

Nonequivalence of Carroll limits in relativistic string theory

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We construct new Carroll strings in flat space by considering the Carroll limit of equivalent relativistic string theories at the classical level. In the limit these Carroll strings are no longer equivalent and have different degrees of freedom. This fact is due to a general phenomenon that the configuration space and canonical Lagrangians are no longer equivalent after a singular redefinition of the variables and of the parameters of the starting Lagrangians.

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I. INTRODUCTION

Carroll symmetry, which was introduced by [1,2] as the limit of velocity of light going to zero, $c \rightarrow 0$, of Poincaré symmetry, has in recent years received a lot of attention because it arises in different physical systems. Carroll structures emerge when one considers asymptotically flat spacetimes in gravity [3–5], where the corresponding symmetry is the BMS group [6,7] which is isomorphic to the conformal Carroll group [8], in the geometry of the black-hole near horizon [9–11], null surfaces [12], relativistic fluids [13], and in condensed matter [14,15]. Very recently Carrollian amplitudes have been computed from strings [16]. An introductory review of some aspects of non-Lorentzian theories can be found in [17]. Dynamical realizations of the conformal Carroll structures [8] have been studied, for example, through the construction of Carrollian particles. A massive nonconformal Carroll particle was introduced as Carroll limit of relativistic massive particle [18] and can also be obtained by the technique of the coadjoint orbits of the Carroll group [12]. The massless Carroll particle was also considered in [18].

The symmetries of massive and massless free Carroll particles are infinite dimensional. In the massless case the

symmetry contains the finite conformal Carroll group introduced in [19,20] and also the BMS symmetry [6,7]. In the case of two nonconformal interacting particles the symmetry algebra is finite dimensional [18]. Recently two models of two interacting conformal Carroll particles [21], which can be obtained as the Carrollian limit of two relativistic conformal particles [22], have been constructed. The first model describes particles that do not move and exhibits infinite dimensional symmetries which are reminiscent of the BMS symmetries. A second model was also proposed, where the particles have nonzero velocity and therefore, as a consequence of the limit $c \rightarrow 0$, are tachyons. Infinite dimensional symmetries are also present in this model.

Some preliminary work on Carroll strings has been already performed in [23] by considering the formulation in the phase space configuration of the Nambu Goto string and by [11] where the near horizon properties of the string are investigated. We extend the work contained in [23] by studying alternative but equivalent formulations of the relativistic string and show that the particle Carroll limit gives rise to Carroll theories with different number of degrees of freedom but with the same Carroll algebra. Analogous considerations can be done for the Carroll string limit [23].

We also consider the *tachyonic* relativistic strings which, in the Carroll limit, produce also new Carroll strings in flat space time. As a warmup exercise, in the Appendix we recall the Carroll limit of the different formulations of the relativistic massive particle, including the tachyon case.

The organization of the paper is as follows. In Sec. II we discuss the nonequivalence of the configuration and canonical Lagrangians in a singular limit, in Sec. III we analyze different formulations of the bosonic relativistic

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string and their Carroll particle limits and in the Appendix we show the nonequivalence for the Carroll limits of different formulations of the relativistic particle.

II. NONEQUIVALENCE OF THE CONFIGURATION SPACE AND CANONICAL LAGRANGIANS IN A SINGULAR LIMIT

Since in this paper we are interested in discussing the nonequivalence of Carroll limits, we start by considering a general dynamical system described by a set of variables discrete or continuous and by a Lagrangian depending on these variables and on their time derivatives. In order to fix the ideas we will consider discrete variables but the considerations that we will do can be easily extended to the continuum case. The equations of motion for such a system can be obtained by looking for the extrema of the action functional

$$S = \int dt L(q_i, \dot{q}_i, t), \quad i = 1 \cdots N. \quad (1)$$

Such a system can also be described by introducing the Hamiltonian through a Legendre transformation and eliminating the velocities in terms of the momenta. This inversion is not always possible in the full phase space leading to constraints that determine a submanifold where this inversion is possible. In this instance we suppose to follow the Dirac procedure solving the second class constraints (or using Dirac brackets) and adding to the canonical Hamiltonian the first class constraints times arbitrary functions [24]. In any case there exists a well-defined procedure to get the Hamiltonian from the Lagrangian [25], and this is the point we will be interested in. Therefore, in the following general discussion, for sake of simplicity, we will suppose that there are no constraints.

