Noninvertible symmetries in 2D from type IIB string theory

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We propose a top-down approach to noninvertible symmetries in two-dimensional quantum field theories and their three-dimensional (3D) associated symmetry topological field theories. We focus on the gauge theory engineered on D1-branes probing a particular Calabi-Yau 4-fold singularity. We show how to derive the symmetry topological field theory, a 3D Dijkgraaf-Witten theory, from the IIB supergravity under dimensional reduction. We also identify branes behind the noninvertible topological lines by dimensionally reducing their world volume actions. The action of noninvertible lines on charged local operators is then realized as the Hanany-Witten transition.

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I. INTRODUCTION

Global symmetry is one of the most important concepts in quantum field theories (QFTs). It provides powerful tools to investigate QFTs, even those strongly coupled or without Lagrangian. A modern approach to understanding global symmetries is through their associated topological symmetry operators or defects [1]: for a D-dimensional QFT with a q-form global symmetry whose symmetry group is G, a topological operator $U(M_{D-q-1})_g$ is associated with the group element g and supported on the codimension-q manifold M_{D-q-1}. An operator charged under this q-form symmetry is supported on q-dimensional manifold N_q, linking with the M_{D-q-1}. It carries a representation of the group G and thus transforms accordingly when acted on by a topological operator $U(M_{D-q-1})_g$. The group multiplication law leads to the simple fusion rule between symmetry operators as $U(M_{D-q-1})_g \times U(M_{D-q-1})_h = U(M_{D-q-1})_{gh}$. The existence of the group element g^{-1} gives rise to the invertibility of the symmetry operator: $U(M_{D-q-1})_g \times U(M_{D-q-1})_{g^{-1}} \equiv U(M_{D-q-1}) \times U^{-1}(M_{D-q-1}) = 1$. Relaxing the group multiplication law and considering nontrivial fusion rules for symmetry operators U_i's as $U_i(M_{D-q-1}) \times U_j(M_{D-q-1}) = \sum_k c_{ij}^k U_k(M_{D-q-1})$, one ends up with symmetries which are not grouplike, known as *noninvertible symmetries.*[1]

In the context of QFTs engineered from singularities in string theory, e.g., via geometric engineering or brane probes, generalized global symmetries admit elegant top-down realizations. On the one hand, the charged defects are built by branes wrapping noncompact cycles of the internal geometry, extending from the singularity (where the QFT is engineered) to "infinity" [7–10]. On the other hand, it was recently pointed out in Refs. [11–13] (see also Refs. [14–18]) that generalized symmetry operators arise from wrapped branes "at infinity."[2] In particular, in the case of noninvertible symmetries, the topological field theory (TFT) living on the symmetry operator, responsible for the nontrivial fusion rules, can be directly obtained from the topological sector of the brane action on its world volume via dimensional reduction on the wrapped cycles at infinity.

Despite many top-down approaches and brane constructions for noninvertible symmetries being introduced in the literature, to our knowledge, they almost exclusively focus on QFTs in $D > 2$ dimensions. To some extent, this is a bit surprising since noninvertible symmetries are most ubiquitous in two dimensions.[3] In this paper, we fill this small gap by explicitly constructing brane origins for noninvertible symmetries in two-dimensional (2D) QFTs with string theory realization.

2D QFT on D1-branes probing singularities. The 2D QFTs we will focus on are gauge theories engineered on D1-branes probing the conical singularity of a Calabi-Yau 4-fold (CY_4). The IIB string theory background reads

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[1]We refer the reader to Refs. [2–6] for recent reviews.

[2]In addition to branes, generalized symmetry operators can also arise from purely geometric configuration. See, e.g., Ref. [19] and Appendix A in Ref. [14].

[3]In two dimensions, noninvertible symmetries have a long history. See, e.g., Refs. [20–27] for a partial list of seminal papers.

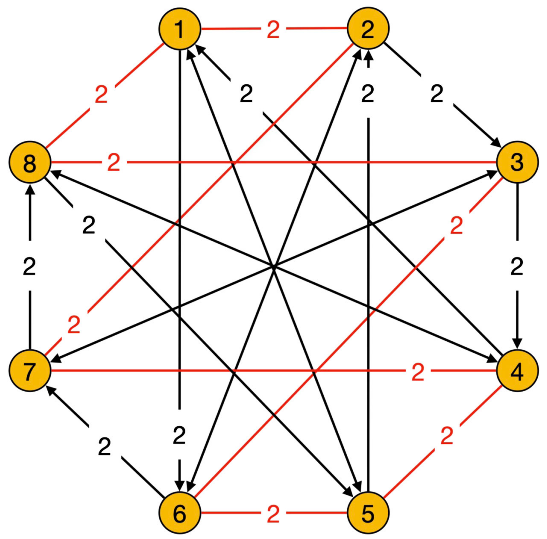

FIG. 1. Quiver diagram for a 2D gauge theory phase associated with $Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$ probed by N D1-branes [40]. Yellow circles denote $U(N)$ gauge groups. Oriented black lines and unoriented red lines denote bifundamental chiral and Fermi superfields, respectively.

$$\mathbb{R}^{1,1}\times Y, \tag{1.1}$$

where $\mathbb{R}^{1,1}$ supports the world volume of a stack of N D1-branes and Y is a local noncompact CY_4. In the case when Y is toric, an infinite class of 2D theories has been explicitly constructed, using an elegant T-dual IIA intersecting brane configuration known as *brane brick models* [28–31].[4] The resulting 2D QFTs are $U(N)^K$ quiver gauge theories,[5] which can be fully specified by quiver diagrams (encoding the field content and the gauge interaction) and superpotentials (encoding the matter interaction).[6]

To illustrate our idea explicitly, in this paper, we focus on the 2D gauge theory associated with a specific conical CY_4,

$$\mathrm{Cone}(Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)), \tag{1.2}$$

which is the cone over a smooth 7-manifold known as $Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$. The 2D gauge theory is constructed in Ref. [40], and its quiver diagram is shown in Fig. 1. The 7-manifold $Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$ falls in an infinite class of Sasaki-Einstein 7-manifolds denoted as $Y^{p,k}(\mathbb{P}^1\times\mathbb{P}^1)$, which are lens space $S^3/\mathbb{Z}_p$ bundles over $\mathbb{P}^1\times\mathbb{P}^1$ [42]. We leave the systematic treatment of noninvertible and other global symmetries for general brane brick models in the forthcoming work [43].

3D symmetry TFT from string theory. We will use the *symmetry TFT* framework to build noninvertible symmetries for our interested 2D gauge theory. Symmetry TFT is a $(D+1)$-dimensional TFT capturing the topological nature of generalized global symmetries in a D-dimensional QFT [21,44–56]. It has a physical boundary and a topological boundary. The local information (local operators and their correlation functions) of the interested D-dimensional QFT is realized on the physical boundary, also known as the *relative QFT* [52]. On the other hand, gapped boundary conditions are defined on the topological boundary, which specifies the global structure of the D-dimensional QFT.

