Dimensionally reducing the classical Regge growth conjecture

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We explore the classical Regge growth conjecture in the 4d effective field theory that results from compactifying D-dimensional general relativity on a compact, Ricci-flat manifold. While the higher dimensional description is given in terms of pure Einstein gravity and the conjecture is automatically satisfied, it imposes several nontrivial constraints in the 4d spectrum. Namely, there must be either none or an infinite number of massive spin-2 modes, and the mass ratio between consecutive Kaluza-Klain spin-2 replicas is bounded by the 4d coupling constants.

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I. INTRODUCTION

In the recent work [1]—see [2] for related ideas—it was studied if a gravitational effective field theory (EFT) that includes a massive spin-2 particle in its spectrum was compatible with the classical Regge growth (CRG) conjecture [3]. This conjecture states that the classical (tree-level) S-matrix $\mathcal{A}(s,t)$ of any consistent theory can never grow faster than s^2 in the Regge limit, that is, at large s and fixed and physical t, with s and t being the usual Mandelstam variables. In terms of equations, it states

$$\lim_{s\to\infty, t<0\,\text{fixed}} \frac{\mathcal{A}(s,t)}{s^3} = 0, \tag{1}$$

where by $s\to\infty$ we mean $\Lambda \gg s \gg |t|$, with Λ being the cutoff of the EFT considered.

The main conclusion of [1] was to show the incompatibility of the setup described above with the CRG conjecture. If the CRG holds, a gravitational EFT which includes a massive spin-2 particle—and no other higher spin particles—would be in the swampland [4].[1]

This result was in line with the so-called spin-2 swampland conjecture [5–7] and with the recent works studying the (in)consistency of massive gravity; see [2,8–14] for a biased selection and [15,16] for reviews.

Regarding the state of the CRG conjecture, though a complete demonstration is still lacking, there is strong evidence in favor of it. Using the duality between Anti-de Sitter space (AdS) and conformal field theories, it has been proven in [17] that, in the dual picture, the CRG conjecture follows from the chaos bound of [18]. In flat space, it was shown in [19] that the scattering of scalar particles in dimensions bigger or equal to five satisfies it. Here, as we did in [1], we will limit ourselves to assuming its validity, studying the consequences that derive from it.

This being said, the logical next step after [1] is to consider an EFT with not only one but any number of massive spin-2 particles in the spectrum. This is a common ingredient in theories with extra dimensions, where in the 4d EFT the graviton comes typically accompanied by an infinite tower of massive spin-2 particles, its Kaluza-Klain (KK) replicas.[2] In this paper, we would like to understand how the CRG conjecture is satisfied in these scenarios.

To do so, we will focus on a very concrete but general model: we will study general relativity (GR) dimensionally reduced to four dimensions. Of course, GR in $\mathbb{R}^{1,D-1}$ trivially satisfies the CRG conjecture; the $2\to 2$ scattering of a GR graviton scales with s in the Regge limit at most as $\mathcal{A}\sim s^2$. The point is that when GR is compactified to 4d (we go from $\mathbb{R}^{1,D-1}$ to $\mathbb{R}^{1,3}\times X_{D-4}$), the description is given not only in terms of a graviton but also it includes an infinite tower of massive spin-2 particles. This provides an arena where the CRG conjecture can be *tested* in the presence of several massive spin-2 states.

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[1]This conclusion is in tension with [2], where a particular choice of constants for de Rham-Gabadadze-Tolley massive gravity was argued to be consistent with the CRG conjecture. This discrepancy, which deserves further clarification, does not affect the results of this paper, since they can be derived independently of [1,2].

[2]When the length of the internal l_{internal} and external l_{external} dimensions satisfy $l_{\text{external}} \gg l_{\text{internal}}$, the mass of the Kaluza-Klain spin-2 replicas usually becomes much bigger than the energy scale probed in the EFT and the massive spin-2 states can be ignored in the low-energy description.

What we will see in this work is that the CRG conjecture imposes nontrivial constraints[3] on the particle content of the 4d description. They teach us how the CRG requirement can be fulfilled in a 4d EFT containing massive spin-2 particles. Namely, as we will see in Sec. IV, in the spectrum of the 4d effective field theory:

(i) There must be either none or an infinite number of massive spin-2 particles. This is in line with the absence of consistent finite truncations of the graviton tower, already discussed in the literature [20–23].

(ii) The mass ratio between consecutive massive spin-2 replicas is bounded by the 4d couplings constants. Similar results imposing unitarity were derived in [23,24].

Before presenting these results, we will start by the beginning, briefly recalling the work done in [1].

II. A SINGLE MASSIVE SPIN-2 PARTICLE

In [1] it was studied the tree-level $2 \to 2$ scattering of a massive spin-2 particle in a theory containing neither other massive spin-2 states nor higher-spin particles. We wanted to check if this setup was compatible with the CRG conjecture. To do so:

(i) We assumed that the spin-2 particle could couple to a graviton, a (massive or massless) scalar particle, and a massive spin-1 particle.[4]

(ii) We considered both parity-even and parity-odd interactions.

(iii) We included all contact terms with an arbitrary but finite number of derivatives.

Exchange diagrams and contact terms are the two sources of contributions to any classical two-to-two scattering amplitude. Both can be computed directly using on-shell methods, in a Lagrangian independent way, as explained in [25].

(i) Exchange diagrams can be built from the on-shell cubic couplings. First, one has to list all possible on-shell three-point interactions between two massive spin-2 particles and the exchanged particle. Then, two sets of these vertices (multiplied by arbitrary constants) are connected through the correspondent propagator. In four dimensions, we found 24 independent exchange pieces, reproducing the results of [26–30].

(ii) Contact terms are a bit more tricky since, in principle, one can construct infinitely many of them, introducing more and more derivatives. In [1], adapting the ideas developed in [28], we included all contact interactions with an arbitrary, but finite, number of derivatives.

[3]These constraints will be automatically satisfied once a valid internal geometry is specified.

[4]Symmetries forbid interactions between two identical massive spin-2 particles and one massless spin-1 field or one fermion.

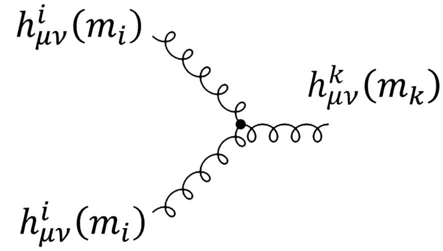

FIG. 1. Cubic interaction between two identical ($h^i_{\mu\nu}$ with mass m_i), one different ($h^k_{\mu\nu}$ with mass $m_k \neq m$) massive spin-2 particles.

With all these ingredients, we showed in [1] that a gravitational theory of a single massive spin-2 particle, coupled to any other state of spin < 2, can never be made consistent with the CRG conjecture.

III. SEVERAL MASSIVE SPIN-2 PARTICLES

We will now explain how to generalize the results of the previous section to incorporate any number of massive spin-2 particles in the spectrum. As we will see, the modifications are conceptually quite simple but technically very involved.

The *only* novelty concerning the previous computation lies in the number of the allowed cubic couplings. Besides the 24 previous pieces, we must include interactions between two identical and one different massive spin-2 particles, as shown in Fig. 1.