In the Hamiltonian formalism one requires the equations of motion that are equivalent to the Lagrange equations coming from the variation of Eq. (1). As it is well known, it is possible to derive the Hamilton equations from a variational principle, introducing an extended Lagrangian and a corresponding action defined in a space spanned by $q_i, \dot{q}_i, p_i$:

$$S_E = \int dt L_E(q_i, \dot{q}_i, p_i, t) = \int dt \left(\sum_{i=1}^N p_i \dot{q}_i - H(q_i, p_i, t) \right), \quad i = 1 \cdots N. \quad (2)$$

We would like to emphasise the fact that the two formulations corresponding to Eqs. (1) and (2) are completely equivalent. The equations of motion of both Lagrangians coincide when we use the Legendre transform.

Due to this equivalence, if we perform some transformation of the variables in L , this will induce a

corresponding variation in L_E such that the two will remain equivalent. However the situation changes completely if our transformation is singular. Just to exemplify, suppose that we scale a single variable $q_i \rightarrow q_i/\omega$ and suppose that L is finite, maybe by some ω redefinition of certain parameters appearing in the Lagrangian, for $\omega \rightarrow \infty$. Then, when we evaluate $L_E(\omega)$, starting from $L(\omega)$ and send $\omega \rightarrow \infty$, in general we have to make some further ω -dependent transformation on $L_E(\omega)$ to make it finite and in general inequivalent:

$$\lim_{\omega \rightarrow \infty} L(\omega) \not\approx \lim_{\omega \rightarrow \infty} \tilde{L}(\omega), \quad (3)$$

where $\tilde{L}(\omega)$ is L_E transformed in such way to get a finite result. Note that in some cases to get a finite limit it is also necessary to add some additional terms to the Lagrangian [27,28].

A further observation concerns the case in which the Lagrangian L is invariant under a continuous group of transformations, G , then, as before, we will rescale one or more variables in such a way that $\lim_{\omega \rightarrow \infty} L(\omega)$ is finite. Then, in general, the resulting Lagrangian is invariant under a contraction of the Lie algebra of the original symmetry. Of course, the Lagrangian L_E has the same group of symmetries G as L and the question arises about the symmetries of $\lim_{\omega \rightarrow \infty} \tilde{L}(\omega)$. Generally speaking the two limiting cases are invariant under the same contracted symmetry group, but corresponding to two different realizations.

III. CARROLL STRINGS FROM RELATIVISTIC STRINGS

A. Canonical action from a tension full string

Let us consider the relativistic Nambu Goto string in the phase space formulation; the action is given by

$$S = \int d\tau \int_0^{2\pi} d\sigma \mathcal{L}_E \quad (4)$$

where the Lagrangian density is given by

$$\mathcal{L}_E = p \cdot x - \frac{e}{2} (p^2 + T^2 x'^2) - \mu p \cdot x', \quad (5)$$

where $p^\mu \equiv p^\mu(\tau, \sigma)$, $x^\mu \equiv x^\mu(\tau, \sigma)$, T is the string tension and $e(\tau, \sigma), \mu(\tau, \sigma)$ are Lagrange multipliers.

In this case, to perform the particle Carroll limit, we introduce the invertible change

$$x^0 = \frac{t}{\omega}, \quad p^0 = \omega E, \quad e = \frac{e}{\omega^2}, \quad T = \omega T \quad (6)$$

and we consider $\omega \rightarrow \infty$, as already done in [23], and we obtain

$$\lim_{\omega \rightarrow \infty} \tilde{\mathcal{L}}_E(\omega) = -E\dot{t} + \mathbf{p} \cdot \dot{\mathbf{x}} - \frac{e}{2}(-E^2 + T^2 \mathbf{x}'^2) - \mu(-Et' + \mathbf{p} \cdot \mathbf{x}'). \quad (7)$$

The corresponding first class constraints are

$$\Pi_e = \frac{\partial \mathcal{L}}{\partial e} = 0, \quad \Pi_\mu = \frac{\partial \mathcal{L}}{\partial \mu} = 0, \quad \text{primary}, \quad (8)$$

$$-E^2 + T^2 \mathbf{x}'^2 = 0, \quad Et' - \mathbf{p} \cdot \mathbf{x}' = 0, \quad \text{secondary}. \quad (9)$$

The model obtained is the electric Carroll string due to the presence of the energy E and not the momenta in the quadratic constraint [29,30].