For QFTs engineered on conical singularities of a local noncompact internal geometry Y in string theory, the associated symmetry TFT can be derived from the topological sector of the dimensional reduction for the 10-dimensional (10D) (11-dimensional for M-theory) supergravity on the asymptotic boundary ∂Y [57] (see also Refs. [14,15,19,56,58–60]). Various string theory fluxes under dimensional reduction give rise to gauge fields in the symmetry TFT. For our interested case in this paper, namely, D1-branes probing $\mathrm{Cone}(Y^{(2,0)}(\mathbb{P}^1\times\mathbb{P}^1))$, the dimensional reduction to obtain a three-dimensional (3D) symmetry TFT is performed in the IIB string theory background

$$M_3\times Y^{(2,0)}(\mathbb{P}^1\times\mathbb{P}^1), \tag{1.3}$$

where $M_3\cong M_2\times\mathbb{R}_{r\geq 0}$ is the 3D manifold for the symmetry TFT bulk. The physical boundary corresponds to $r=0$ where D1-branes are localized, while the topological boundary arises at $r=\infty$ where boundary conditions of various IIB fluxes are picked. The detailed computation will be discussed in Sec. II, where we show the resulting 3D symmetry TFT is a twisted $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ 3D Dijkgraaf-Witten theory

$$S_3=\frac{2\pi}{2}\int_{M_3} a_1\delta\hat{a}_1+b_1\delta\hat{b}_1+c_1\delta\hat{c}_1+a_1b_1c_1. \tag{1.4}$$

Noninvertible symmetry operators from branes. Recall that picking a topological boundary condition for the symmetry TFT corresponds to fixing a global structure of its associated D-dimensional QFT. This procedure is called picking a *polarization*, and the resulting QFT with a well-defined global structure is referred to as an *absolute QFT* (see, e.g., Refs. [19,52,61,62]). From this bulk perspective, gauging a symmetry in a QFT to get another QFT is translated in changing from one polarization to another.

Global symmetries of the resulting absolute QFT can be obtained by investigating the behavior of bulk operators

[4] See Refs. [32–40] for more details.

[5] Strictly speaking, there also exist gauge theory phases whose gauge factors $U(N_i)$ can have different ranks. These are referred to as *nontoric* phases [30], which can be derived by performing the $\mathcal{N}=(0,2)$ triality [41] from toric phases.

[6] Brane brick models enjoy $\mathcal{N}=(0,2)$ supersymmetry. However, at the level of generalized global symmetries we discuss in this paper, supersymmetry matters little.

under the topological boundary condition. Operators trivialized when touching the gapped boundary (due to the possible Dirichlet condition), giving rise to charged defects. In contrast, those not trivialized are still topological operators, generating global symmetries for the absolute QFT.

Based on this general idea, one can start with the symmetry TFT (1.4) and write down gauge-invariant line operators by purely field-theoretic consideration, much as in Refs. [63,64]. However, since the Dijkgraaf-Witten theory (1.4) is derived from string theory, one naturally asks whether there is a direct top-down approach to topological line operators. The answer is indeed yes. As we will discuss in Sec. II, all line operators, whether invertible or not, have their corresponding brane origin. Line operators are obtained exactly from the brane world volume action via dimensional reduction on various cycles wrapped by branes.

Having obtained line operators in the 3D bulk from branes, building noninvertible symmetries in 2D gauge theory associated with $Y^{(2,0)}(\mathbb{P}^1\times\mathbb{P}^1)$ then translates in writing down polarizations under which the noninvertible bulk lines are still noninvertible when touching the gapped boundary. We will show in Sec. III that these polarizations indeed exist, and the resulting noninvertible symmetry is the well-known $\mathbb{Z}_2\times\mathbb{Z}_2$ Tambara-Yamagami fusion category [65]. For example, the polarization corresponding to the boundary condition

$$a_1, \hat{b}_1, \hat{c}_1 \text{ Dirichlet; } \hat{a}_1, b_1, c_1 \text{ Neumann} \tag{1.5}$$

has the noninvertible fusion rules

$$\begin{aligned}\mathcal{N}_{\mathrm{D3}}\times\mathcal{N}_{\mathrm{D3}} &= 1+\eta_{\mathrm{F1}}+\eta_{\mathrm{D1}}+\eta_{\mathrm{F1}}\eta_{\mathrm{D1}},\\ \eta_{\mathrm{F1}}\times\eta_{\mathrm{F1}} &= \eta_{\mathrm{D1}}\times\eta_{\mathrm{D1}}=1,\\ \eta_{\mathrm{F1}}\times\mathcal{N}_{\mathrm{D3}} &= \eta_{\mathrm{D1}}\times\mathcal{N}_{\mathrm{D3}}=\mathcal{N}_{\mathrm{D3}},\end{aligned} \tag{1.6}$$

where $\mathcal{N}_{\mathrm{D3}}$ is the noninvertible line from D3-brane, while η_{F1} and η_{D1} are invertible $\mathbb{Z}_2$ lines from F1- and D1-strings, respectively.

In addition to polarizations enjoying noninvertible symmetries, we also find polarizations where all topological line operators become invertible symmetry lines. That is to say, the noninvertible symmetries we construct in this paper are *nonintrinsic* [66–68].

II. 3D DIJKGRAAF-WITTEN THEORY AND ITS LINE OPERATORS FROM IIB

In this section, we present how to obtain the 3D symmetry TFT and its line operators for 2D gauge theory associated with $\mathrm{Cone}(Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1))$ from IIB string theory via dimensional reduction. In particular, we find the following top-down approach to the field theory content: 3D Dijkgraaf-Witten theory is obtained from IIB supergravity, while line operators in the 3D bulk are derived from branes world volume actions, via the dimensional reduction.

A. 3D Dijkgraaf-Witten theory from the IIB supergravity

To derive the 3D symmetry TFT, we focus on the topological sector of the reduction for IIB string theory on the asymptotic boundary of the Calabi-Yau 4-fold, which in this case is just the base manifold $L_7\equiv Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$ at infinity. In particular, we treat the various IIB supergravity fluxes as elements in differential cohomology uplifts of (see, e.g., Refs. [57,61])

$$H^*(L_7,\mathbb{Z})=\{\mathbb{Z},0,\mathbb{Z}^2\oplus\mathbb{Z}_2,0,\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2,\mathbb{Z}^2,\mathbb{Z}_2,\mathbb{Z}\}. \tag{2.1}$$

The above cohomology classes for $L_7=Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$ can be found in Ref. [42].