In four dimensions, there are seven parity-even and nine parity-odd independent on-shell three-point functions of this kind, with some of them already computed in [23]. We give the complete list with the explicit expressions in Appendix A, where we also derive a Lagrangian basis for the party-even contributions. For practical proposes, let us denote this set of interactions by

$$\mathcal{A}(h^i(m_i), h^i(m_i), h^k(m_k)) \equiv \sum_{j=1}^{16} c^{ii}_{k,j} f_j(h^i, h^i, h^k), \tag{2}$$

where $c^{ii}_{k,j}$ are arbitrary constants and $f_j(h^i, h^i, h^k)$ are the (9 parity-even and 7 parity-odd) on-shell cubic amplitudes, given in Appendix A.

This is the first step in accounting for several massive spin-2 particles, but it is not the end of the story. What we have just described corresponds to a theory in which any pair of identical massive spin-2 particles $\{h^i_{\mu\nu}, h^i_{\mu\nu}\}$ only interacts with one different massive spin-2 state $\{h^k_{\mu\nu}\}$. If we want to include the possibility that they couple to any number of different massive spin-2 particles, we need to replace (2) with

$$\sum_{k=1}^{K^{ii}} \mathcal{A}(h^i(m_i), h^i(m_i), h^k(m_k)) = \sum_{k=1}^{K^{ii}} \sum_{j=1}^{16} c^{ii}_{k,j} f_j(h^i, h^i, h^k), \tag{3}$$

where the index $k = 1, \ldots K^{ii}$ describes the coupling with K^{ii} distinguishable ($m_i \neq m_k$) massive spin-2 particles.

Generalizing [1] to include any number of massive spin-2 fields would correspond to take $K^{ii} = \text{arbitrary}$ and $c^{ii}_{k,j} = \text{arbitrary}$, since this is the most general possibility. Unfortunately, this case is technically very complicated, and little can be done explicitly. It would require introducing an arbitrarily large number of new constants in the equations of [1], which were already very complex.

To understand whether the CRG conjecture can be satisfied in the presence of several massive spin-2 particles, we find starting with a simpler model more illuminating. As a proof of concept example, we will study the 4d effective field theory obtained after dimensionally reducing GR. In this case, $c^{ii}_{k,j}$ are not arbitrary: they are completely fixed once the internal manifold is specified—GR has no free parameters—and there will be relations among them. Similar ideas studying the unitarity of GR under dimensional reductions were derived in [23], which we will use in this note.

IV. PROOF OF CONCEPT: GENERAL RELATIVITY

We will start by commenting and motivating again why this example is interesting. The framework described here is a summary of [23,31], to which we refer the reader for a more detailed discussion—we will only introduce the minimal ingredients to make the note self-contained.

Consider the Einstein-Hilbert action in $D > 4$ dimensions:

$$\mathcal{L} = \frac{M_D^{D-2}}{2} \sqrt{-G} R(G), \tag{4}$$

with M_D being the D-dimensional Plank mass. In $\mathbb{R}^{1,D-1}$, this theory is, of course, consistent with the CRG conjecture [3]. Dimensionally reducing it to 4d while keeping all massive modes just means selecting a different background for the theory. Consequently, one would expect the CRG conjecture to continue to be satisfied in the 4d picture. The interesting point is that, while in D-dimensions we have a description in terms of pure (Einstein) gravity, in 4d the dimensional reduction of the graviton *produces* a graviton but also a tower of massive spin-2, spin-1, and scalar particles. We can then take any Kaluza-Klain massive spin-2 copies of the graviton and compute its $2 \to 2$ scattering. In a generic theory with no other massive spin-2 particles, we saw in [1] that this scattering would violate the CRG bounds. In contrast, we will see below how the CRG conjecture is satisfied in this setup, imposing restrictions in the *effective* spectrum.

A. Dimensionally reduced theory

We will study the Lagrangian (4) in the direct product space $\mathcal{M}_D = \mathbb{R}^{1,3} \times X_{D-4}$, with metric

$$ds^2 = G_{AB} dX^A dX^B = \eta_{\mu\nu} dx^\mu dx^\nu + g_{ab} dy^a dy^b, \tag{5}$$

where $A = 1, \ldots, D$, $\mu = 1, \ldots 4$, $a = D-4, \ldots, D$ and we require X_{D-4} to be a closed, smooth, connected, orientable Ricci-flat[5] Riemannian manifold. To obtain the interactions in the lower dimensional description, one first needs to expand the metric around the background $\bar{G}_{AB}$:

$$G_{AB} = \bar{G}_{AB} + \frac{2}{M_D^{\frac{D-2}{2}}} \delta G_{AB}, \tag{6}$$

and then expand the fluctuations using the usual Hodge decomposition. Skipping some field redefinitions and showing only the contributions relevant to our computations, we have

$$\delta G_{\mu\nu}(x,y) = \sum_n h^n_{\mu\nu}(x) \psi_n(y) + \frac{1}{\sqrt{V}} h^0_{\mu\nu}(x), \tag{7a}$$

$$\delta G_{\mu a}(x,y) = \sum_i A^i_\mu(x) Y_{a,i}(y) + \ldots, \tag{7b}$$

$$\delta G_{ab}(x,y) = \frac{1}{D-4} \frac{1}{\sqrt{V}} \phi^0(x) g_{ab} + \ldots, \tag{7c}$$

where

$$V = \int_{X_{D-4}} \sqrt{-g} dy^a \equiv \int_{X_{D-4}} d\mathrm{vol}_{X_{D-4}} \tag{8}$$

and $\{\psi_n, Y_{n,i}\}$ satisfy

$$\Delta \psi_n \equiv -\Box \psi_i = m_n^2 \psi_i, \tag{9a}$$

$$\int_{X_{D-4}} \psi_n \psi_m d\mathrm{vol}_{X_{D-4}} = \delta_{nm}, \qquad m_n^2 > 0; \tag{9b}$$

$$\Delta Y_{a,i} \equiv -\Box Y_{a,i} + R^b_a Y_{b,i} = m_i^2 Y_{a,i}, \tag{9c}$$

$$\int_{X_{D-4}} Y^a_i Y_{a,j} d\mathrm{vol}_{X_{D-4}} = \delta_{ij}, \qquad m_i^2 \geq 0, \tag{9d}$$

with R_{ab} being the internal Ricci curvature. We refer again to [23,31] for a more detailed discussion of all the quantities and definitions. Plugging all these expressions

[5] This is necessary to solve the vacuum equations.

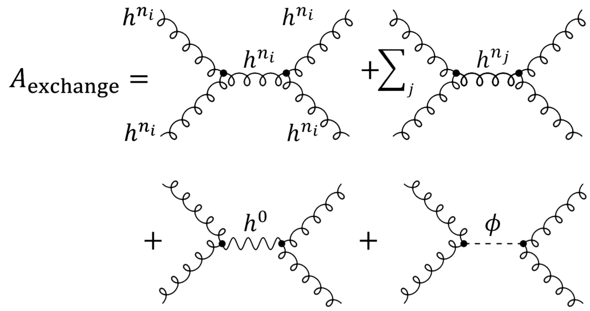

FIG. 2. Exchange contributions to the $2 \to 2$ scattering of a massive particle h^{n_i}. The particle is a part of the KK tower of the graviton in 4d. In the picture, $n_j \neq n_i$.

into (4) and expanding the action, one can obtain the spectrum and the interactions of the dimensionally reduced theory. Let us summarize the main results we will need.