The Lagrangian and the constraints, Eq. (7) and Eqs. (9), are invariant under the Carroll transformations

$$\delta t = \boldsymbol{\beta} \cdot \mathbf{x} + a_t, \quad \delta x^i = \epsilon^{ijk} \theta^j x^k + a^i, \\ \delta p^i = \epsilon^{ijk} \theta^j p^k + \beta^i E, \quad \delta E = 0, \quad (10)$$

and assuming $\delta_C e = \delta_C \mu = 0$. The Lagrangian (7) has an infinite set of symmetries that contain Carroll [23].

As already shown in [23] by going in the conformal gauge $e = 1$, $\mu = 0$, the dynamics of this Carroll string is trivial, since the spatial coordinates of the string are constant in τ . However the momentum density is not constant.

The number of degrees of freedom is given by $N_{\text{dof}} = 1/2[2(D+2) - 2 \times 4] = D - 2$ which corresponds to the transverse degrees of freedom of string.

Notice that the Lagrangian Eq. (7) is related to the Lagrangian of the nonvibrating nonrelativistic string by the mapping [31]

$$E \rightarrow \mathbf{P}, \quad \mathbf{P} \rightarrow -E, \quad \mathbf{x} \rightarrow -t. \quad (11)$$

B. Lagrangian in configuration space and with einbeins

If we eliminate the momenta of the relativistic string Lagrangian (5) we obtain the corresponding Lagrangian in the configuration space formulated using einbeins variables

$$\mathcal{L} = \frac{1}{2e} \dot{\mathbf{x}}^2 - \frac{\mu}{e} \dot{\mathbf{x}} \cdot \mathbf{x}' + \frac{1}{2} \frac{\mu^2}{e} x'^2 - \frac{T^2}{2} e x'^2. \quad (12)$$

In this case, the Carroll limit is

$$x^0 = \frac{t}{\omega}, \quad \omega \rightarrow \infty. \quad (13)$$

We obtain

$$\lim_{\omega \rightarrow \infty} \mathcal{L}(\omega) = \frac{1}{2e} \dot{\mathbf{x}}^2 - \frac{\mu}{e} \dot{\mathbf{x}} \cdot \mathbf{x}' + \frac{1}{2} \frac{\mu^2}{e} x'^2 - \frac{T^2}{2} e x'^2. \quad (14)$$

Note that the tension is not rescaled.

We have

$$\Pi_e = 0, \quad \Pi_\mu = 0, \quad E = \frac{\partial \mathcal{L}}{\partial t} = 0, \quad \text{primary constraints}, \quad (15)$$

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = \frac{1}{e}(\dot{\mathbf{x}} - \mu \mathbf{x}'), \quad (16)$$

$$0 = \frac{\partial \mathcal{L}}{\partial e} = -\frac{1}{2}[(\dot{\mathbf{x}} - \mu \mathbf{x}')^2 + T^2 \mathbf{x}'^2], \quad (17)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \mu} = -e(\dot{\mathbf{x}} - \mu \mathbf{x}') \cdot \mathbf{x}'. \quad (18)$$

The canonical Hamiltonian is

$$H_c = \frac{e}{2}(\mathbf{p}^2 + T^2 \mathbf{x}'^2) + \mu \mathbf{p} \cdot \mathbf{x}'. \quad (19)$$

The secondary constraints are

$$\mathcal{H} = \mathbf{p}^2 + T^2 \mathbf{x}'^2 = 0, \quad \mathcal{T} = \mathbf{p} \cdot \mathbf{x}' = 0. \quad (20)$$