The relevant topological action inherited from the IIB string theory, roughly speaking, consists of two parts. The quadratic part comes from the kinetic terms for IIB fluxes, and the cubic part comes from the 10D Chern-Simons coupling $-\int C_4\wedge dB_2\wedge dC_2$.[7] Consider an IIB string theory background without 7-branes. The topological action that we start with reads

$$\frac{S_{11}}{2\pi}=\int_{N_4\times L_7}\frac{1}{2}\breve{F}_6\star\breve{F}_6-\breve{F}_6\star\breve{H}_3\star\breve{G}_3, \tag{2.2}$$

which lives in 11-dimensional (11D) spacetime $N_4\times L_7$. The 4-manifold N_4 satisfies $\partial N_4=M_3$, an auxiliary bulk manifold whose boundary is the 3-manifold where the symmetry TFT lives. Note that all terms are 12 dimensional since we have uplifted IIB fluxes as differential cohomology elements. $\breve{F}_6$ is the differential cohomology element whose connection part is the IIB self-dual D3-brane 5-form flux F_5. $\breve{H}_3$ and $\breve{G}_3$ are differential cohomology uplift for F1- and D1-string flux dB_2 and dC_2.[8]

According to (2.1), we expand differential cohomology elements as

[7] The topological action of IIB string theory for symmetry TFT computation has been investigated in, e.g., Refs. [8,19,56,57,69,70]. We also refer the reader to the recent work [15] for a more systematic discussion.

[8] The $\star$ symbol defines a bilinear product operation on Cheeger-Simons characters $\breve{H}^{k_1}(M_d)\times\breve{H}^{k_2}(M_d)=\breve{H}^{k_1+k_2}(M_d)$ [71,72]. In particular, when $k_1+k_2=d+1$, the integral describes a perfect pairing $\breve{H}^{k_1}(M_d)\times\breve{H}^{d+1-k_1}(M_d)\to\mathbb{R}/\mathbb{Z}$. We refer the reader to Ref. [73] for a nice review of differential cohomology.

$$\breve{F}_6 = \breve{f}_6\star\breve{1} + \sum_{\alpha=1}^{2}\breve{F}_4^{(\alpha)}\star\breve{u}_{2(\alpha)} + \breve{F}_2\star\breve{u}_4 + \sum_{\alpha=1}^{2}\breve{F}_{1(\alpha)}\star\breve{u}_5^{(\alpha)} + \breve{A}_4\star\breve{t}_2 + \sum_{i=1}^{2}\breve{A}_2^{(i)}\star\breve{t}_{4(i)},$$

$$\breve{G}_3 = N\breve{\mathrm{vol}}_{M_3}\star\breve{1} + \sum_{\alpha=1}^{2}\breve{G}_1^{(\alpha)}\star\breve{u}_{2(\alpha)} + \breve{C}_1\star\breve{t}_2,$$

$$\breve{H}_3 = \breve{h}_3\star\breve{1} + \sum_{\alpha=1}^{2}\breve{H}_1^{(\alpha)}\star\breve{u}_{2(\alpha)} + \breve{B}_1\star\breve{t}_2, \tag{2.3}$$

where the generators for various cohomology classes are denoted as

$$\breve{1} \leftrightarrow H^0(L_7,\mathbb{Z}) = \mathbb{Z},$$
$$\breve{u}_{2(\alpha)}, \alpha = 1,2 \leftrightarrow \text{nontorsional}\, H^2(L_7,\mathbb{Z}) = \mathbb{Z}^2,$$
$$\breve{u}_4 \leftrightarrow \text{nontorsional}\, H^4(L_7,\mathbb{Z}) = \mathbb{Z},$$
$$\breve{u}_5^{(\alpha)}, \alpha = 1,2 \leftrightarrow H^5(L_7,\mathbb{Z}) = \mathbb{Z}^2,$$
$$\breve{\mathrm{vol}} \leftrightarrow H^7(L_7,\mathbb{Z}) = \mathbb{Z},$$
$$\breve{t}_2 \leftrightarrow \text{torsional}\, H^2(L_7,\mathbb{Z}) = \mathbb{Z}_2,$$
$$\breve{t}_{4(i)}, i = 1,2 \leftrightarrow \text{torsional}\, H^4(L_7,\mathbb{Z}) = \mathbb{Z}_2\oplus\mathbb{Z}_2,$$
$$\breve{t}_6 \leftrightarrow H^6(L_7,\mathbb{Z}) = \mathbb{Z}_2. \tag{2.4}$$

Fields from torsional parts give rise to finite symmetries, while those from nontorsional parts correspond to continuous symmetries. In this work, we only focus on finite symmetries and their descendent noninvertible symmetries,[9] so we only turn on the fields as coefficients of the torsional generators t_p, where $p = 2,4,6$.

Substituting the torsional part of (2.3) into the 11D topological action (2.2), we derive the 3D symmetry TFT for finite symmetries,

$$\frac{S_3}{2\pi} = \int_{N_4}\sum_{i,j=1}^{2}\Lambda_{ij}\breve{A}_2^{(i)}\star\breve{A}_2^{(j)} - \sum_i \Delta_i\breve{A}_2^{(i)}\star\breve{B}_1\star\breve{C}_1 = \int_{M_3}\sum_{i,j}\Lambda_{ij}a_1^{(i)}\delta a_1^{(j)} - \sum_{i=1}^{2}\Delta_i a_1^{(i)}b_1c_1, \tag{2.5}$$

where we use respective lower-case letters to express fields in terms of the ordinary cohomology elements and omit the "∪" product symbol for simplicity. The coefficients in the action are given by the linking numbers within the 7-manifold L_7,

$$\Lambda_{ij} \equiv \frac{1}{2}\int_{L_7}\breve{t}_{4(i)}\star\breve{t}_{4(j)} \mod 1,$$
$$\Delta_i \equiv \int_{L_7}\breve{t}_{4(i)}\star\breve{t}_2\star\breve{t}_2 \mod 1, \tag{2.6}$$

whose derivation requires expressing p-dimensional torsional generators $\breve{t}_p$ in terms of various compact $(8-p)$-cycles in the toric Calabi-Yau 4-fold [58]. The linking number computation then translates into reading quadruple intersection numbers between codimension-2 divisors in the toric varieties.[10]

It is easy to see the action (2.5) is not complete. Notice that the quadratic term comes from the noncommutativity for the boundary profile of the self-dual 5-form; thus, other noncommutative fluxes should also be captured in the resulting 3D TFT [8]. This leads to adding quadratic terms for b_1 and c_1 from the noncommutativity F1-NS5 and D1-D5 pairs when wrapping torsional cycles linking to each other.[11] The resulting TFT reads

$$\frac{S_3}{2\pi} = \int_{M_3}\sum_{i,j}\Lambda_{ij}a_1^{(i)}\delta a_1^{(j)} + \Omega(-c_1\delta\hat{c}_1 + b_1\delta\hat{b}_1) - \sum_{i=1}^{2}\Delta_i a_1^{(i)}b_1c_1, \tag{2.7}$$

where $\hat{c}_1$ and $\hat{b}_1$ are from IIB fluxes $\hat{G}_7$ and $\hat{H}_7$ (10D Hodge-dual of G_3 and H_7) via reduction on torsional 5-cycles γ_5 associated to the generator $\breve{t}_6$.[12] Ω denotes the linking number between the $\breve{t}_2$ and $\breve{t}_6$ generators:

$$\Omega \equiv \int_{L_7}\breve{t}_2\star\breve{t}_6 \mod 1. \tag{2.8}$$

Computing the linking number (2.6), (2.8) and redefining the notation as

$$a_1^{(1)} \to a_1, \qquad a_1^{(2)} \to \hat{a}_1, \tag{2.9}$$

we end up with an elegant result

$$S_3 = \frac{2\pi}{2}\int_{M_3} a_1\delta\hat{a}_1 + b_1\delta\hat{b}_1 + c_1\delta\hat{c}_1 + a_1b_1c_1, \tag{2.10}$$

[9]For brane interpretation of continuous symmetry operators, we refer the reader to Ref. [18].