1. Spectrum

From the quadratic terms, one can see that the four-dimensional theory contains the following:

(i) One massless graviton, $h^0_{\mu\nu}$.

(ii) A tower of massive spin-2 particles $h^n_{\mu\nu}$ with squared masses $m_n^2 > 0$. They come from the eigenfunctions of the scalar Laplacian on X_{D-4}.

(iii) A tower of spin-1 fields A^i_μ with squared masses $m_i^2 \geq 0$. They come from the eigenfunctions of the vector Laplacian on X_{D-4}. This tower includes the killing vectors, which are massless.

(iv) A massless scalar field ϕ^0 controlling the internal volume.

There are more scalar fields in the spectrum coming from the terms omitted in the decomposition of δG_{ab}. We will ignore them since they will not play any role in our computation; see Appendix B 1 for the details.

Finally, let us remember that the relation between the higher dimensional (M_D) and the lower dimensional (M_d) Planck mass is given by

$$M_d^{d-2} = V M_D^{D-2}. \tag{10}$$

2. Cubic interactions

We are interested in the (classical) scattering $h^{n_i} h^{n_j} \to h^{n_k} h^{n_l}$. Therefore, we will only need the three-point functions involving two (on-shell) massive spin-2 fields to construct the exchange diagrams. We relegate the explicit expressions to Appendix B 1, while we list here the relevant interactions:

(i) Three massive spin-2 particles: $\mathcal{A}(h^{n_1}, h^{n_2}, h^{n_3})$.

(ii) Two identical massive spin-2 particles—otherwise this interaction vanishes, see (B4)—and the graviton: $\mathcal{A}(h^{n_1}, h^{n_1}, h^0)$.

(iii) Two distinct massive spin-2 particles—otherwise this interaction vanishes, see (B5)—and a spin-1 particle: $\mathcal{A}(h^{n_1}, h^{n_2}, A^i_\mu)$.

(iv) Two massive spin-2 particles and a scalar field: $\mathcal{A}(h^{n_1}, h^{n_2}, \phi)$.

A pictorial representation of the exchange contributions to the $h^{n_i} h^{n_i} \to h^{n_i} h^{n_i}$ scattering can be seen in Fig. 2. It is also worth defining in this section the triple overlap integrals $g_{n_1 n_2 n_3}$:

$$g_{n_1 n_2 n_3} = \int_{X_{D-4}} \psi_{n_1} \psi_{n_2} \psi_{n_3} d\mathrm{vol}_{X_{D-4}}, \tag{11}$$

the ψ_{n_1} introduced in expression (7a), which will be used later on.

3. Contact terms

Finally, to compute any $h^{n_i} h^{n_j} \to h^{n_k} h^{n_l}$ scattering, we need the (on-shell) 4-point interactions. Repeating the previous game, one has to insert the decompositions introduced in Sec. IV A into the action (4) and collect the terms involving four massive spin-2 particles. We write the explicit form of this interaction, which we denote by $\mathcal{A}_{\text{contact}}(h^{n_1}, h^{n_2}, h^{n_3}, h^{n_4})$, in Appendix B 2. For posterior uses, we define here the quartic overlap integrals

$$g_{n_1 n_2 n_3 n_4} = \int_{X_{D-4}} \psi_{n_1} \psi_{n_2} \psi_{n_3} \psi_{n_4} d\mathrm{vol}_{X_{D-4}}, \tag{12}$$

which, as discussed in [23], can be written in terms of the cubic overlap integrals

$$\begin{aligned} g_{n_1 n_2 n_3 n_4} &= \sum_i g_{n_1 n_2 n_i} g_{n_3 n_4 n_i} + \frac{1}{V} \delta(n_1 n_2) \delta(n_3 n_4) \\ &= \sum_i g_{n_1 n_3 n_i} g_{n_2 n_4 n_i} + \frac{1}{V} \delta(n_1 n_3) \delta(n_2 n_4) \\ &= \sum_i g_{n_1 n_4 n_i} g_{n_2 n_3 n_i} + \frac{1}{V} \delta(n_1 n_4) \delta(n_2 n_3). \end{aligned} \tag{13}$$

B. Results

Having introduced all the necessary ingredients, we are finally in the position to test the CRG conjecture in the EFT obtained from the dimensional reduction of GR. Let us briefly recall the steps to follow:

(1) Compute the tree-level $h^{n_i} h^{n_j} \to h^{n_k} h^{n_l}$ scattering. Using the language presented in the previous section, it reads

$$\mathcal{A}(h^{n_i}, h^{n_j}, h^{n_k}, h^{n_l}) = \mathcal{A}_{\text{contact}} + \mathcal{A}_{\text{exhange}},$$

where, to construct the exchange diagrams, we take two sets of the three-point functions introduced in

Sec. IV A 2, "remove" the exchanged leg, and connect them through the correspondent propagator.

(2) Expand the total amplitude in the limit $s \gg t$, where s and t are the usual Mandelstam variables

$$\lim_{s\gg t}\mathcal{A}(h^{n_i},h^{n_j},h^{n_k},h^{n_l}) = \mathcal{A}_0(t)s^0 + \mathcal{A}_1(t)s^1 + \mathcal{A}_2(t)s^2 + \mathcal{A}_3(t)s^3 + \ldots . \tag{14}$$

In Appendix C, we recall the definition of the Mandelstam variables and set the conventions for the kinematics.

(3) Finally, from the previous expansion, we impose that

$$\mathcal{A}_n(t) = 0, \qquad \{n \geq 3, \ \forall \ t\}. \tag{15}$$

We will do this for any of the $5^4 = 625$ choices of polarization of the scattered spin-2 particles.[6]

Taking $n_i = n_j = n_k = n_l$, Eq. (15) requires

$$\forall\, n_i, \ 4m_{n_i}^2 g_{n_i n_i n_i n_i} - \sum_k 3m_{n_k}^2 g_{n_i n_i n_k}^2 = 0, \tag{16}$$

which can also be written, using expansion (13), as

$$\forall\, n_i, \ \sum_k (4m_{n_i}^2 - 3m_{n_k}^2) g_{n_i n_i n_k}^2 + 4m_{n_i}^2 V^{-1} = 0. \tag{17}$$

These relations, which will be automatically satisfied by any valid internal geometry, have several consequences regarding the 4d spectrum. They are not completely new since they also appear when one demands the dimensional reduced theory to be unitary [23,24]—actually, in [23], they were even able to find stronger conditions.[7]

From (17), it can be deduced that there must be an infinite number of KK modes in the spectrum. Since the term outside the sum is positive definite, the sum itself must produce a negative contribution that compensates it. This implies that for any h^{n_i} there must exist some h^{n_l} to which the h^{n_i} couples (that is, $g_{n_i n_i n_l} \neq 0$) and such that

$$\frac{2}{\sqrt{3}} m_{n_i} < m_{n_l}. \tag{18}$$

Taking $m_{n_i} = m_{n_1}$, this equation tells us that there must exist another spin-2 particle with mass $m_{n_2} > m_{n_1}$ in the spectrum. We can then apply the same strategy to m_{n_2} and conclude that the spectrum must contain a third spin-2 state with mass $m_{n_3} > m_{n_2}$. Repeating this reasoning, we see that any finite truncation of the KK tower of the graviton is incompatible with the CRG conjecture. This goes in the lines of [20]—see also [21,22]—who first showed the inconsistencies of a truncated KK spin-2 spectrum by using the breaking of the massive gauge invariances.