Assuming $T \neq 0$, the first constraint of Eq. (20) implies

$$\mathbf{p}^2 = \mathbf{x}'^2 = 0. \quad (21)$$

Notice that the constraints (21) are irregular ones [32]. If we substitute them by the linearized constraints

$$\mathbf{p} = \mathbf{x}' = 0, \quad (22)$$

the constraints (22) become now second class. Therefore taking into account the three first class constraints of Eq. (15) and the $D - 1$ second class ones of (22), the number of degrees of freedom becomes $N_{\text{dof}} = 1/2[2(D+2) - 2 \times 3 - 2 \times (D-1)] = 0$. The model has no local degrees of freedom.

The same result is obtained by performing the Carroll particle limit Eq. (13) limit in the Nambu-Goto Lagrangian

$$\mathcal{L} = -T \sqrt{(\dot{\mathbf{x}} \cdot \mathbf{x}')^2 - \dot{\mathbf{x}}^2 \mathbf{x}'^2}, \quad (23)$$

which gives

$$\mathcal{L} = -T \sqrt{(\dot{\mathbf{x}} \cdot \mathbf{x}')^2 - \dot{\mathbf{x}}^2 \mathbf{x}'^2}, \quad (24)$$

from which the constraints (20) can be directly obtained. The Carroll transformations are given by Eqs. (10) for the space time variables.

Therefore no magnetic model, with a quadratic constraint in the spatial momenta, is produced in the Carroll limit if we do not scale einbein variables and the tension of the string. From now on we will call this case as the Electric Carroll limit.

C. Canonical Lagrangian from a tachyonic string

Now following [33–35] in order to obtain the magnetic Carroll limit we should consider the tachyonic string.

To perform this limit [36] in the phase space formulation it is necessary to change $T^2 \rightarrow -T^2$

$$\mathcal{L}_E = p \cdot x - \frac{e}{2}(p^2 - T^2 x'^2) - \mu p \cdot x'. \quad (25)$$

The Carroll limit (6) gives

$$\lim_{\omega \rightarrow \infty} \tilde{\mathcal{L}}_E(\omega) = -E\dot{t} + \mathbf{p} \cdot \dot{\mathbf{x}} - \frac{e}{2}(-E^2 - T^2 \mathbf{x}'^2) - \mu(-E\dot{t}' + \mathbf{p} \cdot \mathbf{x}'). \quad (26)$$

As it is clear from the form of the quadratic constraint $-E^2 - T^2 \mathbf{x}'^2 = 0$, if we assume $T \neq 0$ we have the constraints $E^2 = \mathbf{x}'^2 = 0$ which are irregular constraints [32].

The linear constraint for the spatial coordinates is $\mathbf{x}' = 0$. The counting of degrees of freedom goes as $N_{\text{dof}} = 1/2[2(D+2) - 2(3+D-1)] = 0$. There are no local degrees of freedom.

D. Tachyonic string Lagrangian in the configuration space

The tachyonic string is obtained from (12) again if we change $T^2 \rightarrow -T^2$ obtaining

$$\mathcal{L} = \frac{1}{2e} \dot{x}^2 - \frac{\mu}{e} \dot{x} \cdot x' + \frac{1}{2} \frac{\mu^2}{e} x'^2 + \frac{T^2}{2} e x'^2. \quad (27)$$

By performing the Carroll limit (13) we obtain

$$\lim_{\omega \rightarrow \infty} \mathcal{L}(\omega) = \frac{1}{2e} \dot{x}^2 - \frac{\mu}{e} \dot{x} \cdot x' + \frac{1}{2} \frac{\mu^2}{e} x'^2 + \frac{T^2}{2} e x'^2. \quad (28)$$

Proceeding as in the case of Sec. III B we obtain as secondary constraints

$$\mathcal{H} = p^2 - T^2 \mathbf{x}'^2 = 0, \quad \mathcal{T} = \mathbf{p} \cdot \mathbf{x}' = 0, \quad (29)$$

which are first class. In this case we obtain a magnetic model since the quadratic constraint depends on the spatial momenta.