[10]See Chap. 7 in Ref. [74] for how to compute intersection numbers in toric varieties.

[11]We thank Inaki Garcia Etxebarria for valuable discussions on this point.

[12]We would like to stress that differential cohomology and flux noncommutativity is not the only way to read the quadratic terms in the symmetry TFT. In fact, it is also possible to derive these terms directly from the supergravity kinetic terms. See Refs. [60,75] and Appendix B in Ref. [56] for more details.

which is just a 3D $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ Dijkgraaf-Witten theory with a simple twist $a_1 b_1 c_1$.

Each TFT field serves as the background gauge field for a factor within the defect group [7,19], which now can be straightforwardly read as

$$\mathbb{D} = (\mathbb{Z}_2^a \times \mathbb{Z}_2^{\hat{a}}) \oplus (\mathbb{Z}_2^b \times \mathbb{Z}_2^{\hat{b}}) \oplus (\mathbb{Z}_2^c \times \mathbb{Z}_2^{\hat{c}}). \quad (2.11)$$

We use $\times$ to denote the group factors with nontrivial Dirac pairing between their defects or, equivalently, those background gauge fields canonical conjugate to each other under the TFT quantization. $\oplus$, on the contrary, means group factors without any noncommutativity between their fluxes.

B. Line operators from brane world volume actions

Having derived a 3D Dijkgraaf-Witten theory as the symmetry TFT for finite symmetries in the 2D QFT, the next natural question is as follows: what is the spectrum of line operators in this 3D TFT, and how do these operators translate in (noninvertible) topological defect lines in the 2D QFT? Field theoretically, this question has been intensively investigated in, e.g., Ref. [64]. In this section, we will provide a top-down treatment where topological defect lines, no matter whether invertible or not, enjoy elegant origins as branes in the IIB string theory.

The first step is to determine the candidate of branes that are responsible for line operators in the Dijkgraaf-Witten theory (2.10). Recall finite gauge fields in (2.10) are reduced from various IIB fluxes, each of which couples to a certain type of branes. For example, $\hat{c}_1$ is the expansion field from the $\breve{t}_6$ reduction of $\breve{\hat{G}}_7$ (as a differential cohomology element) on $\mathrm{Tor}H^6(L_7, \mathbb{Z})$, which corresponds to the IIB flux $\hat{G}_7$ (electrically) coupled to D5-branes. More precisely, according to the universal coefficient theorem $\mathrm{Tor}H_n = \mathrm{Tor}H^{n+1}$, we have the correspondence between torsional cohomology generators and torsional cycles

$$\gamma_n^{(i)} \leftrightarrow t_{n+1(i)}. \quad (2.12)$$

This translates in the $\hat{c}$ case as

$$\hat{c} \leftrightarrow \text{D5s on } \gamma_5, \quad (2.13)$$

where $\gamma_5 \in \mathrm{Tor}H_5(L_7, \mathbb{Z})$ is the torsional 5-cycle dual to the $\breve{t}_6$ generator. Similarly, one can derive brane patterns associated with each TFT field. The dimensional reduction of the D5-brane topological coupling then gives the corresponding naive magnetic line operator in the 3D TFT:

$$\exp\left(2\pi i \int_{M_6} C_6\right) \to \exp\left(2\pi i \int_{M_1\times\gamma_5} \breve{\hat{G}}_7\right) = \exp\left(\pi i \int_{M_1} \hat{c}_1\right). \quad (2.14)$$

However, this invertible line is not the full construction of the magnetic operator dependent on $\hat{c}$ because it is not gauge invariant within our interested Dijkgraaf-Witten theory (see also, e.g., Ref. [64]). Note that C_6 does not carry the full topological information of the D5-brane but only the leading term of the Wess-Zumino part of the D5-brane action. To encode the full topological effect of the D5-brane on the 3D Dijkgraaf-Witten theory (or equivalently, on the 2D QFT on the physical boundary of the 3D bulk), we consider the following action:

$$S_{\mathrm{D5}}^{\mathrm{top}} = \int \mathcal{D}\hat{f}_4 \mathcal{D}f_2 \exp\left(2\pi i \int_{N_2\times\gamma_5} \hat{f}_4 df_2 + \hat{G}_7 - F_5(B_2 - f_2) - \frac{1}{2} G_3 (B_2 - f_2)^2\right). \quad (2.15)$$

This is a topological action on an auxiliary 7D bulk $N_2 \times \gamma_5$, where N_2 satisfies $\partial N_2 = M_1$, i.e., an auxiliary 2-manifold whose boundary is the topological line supporting the operator in the resulting 3D TFT. In this topological action, f_2 is the field strength of the dynamical gauge field from the F1 open string fluctuation, and $\hat{f}_4$ is its Hodge dual on the D5-brane world volume. The first term thus carries the relevant information from the Dirac-Born-Infeld part of the D-brane action [15]. The other three terms come from the Wess-Zumino part of the brane action [76], where the leading term $\hat{G}_7$ is the origin for the naive magnetic operator which we discussed in (2.14). F_5 and G_3 are fluxes for the induced lower-dimensional D3- and D1-brane charges, respectively, while B_2 is the regular notation for the NS-NS field electrically coupled to F1-strings. Note that the path integral is only performed over f_2 and $\hat{f}_4$, which are dynamical degrees of freedon on the D5-brane world volume.

To perform the dimensional reduction on the torsional cycle γ_5, as what we did in computing the symmetry TFT, we promote the topological action in terms of the differential cohomology elements[13]

$$S_{\mathrm{D5}}^{\mathrm{top}} \to \int \mathcal{D}\breve{\hat{f}}_5 \mathcal{D}\breve{f}_3 \exp\left(2\pi i \int_{N_2\times\gamma_5} \breve{\hat{f}}_5 \star \breve{f}_3 + \breve{\hat{G}}_8 - \breve{F}_5 \star (\breve{H}_3 - \breve{f}_3) - \frac{1}{2} \breve{G}_3 \star (\breve{H}_3 - \breve{f}_3)^2\right). \quad (2.16)$$

The expansions of $\breve{F}_5$ and $\breve{H}_3$ are already given in (2.3),[14] while the expansion for $\breve{\hat{G}}_8, \breve{\hat{f}}_5$, and $\breve{f}_3$ can be defined as

[13] The $(\breve{H}_3 - \breve{f}_3)^2$ here means an order-5 differential cohomology element from the star product between the differential cohomology element $\breve{H}_3 - \breve{f}_3$ and its connection part.