On the other side, Eq. (16) is useful to see[8] that the mass ratio of consecutive spin-2 modes is bounded by the 4d couplings. Since the first term in (16) is positive definite, the second term cannot be "too negative." In other words, for $\forall \ h^{n_i}$, there must exist some h^{n_k} to which h^{n_i} couples (that is, $g_{n_i n_i n_k} \neq 0$) and such that

$$\frac{m_{n_k}^2}{m_{n_i}^2} \leq \frac{4}{3}\frac{g_{n_i n_i n_i n_i}}{g_{n_i n_i n_k}^2} > 1, \tag{19}$$

constraining the mass ratios of consecutive spin-2 KK particles.[9]

A consequence of the relation (19) is that it seems to be difficult to generate a consistent gravitational 4d theory in which part of the graviton KK tower can be integrated out, leaving a finite—bigger than zero—number of massive spin-2 particles in the spectrum. For this to make sense, the mass m_Λ of the lightest integrated particle should satisfy $m, E \ll \Lambda_{\rm EFT} \ll m_\Lambda$, where m is the mass scale of the particles kept in the theory, E is the energy at which the theory is being probed, and $\Lambda_{\rm EFT}$ is the cutoff of the EFT. What we learn from (19) is that the 4d coupling constants bound the gap between the mass of the spin-2 replicas, so the couplings should be appropriately tuned to achieve the desired *mass separation*. Once chosen, one should find the concrete geometry producing these values for the 4d couplings, which can be a very nontrivial task.

Equation (17)—or its equivalent expression (16)—and its consequences are the main result of this paper. They teach us that, even if we start with a theory compatible with the CRG conjecture, as it is GR, when it is dimensionally reduced to 4d, the spectrum of the resulting theory satisfies two nontrivial constraints:

(1) Either there is none or an infinite number of massive spin-2 modes.
(2) The gap in the mass ratio of consecutive KK spin-2 states is bounded by the coupling constants of the theory.

We derived all these conditions by looking at the $h^{n_i}h^{n_i} \to h^{n_i}h^{n_i}$ amplitude, so one could wonder about

[6]Not all choices are independent; some of them will be related by crossing symmetry.

[7]This is because, while the CRG conjecture *cares* about terms scaling with the energy at order $s^3 \sim E^6$ or higher, unitarity in this context places conditions on terms scaling with the energy at order E^4 or higher.

[8]This can also be seen from (17) since they are equivalent; Eq. (16) just gives a cleaner expression.

[9]If $m_{n_k} = m_{n_{i+1}}$, then this is the maximum allowed gap between consecutive massive spin-2 states. If not, this means that $m_{n_k} > m_{n_{i+1}}$—since $m_{n_k} > m_{n_i}$—and so the gap is even smaller.

the more general $h^{n_i}h^{n_j} \to h^{n_k}h^{n_l}$ interaction. We also studied the CRG conjecture for this case. Nevertheless, the results and constraints derived from it are less powerful and interesting than the ones we already presented. In any case, the reader interested can find an ancillary *Mathematica* notebook with the code used.[10]

Before moving to the conclusions, it is worth pausing here for a moment to make a couple of comments.

As pointed out throughout the section, the constraints (16) and (17), imposed by the CRG conjecture, had already appeared in the literature. They are also a requirement for the dimensionally reduced theory to be unitary [23,24], which actually demands more stringent conditions. On the other hand, a condition similar to (18) was derived in [20] by studying the gauge invariances of a dimensionally reduced theory when the KK tower of the graviton is truncated. The novelty here is that we have derived all these requisites using the CRG conjecture. This is a nontrivial check of the conjecture: for the first time, it has been tested under dimensional reduction.

We have studied the dimensional reduction of GR on a compact, Riemannian, Ricci-flat internal manifold down to a 4d flat space. A natural question is thus how the conclusions would change if we modified any of the ingredients: including matter or higher-derivative corrections, choosing a different external space, and so on. These considerations can be taken into account all at once by studying the most generic case, discussed in Sec. III, which is a formidable task. A more doable approach could be to study the changes one by one, for instance, by looking at the scattering of massive spin-2 particles in AdS or by starting from GR coupled to some matter. Based on the results of [1] and on the apparent impossibility of constructing truncations with a finite number of massive spin-2 modes [20,22], we would expect the conclusions obtained here to hold in more general scenarios. We leave the exploration of these ideas for future work.

V. CONCLUSIONS

In this note, we have studied the CRG conjecture in the 4d effective field theory that results from compactifying D-dimensional general relativity (with $D > 4$) on a closed, Ricci-flat manifold. To do so, we have used the tools and the framework developed in [23].

Whereas the conjecture is trivially satisfied in the D-dimensional description,[11] the 4d picture consists of a theory of gravity coupled to an infinite number of massive spin-2 particles. We already saw in [1]—see [2] for related work—that any gravitational EFT containing a single massive spin-2 particle cannot be made consistent with the CRG conjecture. The example studied here, in contrast, serves as an arena to see how the CRG conjecture is satisfied in the presence of several massive spin-2 states.

The main result of this work is Eq. (16)—or equivalently Eq. (17)—which is required for the CRG conjecture to hold in the 4d framework. Both conditions are automatically met when choosing a valid internal geometry. From the 4d perspective, they teach us how the CRG conjecture can be realized in a theory containing massive spin-2 particles. These expressions are also part of the conditions for the theory to be unitary [23,24]. Two consequences follow from them:

(i) The 4d spectrum must include either no massive spin-2 fields or an infinite number of them; a finite truncation would not be possible. This was also discussed from other points of view in [20–23].

(ii) The mass ratio between the consecutive KK spin-2 replicas is bounded by the 4d coupling constants. We concrete this point in Sec. IV B. Similar conclusions (actually a bit stronger) were derived in [23] by studying unitarity in the 4d theory.

We see, then, that even if we start with a theory satisfying the CRG conjecture in D dimensions, the conjecture imposes nontrivial conditions in the 4d spectrum. This shows the power of the CRG conjecture to discern between consistent 4d theories, in this case regarding the ones with a higher-dimensional embedding, in line with the spirit of the swampland program [4]—see [7,32] for reviews.

It is important to keep in mind that in this work we have focused on the concrete example of GR dimensionally reduced to a flat 4d background. Therefore, one could wonder about other possibilities: starting with GR plus some matter, including higher-derivative corrections, changing the external space, and so on. We explained in Sec. III how to address the most generic situation, which would simultaneously encode all these possibilities. Unfortunately, this seems to be a highly complex task, so it may be smarter to add more ingredients one by one. These are exciting scenarios that for sure deserve further investigation.

In any case, we actually expect the conclusions presented here to hold in more general contexts. In light of the results of [1] together with this work, it seems pretty unlikely that the CRG conjecture could be satisfied in a gravitational EFT with a finite number of massive spin-2 particles, at least in flat space. We leave the exploration of these avenues and any other potential cases of interest for future work.