Their algebra is given by

$$\{\mathcal{H}(\sigma), \mathcal{H}(\sigma')\} = 4T^2 [\mathcal{T}(\sigma') \partial_\sigma \delta(\sigma - \sigma') - \mathcal{T}(\sigma) \partial_{\sigma'} \delta(\sigma - \sigma')], \quad (30)$$

$$\{\mathcal{H}(\sigma), \mathcal{T}(\sigma')\} = \mathcal{H}(\sigma) \partial_\sigma \delta(\sigma - \sigma'), \quad (31)$$

$$\{\mathcal{T}(\sigma), \mathcal{T}(\sigma')\} = 4T^2 [-\mathcal{T}(\sigma') \partial_\sigma \delta(\sigma - \sigma') + \mathcal{T}(\sigma) \partial_{\sigma'} \delta(\sigma - \sigma')]. \quad (32)$$

We have three primary first class $\Pi_e = 0, \Pi_\mu = 0, E = 0$, and two secondary first class constraints, therefore we have three type gauge transformations. We have an arbitrary

transformation in the time coordinate and two dimensional Euclidean isomorphism. Taking into account the presence of five first class constraints, the number of degrees of freedom is given by $N_{\text{dof}} = 1/2[2(N+2) - 2 \times 5] = D - 3$.

We can check this statement explicitly if we introduce a gauge fixing $t = 0$ to make the first class constraint $E = 0$ second class; then the reduced phase space is $(\mathbf{x}, \mathbf{p})$ and the canonical Lagrangian is given by

$$\mathcal{L} = \dot{\mathbf{x}} \cdot \mathbf{p} - \frac{1}{2} e (\mathbf{p}^2 - T^2 \mathbf{x}'^2) - \mu \mathbf{p} \cdot \mathbf{x}'. \quad (33)$$

Note the difference with respect to the nonvibrating non-relativistic string [31].

The action is still invariant under reparametrizations of world sheet τ and σ coordinates. At least locally, this allows us to identify one of the space coordinates with τ and we may then choose the unit of length such that

$$x^1 = \tau, \quad x^2 = \sigma \quad (34)$$

and assume a 2π periodicity for the variable x^2

$$x^2 \equiv x^2 + 2\pi. \quad (35)$$

With these choices the constraints (29) become second class. Furthermore, we have

$$\mathcal{L} = \dot{\mathbf{x}} \cdot \mathbf{p} = p^1 + \sum_{a=3}^{D-1} \dot{x}^a p^a. \quad (36)$$

The constraints (29) become equivalent to

$$p^2 + \sum_{a=3}^{D-1} p^a x^{a'} = 0, \quad (37)$$

$$p^2 - T^2 \left[1 + \sum_{a=3}^{D-1} (x^{a'})^2 \right] = 0, \quad (38)$$

or

$$p^2 = - \sum_{a=3}^{D-1} p^a x^{a'}, \quad (39)$$

$$p^1 = \pm T \sqrt{1 + \sum_{a=3}^{D-1} (x^{a'})^2 - T^{-2} \sum_{a=3}^{D-1} [(p^a x^{a'})^2 + (p^a)^2]}. \quad (40)$$

By choosing the minus sign solution, from Eq. (36) we can extract the Hamiltonian density

$$H = T \sqrt{1 + \sum_{a=3}^{D-1} (\dot{x}^{a'})^2} - T^{-2} \sum_{a=3}^{D-1} [(p^a x^{a'})^2 + (p^a)^2]. \quad (41)$$

This string is propagating in euclidean space, we could call it an instantonic string. By double dimensional reduction we recover the tachyonic Carroll particle with zero energy [34].

IV. CONCLUSIONS

We have constructed new Carroll strings in flat space by considering the Carroll particle limit of equivalent relativistic string theories at classical level. We have considered two types of limits one that scales only the space-time variables and the other one where we also scale the einbein and the string tension. In the limit these Carroll strings are no longer equivalent and have different degrees of freedom. In the case of ordinary string the only limit that produce a sensible model is the one that scales space-time variables, einbeins and the string tension. The model obtained is electric. When we consider the limit of a tachyonic string the nontrivial model is obtained when we only scale space-time variables, the model obtained is magnetic. This nonequivalence should be a general property for other limits [21,37–39] of free and interacting particles, p-branes and field theories, in particular about relativistic gravity.