[14] Note that in (2.3), the $\breve{F}_6$ is the differential cohomology element via F_5 is its connection part, but here $\breve{F}_5$ is the differential cohomology element itself, regarded as gauge-invariant field strength.

$$\check{\hat{f}}_5 = \hat{\phi}_1 \star \check{t}_{4(1)} + \check{\phi}'_1 \star \check{t}_{4(2)} + \cdots,$$
$$\check{f}_3 = \check{\phi}_1 \star \check{t}_2 + \cdots, \tag{2.17}$$

where we only write down terms relevant to the reduction on the torsional cycle γ_5. Again, using the linking number between various cohomology generators, the resulting one-dimensional TFT reads

$$S_{\rm D5}^{\gamma_5} \propto \int \mathcal{D}\hat{\phi}_0 \mathcal{D}\phi_0 \exp\left(\pi i \int_{M_1} \hat{c}_1\right) \times \exp\left(\pi i \int_{M_1} \hat{\phi}_0 \delta\phi_0 + \phi_0 a_1 - \phi_0 b_0 c_1 + \frac{1}{2}\phi_0^2 c_1\right), \tag{2.18}$$

where $db_0 = b_1$, and we have omitted all other terms decoupled from the dynamical $\hat{\phi}_0$ and ϕ_0. Taking variation of ϕ_0, we get the condition

$$\delta\hat{\phi}_0 = a_1 + \phi_0 c_1 - b_0 c_1, \tag{2.19}$$

substituting which back to (2.18), we integral over c_1 and end up with the topological line operator

$$S_{\rm D5}^{\gamma_5} \propto \mathcal{N}_{\rm D5}(M_1) \equiv \int \mathcal{D}\hat{\phi}_0 \mathcal{D}\phi_0 \exp\left(\pi i \int_{M_1} \hat{c}_1\right) \times \exp\left(\pi i \int_{M_1} \hat{\phi}_0 \delta\phi_0 + \phi_0 a_1 - \hat{\phi}_0 b_1\right). \tag{2.20}$$

This is a noninvertible gauge-invariant magnetic line operator for the 3D Dijkgraaf-Witten theory, matching the result in Ref. [64].

The fusion rule for this line operator is

$$\mathcal{N}_{\rm D5} \times N_{\rm D5} = \left[1 + \exp\left(\pi i \int_{M_1} a_1\right)\right]\left[1 + \exp\left(\pi i \int_{M_1} b_1\right)\right]. \tag{2.21}$$

Further note that $e^{\pi i \int_{M_1} a_1} \equiv \eta_a$ and $e^{\pi i \int_{M_1} b_1} \equiv \eta_b$ are $\mathbb{Z}_2$ topological lines, where we use η to denote invertible lines with subindices showing the TFT field dependence. The right-hand side of the above equation is the condensation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ topological lines on the M_1, which can be regarded as a result of higher gauging [77]. Now, we can write down the full fusion rule involving noninvertible line $\mathcal{N}_{D5}$ and invertible $\mathbb{Z}_2$ lines η_a and η_b as

$$\begin{aligned} \mathcal{N}_{\rm D5} \times \mathcal{N}_{\rm D5} &= 1 + \eta_a + \eta_b + \eta_a \eta_b, \\ \eta_a \times \eta_a &= \eta_b \times \eta_b = 1, \\ \eta_a \times \mathcal{N}_{\rm D5} &= \eta_b \times \mathcal{N}_{\rm D5} = \mathcal{N}_{\rm D5}, \end{aligned} \tag{2.22}$$

which is exactly the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami category [65].

Similarly, we can derive other topological line operators respectively dependent on $a_1, \hat{a}_1, b_1, \hat{b}_1$, and c_1 as we did for $\hat{c}$ and its corresponding D5-brane action. We leave the computation to the interested reader as an exercise and conclude the results in Table I.

It is easy to see the electric $\mathbb{Z}_2$ lines in (2.22) are identified with the brane origin $\eta_a = \eta_{\rm D3}$, $\eta_b = \eta_{\rm F1}$. Furthermore, the noninvertible lines from D3- and NS5-branes also obey the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami fusion category, respectively:

$$\begin{aligned} \mathcal{N}_{\rm D3} \times \mathcal{N}_{\rm D3} &= 1 + \eta_{\rm F1} + \eta_{\rm D1} + \eta_{\rm F1}\eta_{\rm D1}, \\ \eta_{\rm F1} \times \eta_{\rm F1} &= \eta_{\rm D1} \times \eta_{\rm D1} = 1, \\ \eta_{\rm F1} \times \mathcal{N}_{\rm D3} &= \eta_{\rm D1} \times \mathcal{N}_{\rm D3} = \mathcal{N}_{\rm D3}, \end{aligned} \tag{2.23}$$

TABLE I. Line operators in 3D Dijkgraaf-Witten theory (2.10) and their brane origins. The first three brane configurations give rise to invertible electric lines, while the last three wrapped branes correspond to noninvertible magnetic lines.

Line operators in 3D TFT	Branes configuration
$\eta_{\rm D3} = e^{\pi i \int_{M_1} a_1}$	D3-brane on $\gamma_3^{(1)}$
$\eta_{\rm F1} = e^{\pi i \int_{M_1} b_1}$	F1-string on γ_1
$\eta_{\rm D1} = e^{\pi i \int_{M_1} c_1}$	D1-string on γ_1
$\mathcal{N}_{\rm D3} = \int \mathcal{D}\hat{\phi}_0 \mathcal{D}\phi_0 e^{\pi i \int_{M_1} \hat{a}_1} e^{\pi i \int_{M_1} \hat{\phi}_0 \delta\phi_0 + \phi_0 b_1 - \hat{\phi}_0 c_1}$	D3-brane wrapping $\gamma_3^{(2)}$
$\mathcal{N}_{\rm NS5} = \int \mathcal{D}\hat{\phi}_0 \mathcal{D}\phi_0 e^{\pi i \int_{M_1} \hat{b}_1} e^{\pi i \int_{M_1} \hat{\phi}_0 \delta\phi_0 + \phi_0 c_1 - \hat{\phi}_0 a_1}$	NS5-brane wrapping γ_5
$\mathcal{N}_{\rm D5} = \int \mathcal{D}\hat{\phi}_0 \mathcal{D}\phi_0 e^{\pi i \int_{M_1} \hat{c}_1} e^{\pi i \int_{M_1} \hat{\phi}_0 \delta\phi_0 + \phi_0 a_1 - \hat{\phi}_0 b_1}$	D5-brane wrapping γ_5

and

$$\begin{aligned}\mathcal{N}_{\mathrm{NS5}}\times\mathcal{N}_{\mathrm{NS5}}&=1+\eta_{\mathrm{F1}}+\eta_{\mathrm{D3}}+\eta_{\mathrm{F1}}\eta_{\mathrm{D3}},\\ \eta_{\mathrm{F1}}\times\eta_{\mathrm{F1}}&=\eta_{\mathrm{D3}}\times\eta_{\mathrm{D3}}=1,\\ \eta_{\mathrm{F1}}\times\mathcal{N}_{\mathrm{NS5}}&=\eta_{\mathrm{D3}}\times\mathcal{N}_{\mathrm{NS5}}=\mathcal{N}_{\mathrm{NS5}}.\end{aligned}\tag{2.24}$$

III. BRANES BEHIND POLARIZATIONS AND NONINVERTIBLE SYMMETRIES IN TWO DIMENSIONS

The 3D symmetry TFT bulk itself does not fully specify the global symmetry structure of the 2D QFT. At this stage, the 2D QFT associated with the conical singularity probed by D1-branes is a relative QFT [19,52,62]. It does not have a well-defined scalar-valued partition function but carries a partition vector. The corresponding space for the partition vector is regarded as the Hilbert space $\mathcal{H}$ from the 3D TFT quantization (see, e.g., Refs. [21,47,52]). Therefore, in this sense, the 2D QFT is "relative" to the 3D bulk theory.