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[10] As a double-check of this code, we verified that in the high energy limit—that is, in the $\{s \to \infty, t \to \infty\}$ limit—it reproduces the results of [23].

[11] The $2 \to 2$ scattering of a GR graviton in $\mathbb{R}^{1,D-1}$ scales with s in the Regge limit as $\mathcal{A} \sim s^n, n \leq 2$ [3].

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APPENDIX A: CUBIC VERTICES

In this Appendix, we will discuss the possible three-point amplitudes for three massive spin-2 particles, two of which are identical and different from the third one. We divide this Appendix into two sections. In the first part, we list all the parity-even and parity-odd on-shell three-point interactions. In the second part, we give a Lagrangian basis for the parity-even terms.

1. On-shell amplitudes

Here we list all the possible (parity-even and parity-odd) on-shell three-point functions between two identical and one different massive spin-2 particles. In the particular case $d=4$, we also discuss the dimensionally dependent relations, which come from the fact that any set of five or more vectors is linearly dependent in four dimensions.

Notation: we denote by $\mathcal{M}_{i,j,k}(m_i, m_j, m_k)$ the on-shell three-point amplitude involving three particles of spin $\{i,j,k\}$, mass $\{m_i, m_j, m_k\}$, polarization matrices $\{\epsilon_1, \epsilon_2, \epsilon_3\}$, and momentum $\{p_1, p_2, p_3\}$. We define $A_{ij} \equiv \epsilon_i \cdot p_j$ and $B_{ij} = \epsilon_i \cdot \epsilon_j$. ε is the Levi-Civita tensor, $\varepsilon(p_i, p_j, \epsilon_k, \epsilon_l) \equiv \varepsilon^{\mu\nu\alpha\beta} p_{i\mu} p_{j\nu} \epsilon_{k\alpha} \epsilon_{l\beta}$.

a. Parity even

There are in general 8 different parity-even on-shell cubic amplitudes, listed in Table I. Part of this classification was already discussed in [23].

In $d=4$, the Gram matrix of the vectors $\{p_i, \epsilon_{i\mu}\}$ must vanish, from which we obtain

$$(-4m^2 m_k^2 + m_k^4)\mathcal{X}_1 + 2m_k^2 \mathcal{X}_2 + 2m^2 \mathcal{X}_3 + (2m_k^2 - 4m^2)\mathcal{X}_4 + 2m_k^2 \mathcal{X}_5 + 4\mathcal{X}_6 + 4\mathcal{X}_7 = 0, \quad \text{(A1)}$$

which can be used to ignore, for instance, $\mathcal{X}_7$.

TABLE I. All possible parity-even on-shell cubic amplitudes for two identical, one different massive spin-2 particles.

$\mathcal{M}^{\text{even}}_{2,2,2}(m, m, m_k)$
$B_{12}B_{23}B_{13} = \mathcal{X}_1$
$B_{12}^2 A_{31}^2 = \mathcal{X}_2$
$B_{13}^2 A_{23}^2 + B_{23}^2 A_{12}^2 = \mathcal{X}_3$
$B_{13}B_{23}A_{12}A_{23} = \mathcal{X}_4$
$B_{12}B_{23}A_{12}A_{31} + B_{12}B_{13}A_{23}A_{31} = \mathcal{X}_5$
$B_{12}A_{12}A_{21}A_{31}^2 = \mathcal{X}_6$
$B_{23}A_{12}^2 A_{21}A_{31} - B_{13}A_{12}A_{21}^2 A_{31} = \mathcal{X}_7$
$A_{12}^2 A_{21}^2 A_{31}^2 = \mathcal{X}_8$

TABLE II. All possible parity-odd on-shell three-point amplitudes for two identical, one different massive spin-2 particles.

$\mathcal{M}^{\text{odd}}_{2,2,2}(m, m, m_k)$
$B_{13}B_{23}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_2) = \tilde{\mathcal{X}}_1$
$B_{12}B_{23}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_3) - B_{12}B_{13}\varepsilon(p_1, p_2, \epsilon_2, \epsilon_3) = \tilde{\mathcal{X}}_2$
$A_{31}B_{12}(\varepsilon(p_1, \epsilon_1, \epsilon_2, \epsilon_3) + \varepsilon(p_2, \epsilon_1, \epsilon_2, \epsilon_3)) = \tilde{\mathcal{X}}_3$
$A_{21}B_{13}\varepsilon(p_1, \epsilon_1, \epsilon_2, \epsilon_3) - A_{12}B_{23}\varepsilon(p_2, \epsilon_1, \epsilon_2, \epsilon_3) = \tilde{\mathcal{X}}_4$
$A_{21}B_{13}\varepsilon(p_2, \epsilon_1, \epsilon_2, \epsilon_3) - A_{12}B_{23}\varepsilon(p_1, \epsilon_1, \epsilon_2, \epsilon_3) = \tilde{\mathcal{X}}_5$
$A_{31}^2 B_{12}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_2) = \tilde{\mathcal{X}}_6$
$(A_{21}A_{31}B_{13} - A_{12}A_{31}B_{23})\varepsilon(p_1, p_2, \epsilon_1, \epsilon_2) = \tilde{\mathcal{X}}_7$
$A_{31}B_{12}(A_{21}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_3) + A_{12}\varepsilon(p_1, p_2, \epsilon_2, \epsilon_3)) = \tilde{\mathcal{X}}_8$
$A_{21}^2 B_{13}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_3) - A_{12}^2 B_{23}\varepsilon(p_1, p_2, \epsilon_2, \epsilon_3) = \tilde{\mathcal{X}}_9$
$A_{12}A_{21}(B_{23}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_3) - B_{13}\varepsilon(p_1, p_2, \epsilon_2, \epsilon_3)) = \tilde{\mathcal{X}}_{10}$
$A_{12}A_{21}A_{31}(\varepsilon(p_1, \epsilon_1, \epsilon_2, \epsilon_3) + \varepsilon(p_2, \epsilon_1, \epsilon_2, \epsilon_3)) = \tilde{\mathcal{X}}_{11}$
$A_{12}A_{21}A_{31}^2\varepsilon(p_1, p_2, \epsilon_1, \epsilon_2) = \tilde{\mathcal{X}}_{12}$
$A_{21}A_{31}A_{12}(A_{21}\varepsilon(p_1, p_2, \epsilon_1, \epsilon_3) + A_{12}\varepsilon(p_1, p_2, \epsilon_2, \epsilon_3)) = \tilde{\mathcal{X}}_{13}$

b. Parity odd

Regarding the parity-odd terms, in general, there are 13 distinct possibilities, enumerated in Table II.