The relation with other approaches to construct Carroll theories like the seed procedure [30] should be analyzed.

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APPENDIX: RELATIVISTIC PARTICLES

Relativistic timelike particles have tree possible different actions, the configuration action with only spacetime coordinates

$$S = -m \int d\tau \sqrt{-\dot{x}^2}, \quad (A1)$$

where m is the mass of the particle; the metric $g_{\mu\nu} = (-, +, \dots, +)$ and natural units are assumed. The action with space-time variables and an einbein variable (Polyakov) formulation is

$$S = \int d\tau \left(\frac{1}{2e} \dot{x}^2 - \frac{1}{2} e m^2 \right), \quad (A2)$$

while the canonical action is

$$S = \int d\tau \left[p \cdot \dot{x} - \frac{1}{2} e (p^2 + m^2) \right]. \quad (A3)$$

They are equivalent by elimination of the momenta and the einbein variables of the canonical action. The analogous actions for the tachyons are obtained by changing the sign of the square root and changing $m^2 \rightarrow -m^2$ in the Polyakov and canonical action. This equivalence is broken when we consider the Carroll limit. The causal structure admits two types of time intervals that coincide with light-like and space-like intervals.

1. Carroll limit (time-like case)

Let us consider the Carroll limit of a massive relativistic particle, whose action is given by Eq. (A1), trough the substitution ($c = 1$)

$$x^0 \rightarrow \frac{1}{\omega} t. \quad (A4)$$

In the limit $\omega \rightarrow \infty$ the Lagrangian is not real and we do not proceed furthermore.

Let us now consider Eq. (A2) in the limit (A4); if we do not scale the einbein and the mass we get

$$\lim_{\omega \rightarrow \infty} L(\omega) = \frac{1}{2e} (\dot{x}^2 - e^2 m^2). \quad (A5)$$

There are two primary constraints

$$E = \frac{\partial L}{\partial \dot{t}} = 0, \quad \Pi_e = \frac{\partial L}{\partial \dot{e}} = 0, \quad (A6)$$

where E is the energy and Π_e is the momentum of the einbein variable e . There is also a secondary constraint

$$p^2 + m^2 = 0. \quad (A7)$$

which has real solution only when $m = 0$. In this case the $p^2 = 0$ constraint is an irregular one [32].

If we consider the linear constraint $p = 0$ associated to the irregular constraint $p^2 = 0$, [32], the canonical Lagrangian for the linear constraint action becomes

$$L_c = -E\dot{t} + p \cdot \dot{x} - \lambda \cdot p - \mu E, \quad (A8)$$

where λ are $D - 1$ Lagrangian multipliers. There is no a configuration space action. The number of degrees of freedom in configuration space is $N_{\text{dof}} = 1/2[2(D + D - 1 + 1) - 2(D - 1 + D - 1 + 1 + 1)] = 0$. In conclusion the model has no local physical degrees of freedom and its action is Carroll invariant under the following boost transformations

$$\begin{aligned}\delta_C t &= \boldsymbol{\beta} \cdot \mathbf{x}, & \delta_C x^i &= 0, \\ \delta_C p^i &= \beta^i E, & \delta_C E &= 0, \\ \delta_C \lambda &= 0, & \delta_C \mu &= -\boldsymbol{\beta} \cdot \boldsymbol{\lambda}.\end{aligned}\quad (\text{A9})$$

The Carroll limit of the canonical action, Eq. (A3), is defined by

$$x^0 \rightarrow \frac{1}{\omega} t, \quad m = \omega M, \quad e = \frac{\tilde{e}}{\omega^2}. \quad (\text{A10})$$

Note the scaling of the einbein variable and of the mass. The Lagrangian is given by [18]

$$L_c = -E\dot{t} + \mathbf{p} \cdot \dot{\mathbf{x}} + \frac{\tilde{e}}{2} (E^2 - \tilde{m}^2) \quad (\text{A11})$$

and describes a particle with constant energy

$$E^2 = \tilde{m}^2, \quad (\text{A12})$$

zero velocity and no relation among the momentum and the velocity. The model obtained is the electric Carroll particle. The number of degrees of freedom is equal to $N_{\text{dof}} = 1/2[2(D+1) - 2 \times 2] = D - 1$.