To get rid of the "relativeness" upon the 3D bulk and thus obtain a well-defined QFT with a scalar-valued partition function, we need to pick a polarization for the system. From the 3D TFT perspective, this translates to introducing a purely gapped boundary, on which we impose a topological boundary condition. Such a boundary condition can be equivalently presented as a Lagrangian subgroup $L\subset\mathbb{D}$ of the defect group $\mathbb{D}$.[15] With respect to the partition vector space under the 3D TFT quantization, the relative QFT and the gapped boundary condition can be expressed as two boundary states.

Colliding the gapped boundary with the relative QFT boundary, one obtains a genuine 2D system, known as an absolute QFT, that enjoys a scalar-valued partition function. This process can be nicely expressed in terms of the inner product between boundary states $|\mathcal{R}\rangle$ and $|\mathcal{P};B\rangle$ in the partition vector space $\mathcal{H}$:

$$Z_{\mathcal{P}}[B]=\langle\mathcal{R}|\mathcal{P};B\rangle.\tag{3.1}$$

In this expression, $\langle\mathcal{R}|$ denotes the relative QFT (dual) partition vector, $|\mathcal{P};B\rangle$ denotes the boundary state for polarization $\mathcal{P}$ with the flux profile B, and $Z_{\mathcal{P}}[B]$ gives rise to the well-defined partition function with the presence of the background B.[16]

[15] Mathematically, the partition vector space of a relative QFT is captured by the Heisenberg group $\underline{H}^1(M_2,\mathbb{D})$ with coefficients in the defect group $\mathbb{D}$. Picking a polarization corresponds to picking a maximally isotropic subspace of the Heisenberg group. We refer the interested reader to Ref. [19] for a detailed discussion.

[16] We remark that picking polarizations is not always possible for a generic relative QFT. Well-known examples of this type include many 2D chiral CFTs and 6D SCFTs. See, e.g., Refs. [19,52] for more details.

A. "Standard" polarization with only invertible symmetries

Come back to the 3D Dijkgraaf-Witten theory (2.10) and its relative 2D QFT associated with $Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$. The simplest boundary condition one can consider is

$$a_1,b_1,c_1\ \text{Dirichlet};\ \hat{a}_1,\hat{b}_1,\hat{c}_1\ \text{Neumann}.\tag{3.2}$$

This corresponds to the polarization which picks the Lagrangian subgroup L of the defect group (2.11)

$$L=\mathbb{Z}_2^{\hat{a}}\times\mathbb{Z}_2^{\hat{b}}\times\mathbb{Z}_2^{\hat{c}}.\tag{3.3}$$

Therefore, the resulting absolute 2D theory has a $(\mathbb{Z}_2)^3$ global symmetry

$$G=\mathbb{Z}_2^{a}\times\mathbb{Z}_2^{b}\times\mathbb{Z}_2^{c}.\tag{3.4}$$

Based on their behavior under the gapped boundary condition (3.2), line operators in the 3D TFT shown in Table I induce to various local charged operators and topological defect lines in the 2D absolute theory.

For instance, due to the Dirichlet condition of a_1, η_{D3} will terminate on the gapped boundary; i.e., it does not continue to fluctuate along the boundary and thus becomes a local operator after shrinking the 3D TFT bulk. On the contrary, $\mathcal{N}_{\mathrm{D3}}$ is not fully trivialized on the gapped boundary due to the Neumann condition of $\hat{a}_1$. Its line manifold continues along the gapped boundary and thus gives rise to a topological defect line. However, it loses its noninvertible property during this process. To see this, notice that b_1 and c_1 are trivialized on the gapped boundary, leading to

$$\mathcal{N}_{\mathrm{D3}}\to e^{\pi i\int_{M_1}\hat{a}_1}\int\mathcal{D}\hat{\phi}_0\mathcal{D}\phi_0 e^{\pi i\int_{M_1}\hat{\phi}_0\delta\phi_0}\propto e^{\pi i\int_{M_1}\hat{a}_1},\tag{3.5}$$

where the path integral over ϕ_0 and $\hat{\phi}_0$ is now totally decoupled as a overall factor. Therefore, $\mathcal{N}_{\mathrm{D3}}$, under this polarization/absolute QFT, becomes an invertible $\mathbb{Z}_2$ line.

What is the brane configuration behind all this? Recall that the brane origins of η_{D3} and $\mathcal{N}_{\mathrm{D3}}$ are D3-branes wrapping $\gamma_3^{(1)}$ and $\gamma_3^{(2)}$, respectively. The nontrivial linking between these two torsional 3-cycles is responsible for the canonical conjugation between a_1 and $\hat{a}_1$ in the 3D Dijkgraaf-Witten theory, thus telling us the local operator reduced from η_{D3} is charged under the $\mathbb{Z}_2^a\subset L^\vee$ symmetry generated by the invertible line reduced from $\mathcal{N}_{\mathrm{D3}}$. In the 10D IIB string theory picture, the gapped boundary for the 3D TFT is translated into the topological boundary conditions on the asymptotic boundary $Y^{2,0}(\mathbb{P}^1\times\mathbb{P}^1)$ at infinity. Therefore, we have the following correspondence between the brane pattern behind operators under (3.4) and the polarization (3.2):

$$\begin{aligned} &\text{local operator from } \eta_{\mathrm{D3}}: \ \text{D3-branes wrapping cone } (\gamma_3^{(1)}), \\ &\text{invertible } \mathbb{Z}_2^a \text{ line from } \mathcal{N}_{\mathrm{D3}}: \ \text{D3-branes wrapping } \gamma_3^{(2)} \text{ at infinity.} \end{aligned} \tag{3.6}$$

This type of brane pattern falls in the general idea of branes at infinity as generalized symmetry operators introduced in Refs. [11–13]. The charged local operators and topological defect lines for the global symmetry $\mathbb{Z}_2^b$ and $\mathbb{Z}_2^c$ can be read similarly:

$$\begin{aligned} &\text{local operator from } \eta_{\mathrm{F1}}: \ \text{F1-strings wrapping cone } (\gamma_2), \\ &\text{invertible } \mathbb{Z}_2^b \text{ line from } \mathcal{N}_{\mathrm{NS5}}: \ \text{NS5-branes wrapping } \gamma_5 \text{ at infinity,} \\ &\text{local operator from } \eta_{\mathrm{D1}}: \ \text{D1-strings wrapping cone } (\gamma_2), \\ &\text{invertible } \mathbb{Z}_2^c \text{ line from } \mathcal{N}_{\mathrm{D5}}: \ \text{D5-branes wrapping } \gamma_5 \text{ at infinity.} \end{aligned} \tag{3.7}$$