Dimensional-dependent relations in $d=4$—see [28] for the details—impose

$$2(m_k^4 - 4m^2 m_k^2)\tilde{\mathcal{X}}_1 + (4m^2 m_k^2 - m_k^4)\tilde{\mathcal{X}}_2 - (m_k^2 - 2m^2)^2 \tilde{\mathcal{X}}_3 + (2m^2 m_k^2 - 4m^4)\tilde{\mathcal{X}}_4 - 4m^4 \tilde{\mathcal{X}}_5 + 4(m^2 - m_k^2)\tilde{\mathcal{X}}_6 + 4m^2 \tilde{\mathcal{X}}_7 = 0, \quad \text{(A2a)}$$

$$\tilde{\mathcal{X}}_2 - \tilde{\mathcal{X}}_3 = 0, \quad \text{(A2b)}$$

$$\tilde{\mathcal{X}}_{10} - \tilde{\mathcal{X}}_{11} = 0, \quad \text{(A2c)}$$

$$m_k^2 \tilde{\mathcal{X}}_7 - m_k^2 \tilde{\mathcal{X}}_8 - 2m^2 \tilde{\mathcal{X}}_{11} + 2\tilde{\mathcal{X}}_{12} = 0, \quad \text{(A2d)}$$

which we can use to ignore four of the $\tilde{\mathcal{X}}_i$ involved in (A2), writing them as linear combinations of the others.

2. Lagrangian basis

To write a Lagrangian basis, we recall the expression for the linearized version of the Riemann tensor $R_{\alpha\beta\mu\nu}$ for a spin-2 field $h_{\mu\nu}$,

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}[\partial_\mu \partial_\beta h_{\nu\alpha} + \partial_\nu \partial_\alpha h_{\beta\mu} - \partial_\nu \partial_\beta h_{\mu\alpha} - \partial_\mu \partial_\alpha h_{\beta\nu}], \quad \text{(A3)}$$

and define the tensor $F_{\alpha\beta\mu}$ as

$$F_{\alpha\beta\mu} \equiv \partial_\alpha h_{\beta\mu} - \partial_\beta h_{\alpha\mu}. \quad \text{(A4)}$$

Using these two quantities, a Lagrangian basis for the parity-even on-shell three-point amplitudes introduced in Table I is given in Table III below.

TABLE III. Lagrangians describing parity-even three-point amplitudes for two identical, one different massive spin-2 particles.

Lagrangian basis $\mathcal{L}_{2,2,2}(m,m,m_k)$
$h^{\mu}_{1\nu}h^{\nu}_{2\alpha}h^{\alpha}_{3\mu}=\mathcal{L}_1$
$F^{1\mu\alpha}_{\beta}F^{2}_{\nu\mu\alpha}h^{3\beta\nu}=\mathcal{L}_2$
$F^{1\mu\alpha}_{\beta}h^{2\beta\nu}F^{3}_{\nu\mu\alpha}+h^{1\beta\nu}F^{2\mu\alpha}_{\beta}F^{3}_{\nu\mu\alpha}=\mathcal{L}_3$
$h^{1\mu\alpha}h^{2\nu\beta}R^{3}_{\mu\nu\alpha\beta}=\mathcal{L}_4$
$h^{1\mu\alpha}R^{2}_{\mu\nu\alpha\beta}h^{3\nu\beta}+R^{1}_{\mu\nu\alpha\beta}h^{2\nu\beta}h^{3\mu\alpha}=\mathcal{L}_5$
$R^{1\mu\nu\alpha\beta}F^{2\delta}_{\mu\nu}F^{3}_{\alpha\beta\delta}+F^{1\delta}_{\mu\nu}R^{2\mu\nu\alpha\beta}F^{3}_{\alpha\beta\delta}=\mathcal{L}_6$
$F^{1}_{\alpha\beta\delta}F^{2\delta}_{\mu\nu}R^{3\mu\nu\alpha\beta}=\mathcal{L}_7$
$R^{1\mu\nu}_{\alpha\beta}R^{2\gamma\delta}_{\mu\nu}R^{3\alpha\beta}_{\gamma\delta}=\mathcal{L}_8$

The relation with Table I is given by

$$\mathcal{X}_1=\mathcal{L}_1, \tag{A5a}$$

$$\mathcal{X}_2=\left(m^2-\frac{3}{2}m_k^2\right)\mathcal{L}_1-\mathcal{L}_2+\mathcal{L}_3-\mathcal{L}_4, \tag{A5b}$$

$$\mathcal{X}_3=(m_k^2-2m^2)\mathcal{L}_1+2\mathcal{L}_2-\mathcal{L}_5, \tag{A5c}$$

$$2\mathcal{X}_4=(2m^2-m_k^2)\mathcal{L}_1-2\mathcal{L}_2-\mathcal{L}_4+\mathcal{L}_5, \tag{A5d}$$

$$\mathcal{X}_5=m_k^2\mathcal{L}_1-\mathcal{L}_3+\mathcal{L}_4, \tag{A5e}$$

$$4\mathcal{X}_6=3m_k^4\mathcal{L}_1-2m_k^2(\mathcal{L}_3-\mathcal{L}_4)+\mathcal{L}_7, \tag{A5f}$$

$$\begin{aligned}4\mathcal{X}_7=&\,6m_k^2(2m^2-m_k^2)\mathcal{L}_1-4m_k^2\mathcal{L}_2+2(m_k^2-2m^2)\mathcal{L}_3\\&+4(m^2-m_k^2)\mathcal{L}_4+2m_k^2\mathcal{L}_5+\mathcal{L}_6,\end{aligned} \tag{A5g}$$

$$\begin{aligned}8\mathcal{X}_8=&\,5m_k^4(2m^2-m_k^2)\mathcal{L}_1-2m_k^4\mathcal{L}_2+2m_k^2(m_k^2-2m^2)\mathcal{L}_3\\&+m_k^2(4m^2-3m_k^2)\mathcal{L}_4+m_k^4\mathcal{L}_5+m_k^2\mathcal{L}_6\\&+(2m^2-m_k^2)\mathcal{L}_7-\mathcal{L}_8.\end{aligned} \tag{A5h}$$

It is important to bear in mind that Lagrangians are off-shell quantities. Under field redefinitions or integration by parts, they give rise to the same (on-shell) dynamics. This being said, notice that the lhs of Eq. (A5) is defined on-shell. Therefore, the equality only makes sense when the rhs—the linear combination of Lagrangians—is also evaluated on-shell.

APPENDIX B: COUPLINGS FROM GR

1. Cubic couplings

In this Appendix, we write the three-point interactions that result from plugging the decomposition (7) into the Einstein-Hilbert action. These results were published initially in [23,31].

To start, we need to fix the notation. A particle i has momentum p_i and mass m_i. We denote its polarization tensor by ϵ_i. This tensor is symmetric and traceless. Formally, when constructing the interactions, we will write the polarization matrices as a product of vectors $\epsilon_i=\epsilon_{i\mu\nu}\equiv\epsilon_{i\mu}\epsilon_{i\nu}$. This does not mean that the polarization matrices have rank one; it is only a trick to keep track of the contractions more easily. To simplify the expressions, we call $A_{ij}\equiv\epsilon_i\cdot p_j$, $B_{ij}=\epsilon_i\cdot\epsilon_j$, and $p_{ij}=p_i\cdot p_j$. Finally, we put the external legs of the amplitude (the massive spin-2 particles) on-shell, whereas we keep the exchanged particle off-shell.

This being said, the vertices involving (at least) two massive spin-2 particles are as follows.