2. Carroll limit (space-like or tachyon case)

The configuration relativistic action for a tachyon is

$$S = -m \int d\tau \sqrt{\dot{\mathbf{x}}^2} \quad (\text{A13})$$

and the Carroll limit, Eq (A4), gives

$$S = -m \int d\tau \sqrt{\dot{\mathbf{x}}^2}. \quad (\text{A14})$$

The primary constraints are

$$\mathbf{p}^2 - m^2 = 0, \quad E = 0. \quad (\text{A15})$$

Therefore the Lagrangian coincides with the massless Galilean particle with m color and vanishing energy [40].

Let us now consider the relativistic tachyon Lagrangian with einbein variable

$$L_T = \frac{1}{2e} \dot{\mathbf{x}}^2 + \frac{1}{2} e m^2. \quad (\text{A16})$$

By performing the Carroll limit of Eq. (A16) we obtain

$$\lim_{\omega \rightarrow \infty} L_T(\omega) = \frac{1}{2e} (\dot{\mathbf{x}}^2 + e^2 m^2). \quad (\text{A17})$$

There are two primary constraints

$$E = 0, \quad \Pi_e = 0 \quad (\text{A18})$$

and a secondary constraint

$$\mathbf{p}^2 - m^2 = 0, \quad (\text{A19})$$

fixing the modulus of the spatial momentum without any restriction of the parameter m . Note that the constraints coincide with those of the configuration action and therefore in this case the two limits are equivalent.

This particle is a Carroll tachyon with zero energy and momentum $\mathbf{p}$. Also its velocity $\dot{\mathbf{x}}$ is not zero. The model is the magnetic Carroll particle [35]. In conclusion, taking into account the presence of three first class constraints, the number of degrees of the physical freedom is equal to $N_{\text{dof}} = 1/2[2(D+1) - 2 \times 3] = D - 2$.

In analogous way the limit of the tachyon canonical Lagrangian

$$L_{TE} = \mathbf{p} \cdot \dot{\mathbf{x}} + \frac{1}{2} e (\mathbf{p}^2 - m^2) \quad (\text{A20})$$

gives

$$\lim_{\omega \rightarrow \infty} \tilde{L}_{TE} = -E\dot{t} + \mathbf{p} \cdot \dot{\mathbf{x}} + \frac{\tilde{e}}{2} (E^2 + \tilde{m}^2) \quad (\text{A21})$$

where

$$\tilde{e} = e\omega^2, \quad \tilde{m}^2 = \frac{m^2}{\omega^2}. \quad (\text{A22})$$

We get the constraints

$$\Pi_{\tilde{e}} = 0, \quad E^2 + \tilde{m}^2 = 0 \quad (\text{A23})$$

with solutions only for $\tilde{m} = 0$. The particle has zero energy and momentum. The model obtained is the conformal Carroll particle. The number of degrees of freedom is equal to $N_{\text{dof}} = 1/2[2(D+1) - 2 \times 2] = D - 1$.