B. Polarizations with noninvertible symmetries

In the standard polarization $L = \mathbb{Z}_2^{\hat{a}} \oplus \mathbb{Z}_2^{\hat{b}} \oplus \mathbb{Z}_2^{\hat{c}}$, there is a mixed anomaly for the global symmetry $G = \mathbb{Z}_2^a \times \mathbb{Z}_2^b \times \mathbb{Z}_2^c$, inherited from the Dijkgraaf-Witten twist in the 3D symmetry TFT:

$$\pi \int_{M_3} a_1 b_1 c_1. \tag{3.8}$$

According to Ref. [78] (see also Ref. [64]), gauging two of the three $\mathbb{Z}_2$ symmetries, the leftover one will be promoted to a noninvertible symmetry. Let us take gauging $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ as an example. From the symmetry TFT perspective, this gauging process translates into changing the original Dirichlet boundary condition for a_1 and b_1 fields to Neumann boundary conditions. Their canonical conjugates $\hat{a}_1$ and $\hat{b}_1$ then pick Dirichlet boundary conditions accordingly. The resulting gapped boundary condition reads

$$\hat{a}_1, \hat{b}_1, c_1 \text{ Dirichlet}; \ a_1, b_1, \hat{c}_1 \text{ Neumann}, \tag{3.9}$$

which picks a new polarization associated with the Lagrangian subgroup

$$L = \mathbb{Z}_2^a \times \mathbb{Z}_2^b \times \mathbb{Z}_2^{\hat{c}}. \tag{3.10}$$

As we did in the "standard" polarization case, we can investigate the fate of various line operators in Table I under the gapped condition (3.9) to investigate their roles in the resulting 2D absolute QFT. It is easy to see now

$$\eta_{\mathrm{D1}}, \mathcal{N}_{\mathrm{D3}}, \mathcal{N}_{\mathrm{NS5}} \tag{3.11}$$

are terminating on the gapped boundary, thus corresponding to local operators in the 2D QFT, while

$$\eta_{\mathrm{D3}}, \eta_{\mathrm{F1}}, \mathcal{N}_{\mathrm{D5}} \tag{3.12}$$

can continue along the gapped boundary, thus corresponding to the topological defect line. Furthermore, the Neumann boundary condition for a_1, b_1, and $\hat{c}_1$ preserves the noninvertible property for the $\mathcal{N}_{\mathrm{D5}}$ line, so it still reads

TABLE II. A standard polarization with the $(\mathbb{Z}_2)^3$ invertible symmetry, as well as three polarizations with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami (TY) categorical symmetry. The concrete torsional cycles wrapped by branes for various η and $\mathcal{N}$ operators can be found in Table I. The charge operators are built by branes terminating at the asymptotic boundary $Y^{2,0}(\mathbb{P}^1 \times \mathbb{P}^1)$, while the symmetry lines are built by branes at infinity along the asymptotic boundary.

Polarization L	Global symmetry	Charged operators	Symmetry lines
$\mathbb{Z}_2^{\hat{a}} \times \mathbb{Z}_2^{\hat{b}} \times \mathbb{Z}_2^{\hat{c}}$	$\mathbb{Z}_2^a \times \mathbb{Z}_2^b \times \mathbb{Z}_2^c$	$\eta_{\mathrm{D3}}, \eta_{\mathrm{F1}}, \eta_{\mathrm{D1}}$	$\mathcal{N}_{\mathrm{D3}}, \mathcal{N}_{\mathrm{NS5}}, \mathcal{N}_{\mathrm{D5}}$
$\mathbb{Z}_2^a \times \mathbb{Z}_2^b \times \mathbb{Z}_2^{\hat{c}}$	$\mathbb{Z}_2^{\hat{a}} \times \mathbb{Z}_2^{\hat{b}}$ TY category	$\mathcal{N}_{\mathrm{D3}}, \mathcal{N}_{\mathrm{NS5}}, \eta_{\mathrm{D1}}$	$\eta_{\mathrm{D3}}, \eta_{\mathrm{F1}}, \mathcal{N}_{\mathrm{D5}}$
$\mathbb{Z}_2^a \times \mathbb{Z}_2^{\hat{b}} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2^{\hat{a}} \times \mathbb{Z}_2^{\hat{c}}$ TY category	$\mathcal{N}_{\mathrm{D3}}, \eta_{\mathrm{F1}}, \mathcal{N}_{\mathrm{D5}}$	$\eta_{\mathrm{D3}}, \eta_{\mathrm{D1}}, \mathcal{N}_{\mathrm{NS5}}$
$\mathbb{Z}_2^{\hat{a}} \times \mathbb{Z}_2^b \times \mathbb{Z}_2^c$	$\mathbb{Z}_2^{\hat{b}} \times \mathbb{Z}_2^{\hat{c}}$ TY category	$\eta_{\mathrm{D3}}, \mathcal{N}_{\mathrm{NS5}}, \mathcal{N}_{\mathrm{D5}}$	$\eta_{\mathrm{F1}}, \eta_{\mathrm{D1}}, \mathcal{N}_{\mathrm{D3}}$

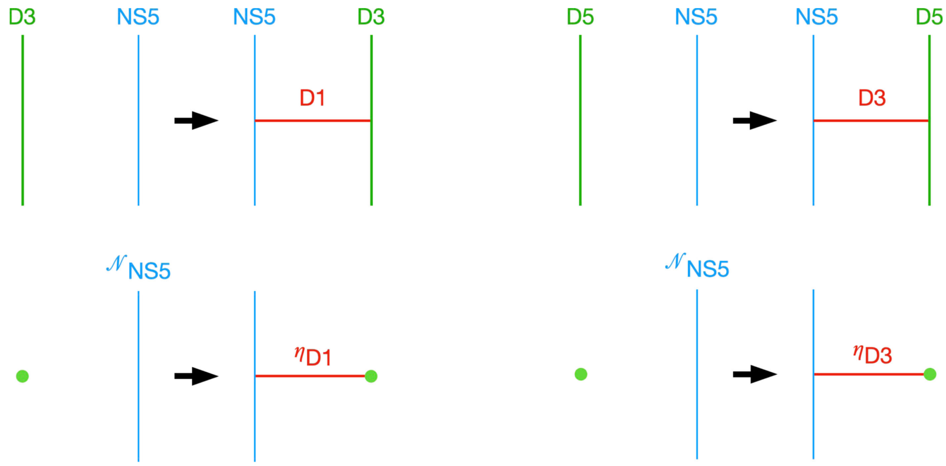

FIG. 2. Bottom: action of noninvertible defect line $\mathcal{N}_{\mathrm{NS5}}$ (in the polarization $L = \mathbb{Z}_2^a \times \mathbb{Z}_2^{\hat{b}} \times \mathbb{Z}_2^c$) on the local operators charged under the invertible symmetry $\mathbb{Z}_2^{\hat{a}}$ and $\mathbb{Z}_2^{\hat{c}}$. The corresponding topological defect lines are generated under this action. Top: this nontrivial action enjoys a string theory origin as the Hanany-Witten transition, where the created branes wrapping cycles at infinity perfectly serve as the topological defect lines.

$$\mathcal{N}_{\mathrm{D5}} = \int \mathcal{D}\hat{\phi}_0 \mathcal{D}\phi_0 e^{\pi i \int_{M_1} \hat{c}_1} e^{\pi i \int_{M_1} \hat{\phi}_0 \delta\phi_0 + \phi_0 a_1 - \hat{\phi}_0 b_1}. \qquad (3.13)$$

Therefore, based on its fusion rule (2.22), we conclude the global symmetry for the polarization (3.10) is

$$G = \mathbb{Z}_2^{\hat{a}} \times \mathbb{Z}_2^{\hat{b}} \text{ Tambara-Yamagami categorical symmetry.} \qquad (3.14)$$

The brane pattern for this global symmetry can be built by wrapping branes in (3.11) terminating at infinity as charged operators while wrapping branes in (3.12) at infinity as topological defect lines.