(i) Three massive spin-2 particles $\{h^{n_1},h^{n_2},h^{n_3}\}$:

$$\begin{aligned}&\mathcal{A}(h^{n_1},h^{n_2},h^{n_3})\\&=-\frac{g_{n_1n_2n_3}}{4M_D^{\frac{D-2}{2}}}B_{23}[4A_{13}(B_{23}A_{12}-2B_{12}A_{32})\\&\quad+(2p_{12}-m_{n_1}^2)B_{12}B_{23}]+5\,\text{permutations},\end{aligned} \tag{B1}$$

where we have implicitly assigned the numbers

$$\{1,2,3\}\equiv\{h^{n_1},h^{n_2},h^{n_3}\}, \tag{B2}$$

(we will also do this for the other interactions) and have introduced the triple overlap integrals

$$g_{n_1n_2n_3}=\int_{X_{D-4}}\psi_{n_1}\psi_{n_2}\psi_{n_3}d\text{vol}_{X_{D-4}}. \tag{B3}$$

(ii) Two massive spin-2 particles and the graviton $\{h^{n_1},h^{n_2},h^0\}$:

$$\begin{aligned}&\mathcal{A}(h^{n_1},h^{n_2},h^0)\\&=\frac{1}{4M_d^{\frac{d-2}{2}}}\delta_{n_1,n_2}[8(B_{23}A_{12}-B_{13}A_{21})(B_{23}A_{12}+B_{12}A_{31})\\&\quad+(B_{12})^2(p_{33}B_{33}-4A_{31}A_{32})]+1\leftrightarrow 2.\end{aligned} \tag{B4}$$

(iii) Two massive spin-2 particles and one spin-1 particle $\{h^{n_1},h^{n_2},A_\mu^{i_3}\}$:

$$\begin{aligned}\mathcal{A}(h^{n_1},h^{n_2},A_\mu^{i_3})=&\,\frac{\sqrt{2}}{M_D^{\frac{D-2}{2}}}g_{n_1n_2i_3}(2B_{12}B_{13}A_{21}-B_{12}^2A_{31}\\&-(1\leftrightarrow 2)),\end{aligned} \tag{B5}$$

with

$$g_{n_1n_2i_3}=\int_{X_{D-4}}\partial^a\psi_{n_1}\psi_{n_2}Y_{a,i_3}d\text{vol}_{X_{D-4}}, \tag{B6}$$

with it being antisymmetric in the first two indices—so necessarily $n_1 \neq n_2$.

(iv) Two massive spin-2 particles and one scalar particle $\{h^{n_1}, h^{n_2}, \phi\}$:

$$\mathcal{A}(h^{n_1}, h^{n_2}, \phi) \propto \phi B_{12}^2. \tag{B7}$$

As commented in [1], for any polarization of the spin-2 particles, this term scales with s as s^n, $n \leq 2$: it does not contribute to the CRG equations. For this reason, for our purposes, it is enough to write the part that gives the dependence on the kinematics.

2. Quartic couplings

Following the notation introduced in the previous section, the on-shell 4-point interaction between any four massive spin-2 particles $\{h^{n_1}, h^{n_2}, h^{n_3}, h^{n_4}\}$ that are part of the KK tower of the graviton is given by

$$\begin{aligned}
&\mathcal{A}_{\text{contact}}(h^{n_1}, h^{n_2}, h^{n_3}, h^{n_4}) \\
&= \frac{1}{M_D^{D-2}}\Bigg[g_{n_1n_2n_3n_4}B_{12}B_{34}((p_{12} - m_{n_1}^2)B_{14}B_{23} \\
&\quad + A_{14}[B_{34}A_{23} - B_{24}A_{32} - B_{23}(A_{41} + 2A_{43})]) \\
&\quad + \sum_i \frac{1}{2} m_{n_i}^2 g_{n_1n_2n_i} g_{n_3n_4n_i} B_{12}B_{34}B_{14}B_{23}\Bigg] \\
&\quad + 23\,\text{permutations},
\end{aligned} \tag{B8}$$

where

$$g_{n_1n_2n_3n_4} = \int_{X_{D-4}} \psi_{n_1}\psi_{n_2}\psi_{n_3}\psi_{n_4} d\text{vol}_{X_{D-4}}. \tag{B9}$$

APPENDIX C: KINEMATICS

In this Appendix, we will write the definitions of the variables used to compute the $2 \to 2$ scattering of Sec. IV B.

The incoming particles are labeled by 1 and 2, and outgoing particles by 3 and 4. Momentum conservation requires

$$p_1 + p_2 = p_3 + p_4, \tag{C1}$$

where we take

$$p_i^\mu = (E_i, p_i \sin\theta_i, 0, p_i \cos\theta_i), \tag{C2}$$

with $E_i^2 = p_i^2 + m_i^2$ and $\theta_1 = 0$, $\theta_2 = \pi$, $\theta_3 = \theta$, $\theta_4 = \theta - \pi$. The Mandelstam variables are

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2, \tag{C3}$$

and we are taking the metric $\eta = \text{diag}(-1, 1, 1, 1)$. They satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2, \tag{C4}$$

with m_i being the mass of the particle i. When $m_1 = m_2 = m_3 = m_4 = m$—and $E_i = E, p_i = p$—the Mandelstam variables have the simple expressions

$$s = 4E^2, \qquad \cos\theta = 1 - \frac{2t}{4m^2 - s}. \tag{C5}$$

To construct the polarization matrices of the spin-2 particles, we first introduce the polarization vectors:

$$\epsilon_1^\mu(p_i) = (0, 0, 1, 0), \tag{C6a}$$

$$\epsilon_2^\mu(p_i) = (0, \cos(\theta_i), 0, -\sin(\theta_i)), \tag{C6b}$$

$$\epsilon_3^\mu(p_i) = \frac{1}{m_i}(p_i, e_i \sin(\theta_i), 0, e_i \sin(\theta_i)), \tag{C6c}$$

from which the polarization matrices can be constructed, adapting the conventions of [33], as

$$\epsilon_{T,1}^{\mu\nu}(p_i) = \frac{1}{\sqrt{2}}(\epsilon_1^\mu(p_i)\epsilon_1^\nu(p_i) - \epsilon_2^\mu(p_i)\epsilon_2^\nu(p_i)), \tag{C7a}$$

$$\epsilon_{T,2}^{\mu\nu}(p_i) = \frac{1}{\sqrt{2}}(\epsilon_1^\mu(p_i)\epsilon_2^\nu(p_i) + \epsilon_2^\mu(p_i)\epsilon_1^\nu(p_i)), \tag{C7b}$$

$$\epsilon_{V,1}^{\mu\nu}(p_i) = \frac{1}{\sqrt{2}}(\epsilon_3^\mu(p_i)\epsilon_1^\nu(p_i) + \epsilon_1^\mu(p_i)\epsilon_3^\nu(p_i)), \tag{C7c}$$

$$\epsilon_{V,2}^{\mu\nu}(p_i) = \frac{1}{\sqrt{2}}(\epsilon_3^\mu(p_i)\epsilon_2^\nu(p_i) + \epsilon_2^\mu(p_i)\epsilon_3^\nu(p_i)), \tag{C7d}$$

$$\epsilon_S^{\mu\nu}(p_i) = \sqrt{\frac{3}{2}}\left(\epsilon_3^\mu(p_i)\epsilon_3^\nu(p_i) - \frac{1}{3}\left(\eta^{\mu\nu} + \frac{1}{m_i^2}p_i^\mu p_i^\nu\right)\right), \tag{C7e}$$

where T, V, and S stand for tensor, vector, and scalar polarizations, respectively.