- [1] J.-M. Levy-Leblond, *Ann. l'I. H. P. Phys. Théor.* **3**, 1 (1965).
- [2] N. Gupta, *Nuovo Cimento A* **44**, 512 (1966).
- [3] G. Barnich and C. Troessaert, *Phys. Rev. Lett.* **105**, 111103 (2010).
- [4] A. Bagchi, *Phys. Rev. Lett.* **105**, 171601 (2010).
- [5] A. Bagchi and R. Fareghbal, *J. High Energy Phys.* **10** (2012) 092.
- [6] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Proc. R. Soc. A* **269**, 21 (1962).
- [7] R. Sachs, *Phys. Rev.* **128**, 2851 (1962).
- [8] C. Duval, G. W. Gibbons, and P. A. Horvathy, *Classical Quantum Gravity* **31**, 092001 (2014).
- [9] R. F. Penna, [arXiv:1812.05643](https://arxiv.org/abs/1812.05643).
- [10] L. Donnay and C. Merteau, *Classical Quantum Gravity* **36**, 165002 (2019).
- [11] A. Bagchi, A. Banerjee, J. Hartong, E. Have, K. S. Kolekar, and M. Mandlik, [arXiv:2312.14240](https://arxiv.org/abs/2312.14240).
- [12] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, *Classical Quantum Gravity* **31**, 085016 (2014).
- [13] L. Ciambelli, C. Merteau, A. C. Petkou, P. M. Petropoulos, and K. Siampos, *J. High Energy Phys.* **07** (2018) 165.
- [14] L. Bidussi, J. Hartong, E. Have, J. Musaeus, and S. Prohazka, *SciPost Phys.* **12**, 205 (2022).
- [15] L. Marsot, P. M. Zhang, M. Chernodub, and P. A. Horvathy, *Phys. Rep.* **1028**, 1 (2023).
- [16] S. Stieberger, T. R. Taylor, and B. Zhu, *J. High Energy Phys.* **04** (2024) 127.
- [17] E. Bergshoeff, J. Figueroa-O'Farrill, and J. Gomis, *SciPost Phys. Lect. Notes* **69**, 1 (2023).
- [18] E. Bergshoeff, J. Gomis, and G. Longhi, *Classical Quantum Gravity* **31**, 205009 (2014).
- [19] A. Bagchi, R. Basu, A. Kakkar, and A. Mehra, *J. High Energy Phys.* **12** (2016) 147.
- [20] B. Chen, R. Liu, and Y.-f. Zheng, *SciPost Phys.* **14**, 088 (2023).
- [21] R. Casalbuoni, D. Dominici, and J. Gomis, *Phys. Rev. D* **108**, 086005 (2023).
- [22] R. Casalbuoni and J. Gomis, *Phys. Rev. D* **90**, 026001 (2014).
- [23] B. Cardona, J. Gomis, and J. M. Pons, *J. High Energy Phys.* **07** (2016) 050.
- [24] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science. Monographs Series 2 (Yeshiva University, New York, 1964).
- [25] The equivalence among the Lagrangian and Hamiltonian formalism has been studied. See for example [26].
- [26] C. Batlle, J. Gomis, J. M. Pons, and N. Roman, *J. Math. Phys. (N.Y.)* **27**, 2953 (1986).
- [27] J. Gomis and H. Ooguri, *J. Math. Phys. (N.Y.)* **42**, 3127 (2001).
- [28] J. Gomis, J. Gomis, and K. Kamimura, *J. High Energy Phys.* **12** (2005) 024.
- [29] M. Henneaux and P. Salgado-Rebolledo, *J. High Energy Phys.* (2021) 1.
- [30] E. A. Bergshoeff, J. Gomis, and A. Kleinschmidt, *J. High Energy Phys.* **01** (2023) 167.
- [31] C. Batlle, J. Gomis, and D. Not, *J. High Energy Phys.* **02** (2017) 049.
- [32] O. Miskovic and J. Zanelli, *J. Math. Phys. (N.Y.)* **44**, 3876 (2003).
- [33] C. Batlle, J. Gomis, L. Mezincescu, and P. K. Townsend, *J. High Energy Phys.* **04** (2017) 120.
- [34] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, *Front. Phys.* **10**, 810405 (2022).
- [35] J. Gomis and A. Kleinschmidt, *Front. Phys.* **10**, 892812 (2022).
- [36] The nonrelativistic limit of a tachyonic relativistic particle in canonical form was done in [33].
- [37] J. Brugues, T. Curtright, J. Gomis, and L. Mezincescu, *Phys. Lett. B* **594**, 227 (2004).
- [38] J. Brugues, J. Gomis, and K. Kamimura, *Phys. Rev. D* **73**, 085011 (2006).
- [39] A. Barducci, R. Casalbuoni, and J. Gomis, *Phys. Rev. D* **98**, 085018 (2018).
- [40] J. M. Souriau, *Structure des systèmes dynamiques* (Dunod, Paris, 1969).