Field theoretically, one would follow the step in Ref. [78] to compute what would be a noninvertible TFT promote of the invertible $\mathbb{Z}_2^c$ line after gauging with the presence of the mixed anomaly (3.8). The result will perfectly coincide with (3.13).[17] The punchline of our top-down approach is that the noninvertible line directly comes from the D5-brane world volume action, as we computed in Sec. II. Its (non)invertible property in absolute QFTs before/after gauging simply results from changing polarizations, which translates into different brane patterns at infinity.

We remark that the noninvertible symmetry in this context is known as the nonintrinsic one [66]. This is because it is related to an invertible symmetry via changing polarizations.[18]

We conclude this subsection by presenting three polarizations that enjoy noninvertible symmetries and their comparison with the standard polarization in Table II.

C. Action of noninvertible lines and the Hanany-Witten transition

One salient property of the noninvertible symmetry defect is its action on the charged operator (see, e.g., Refs. [26,79]). Consider the polarization (3.10) with the noninvertible line $\mathcal{N}_{\mathrm{D5}}$. Moving this line past the local operator charged under $\mathbb{Z}_2^{\hat{a}}$ (respectively, $\mathbb{Z}_2^{\hat{b}}$) will make the charged operator nongenuine and attach to the topological $\mathbb{Z}_2^{\hat{a}}$ (respectively, $\mathbb{Z}_2^{\hat{b}}$) topological line. Namely, it belongs to the defect Hilbert space of the line it attached [24].

This nontrivial action enjoys an elegant string theory origin as the Hanany-Witten transition [80]. Note that the charged operator under $\mathbb{Z}_2^{\hat{a}}$ (respectively, $\mathbb{Z}_2^{\hat{b}}$) comes from the D3-brane wrapping on cone($\gamma_3^{(2)}$) [respectively, NS5-brane wrapping on cone(γ_5)]. When the D5-brane generating the noninvertible $\mathcal{N}_{\mathrm{D5}}$ passes through the above D3-brane (respectively, NS5-brane), a F1 string wrapping γ_1 (respectively, D3-brane wrapping $\gamma_3^{(1)}$) is generated connecting them. What object is generated by the F1-string wrapping γ_1 (respectively, D3-brane wrapping $\gamma_3^{(1)}$)? It is exactly the topological defect line η_{F1} (respectively, η_{D3}) for the $\mathbb{Z}_2^{\hat{a}}$ (respectively, $\mathbb{Z}_2^{\hat{b}}$) symmetry (see Tables I and II). See Fig. 2 for an illustration of how the Hanany-Witten transition translates into nontrivial transitions of charged operators in the polarization, e.g., $L = \mathbb{Z}_2^a \times \mathbb{Z}_2^{\hat{b}} \times \mathbb{Z}_2^c$.

A similar Hanany-Witten transition origin for the nontrivial action of noninvertible defect has been observed in four-dimensional QFTs [11,14]. Still, to our knowledge,

[17]We thank Ho Tat Lam for valuable discussions on this point.
[18]For discussion on intrinsic vs nonintrinsic noninvertible symmetries from higher-dimensional perspective, we refer the reader to Refs. [14,67].

in the context of 2D QFTs, this is the first time the Hanany-Witten interpretation of this action appears in the literature. It is, therefore, natural to conjecture that for noninvertible symmetries with brane origins[19] this correspondence holds for diverse dimensions.[20]

IV. CONCLUSIONS

Our work suggests various natural directions for future investigation. Some obvious and interesting directions include the following:

(i) As we pointed out at the beginning of this paper, there is so far an infinite family of 2D gauge theories arising from D1-branes probing toric Calabi-Yau 4-folds. Investigating their symmetry TFTs, noninvertible, and other generalized symmetries would be interesting. A natural expectation is that this infinite class of 2D gauge theories enjoy noninvertible symmetries. The argument is as follows. Briefly, the IIB Chern-Simons term in general provides mixed anomaly terms in the 3D SymTFT under dimensional reduction on the Sasaki-Einstein 7-manifold. This mixed anomaly term encodes three finite symmetries, generated by D3-, D5- and NS5-branes. Gauging the symmetries of D5- and NS5-branes, D3-branes then generate noninvertible symmetries. We will explore this subject in Ref. [43].

(ii) For a given Calabi-Yau 4-fold, there can be multiple 2D gauge theories associated to it, which are connected by $\mathcal{N} = (0, 2)$ triality [41]. It is natural to ask how the noninvertible symmetries found in this work interplay with the triality. As happens for ordinary global symmetries, we expect the dual QFTs connected by triality to enjoy the same noninvertible global symmetries. This can be understood since they share the same asymptotic boundary geometry in the string theory background. However, at the level of the quivers, it seems not clear how noninvertible symmetries are implemented. It would be interesting to investigate how the quiver mutations (i.e., field-theory trialities) and noninvertible finite symmetries interplay.[21]

(iii) There are four fusion categories associated with the same $\mathbb{Z}_2 \times \mathbb{Z}_2$ TY fusion rule, but distinguished by their associators or F-symbols [65]. Three of those are given by the representations as Rep(D_4), Rep(Q_8), and Rep$(\mathcal{H}_8)$.[22] Identifying which corresponds to the categorical symmetry we derived in this paper would be interesting. This example may shed new light on a general question: given a categorical symmetry with certain fusion rules from string theory, is there any top-down approach to its F-symbols or (generalized) Frobenius-Schur indicators? We expect this 2D example to be a nice starting point to answer this question in diverse dimensions.

(iv) Based on the above direction, it would also be interesting to investigate anomalies and gauging of noninvertible symmetries from string theory perspectives, following the purely field-theoretic consideration [21,23,82–84].

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[19]There are also cases where this nontrivial action is engineered from purely geometric setup; see, e.g., Ref. [19].

[20]Recently, this nontrivial action is formulated in the language of the higher representation and generalized charges in Ref. [15].

[21]We thank the PRD referee for pointing out this direction.

[22]$\mathcal{H}_8$ is the 8-dimensional Kac-Paljutkin Hopf algebra [81].

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