Regarding the propagators, the propagator of a scalar particle with mass M is

$$\frac{-i}{p^2 + M^2 - i\epsilon}. \tag{C8}$$

For a massive spin-1 particle, first we need to introduce the projector

$$\Pi_{\mu\nu}(\tilde{m}) = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{\tilde{m}^2}, \tag{C9}$$

from which one can write the propagator of a massive spin-1 particle with mass m_1 as

$$P_{\mu\nu} = \frac{-i\Pi_{\mu\nu}(m_1)}{p^2 + m_1^2 - i\epsilon}. \tag{C10}$$

Finally, the propagator of a massive spin-2 particle of mass m is

$$P_{\mu_1\mu_2,\nu_1\nu_2} = \frac{-i}{2}\frac{\Pi_{\mu_1\nu_1}\Pi_{\mu_2\nu_2} + \Pi_{\mu_1\nu_2}\Pi_{\mu_2\nu_1} - \frac{2}{3}\Pi_{\mu_1\mu_2}\Pi_{\nu_1\nu_2}}{p^2 + m^2 - i\epsilon}, \tag{C11}$$

with $\Pi_{\nu_1\nu_2} = \Pi_{\nu_1\nu_2}(m)$, whereas for a massless spin-2 (in de Donder gauge) it reads

$$\tilde{P}_{\mu_1\mu_2,\nu_1\nu_2} = \frac{-i}{2}\frac{\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\mu_2\nu_1} - \eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}}{p^2 - i\epsilon}. \tag{C12}$$

[1] S. Kundu, E. Palti, and J. Quirant, Regge growth of isolated massive spin-2 particles and the swampland, J. High Energy Phys. 05 (2024) 139.
[2] S. D. Chowdhury, V. Kumar, S. Kundu, and A. Rahaman, Regge constraints on local four-point scattering amplitudes of massive particles with spin, J. High Energy Phys. 05 (2024) 123.
[3] S. D. Chowdhury, A. Gadde, T. Gopalka, I. Halder, L. Janagal, and S. Minwalla, Classifying and constraining local four photon and four graviton S-matrices, J. High Energy Phys. 02 (2020) 114.
[4] C. Vafa, The string landscape and the swampland, arXiv: hep-th/0509212.
[5] D. Klaewer, D. Lüst, and E. Palti, A spin-2 conjecture on the swampland, Fortschr. Phys. **67**, 1800102 (2019).
[6] C. De Rham, L. Heisenberg, and A. J. Tolley, Spin-2 fields and the weak gravity conjecture, Phys. Rev. D **100**, 104033 (2019).
[7] E. Palti, The swampland: Introduction and review, Fortschr. Phys. **67**, 1900037 (2019).
[8] J. Bonifacio, K. Hinterbichler, and R. A. Rosen, Positivity constraints for pseudolinear massive spin-2 and vector Galileons, Phys. Rev. D **94**, 104001 (2016).
[9] C. de Rham, S. Melville, and A. J. Tolley, Improved positivity bounds and massive gravity, J. High Energy Phys. 04 (2018) 083.
[10] B. Bellazzini, F. Riva, J. Serra, and F. Sgarlata, Beyond positivity bounds and the fate of massive gravity, Phys. Rev. Lett. **120**, 161101 (2018).
[11] C. de Rham, S. Melville, A. J. Tolley, and S.-Y. Zhou, Positivity bounds for massive spin-1 and spin-2 fields, J. High Energy Phys. 03 (2019) 182.
[12] L. Alberte, C. de Rham, A. Momeni, J. Rumbutis, and A. J. Tolley, Positivity constraints on interacting spin-2 fields, J. High Energy Phys. 03 (2020) 097.
[13] Z.-Y. Wang, C. Zhang, and S.-Y. Zhou, Generalized elastic positivity bounds on interacting massive spin-2 theories, J. High Energy Phys. 04 (2021) 217.
[14] B. Bellazzini, G. Isabella, S. Ricossa, and F. Riva, Massive gravity is not positive, Phys. Rev. D **109**, 024051 (2024).
[15] K. Hinterbichler, Theoretical aspects of massive gravity, Rev. Mod. Phys. **84**, 671 (2012).
[16] C. de Rham, Massive gravity, Living Rev. Relativity **17**, 7 (2014).
[17] D. Chandorkar, S. D. Chowdhury, S. Kundu, and S. Minwalla, Bounds on Regge growth of flat space scattering from bounds on chaos, J. High Energy Phys. 05 (2021) 143.
[18] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, J. High Energy Phys. 08 (2016) 106.
[19] K. Häring and A. Zhiboedov, Gravitational Regge bounds, SciPost Phys. **16**, 034 (2024).
[20] M. J. Duff, C. N. Pope, and K. S. Stelle, Consistent interacting massive spin-2 requires an infinity of states, Phys. Lett. B **223**, 386 (1989).
[21] J. P. Gauntlett, S. Kim, O. Varela, and D. Waldram, Consistent supersymmetric Kaluza-Klein truncations with massive modes, J. High Energy Phys. 04 (2009) 102.
[22] J. T. Liu and C. N. Pope, Inconsistency of breathing mode extensions of maximal five-dimensional supergravity embedding, J. High Energy Phys. 06 (2012) 067.
[23] J. Bonifacio and K. Hinterbichler, Unitarization from geometry, J. High Energy Phys. 12 (2019) 165.
[24] C. Csaki, C. Grojean, H. Murayama, L. Pilo, and J. Terning, Gauge theories on an interval: Unitarity without a Higgs, Phys. Rev. D **69**, 055006 (2004).
[25] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, Spinning conformal correlators, J. High Energy Phys. 11 (2011) 071.
[26] K. Hinterbichler, A. Joyce, and R. A. Rosen, Massive spin-2 scattering and asymptotic superluminality, J. High Energy Phys. 03 (2018) 051.
[27] J. J. Bonifacio, Aspects of massive spin-2 effective field theories, Ph.D. thesis, Oxford University, 2017.
[28] J. Bonifacio and K. Hinterbichler, Bounds on amplitudes in effective theories with massive spinning particles, Phys. Rev. D **98**, 045003 (2018).
[29] J. Bonifacio and K. Hinterbichler, Universal bound on the strong coupling scale of a gravitationally coupled massive spin-2 particle, Phys. Rev. D **98**, 085006 (2018).

[30] J. Bonifacio, K. Hinterbichler, and R. A. Rosen, Constraints on a gravitational Higgs mechanism, Phys. Rev. D **100**, 084017 (2019).

[31] K. Hinterbichler, J. Levin, and C. Zukowski, Kaluza-Klein towers on general manifolds, Phys. Rev. D **89**, 086007 (2014).

[32] M. van Beest, J. Calderón-Infante, D. Mirfendereski, and I. Valenzuela, Lectures on the swampland program in string compactifications, Phys. Rep. **989**, 1 (2022).

[33] J. Bonifacio, K. Hinterbichler, A. Joyce, and R. A. Rosen, Massive and massless spin-2 scattering and asymptotic superluminality, J. High Energy Phys. 06 (2018) 075.