 26$, $B = \pi$, and $b_a = 0.4$, can account for the dilaton stabilization across all cases.

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